Publ. Math. Debrecen
61 / 3-4 (2002), 645–661

On zeros of reciprocal polynomials

By PIROSKA LAKATOS (Debrecen)

Abstract. The purpose of this paper is to show that all zeros of the reciprocal polynomial

$$P_m(z) = \sum_{k=0}^{m} A_k z^k \quad (z \in \mathbb{C})$$

of degree $m \geq 2$ with real coefficients $A_k \in \mathbb{R}$ (i.e. $A_m \neq 0$ and $A_k = A_{m-k}$ for all $k = 0, \ldots, \left[\frac{m}{2}\right]$) are on the unit circle, provided that the "coefficient condition"

$$|A_m| \geq \sum_{k=1}^{m-1} |A_k - A_m|$$

is satisfied.

Moreover, if the "coefficient condition" holds, then all zeros e^{iu_j}, $(j = 1, 2, \ldots, m)$ can be arranged such that

$$\left| e^{i\frac{2\pi j}{m+1}} - e^{iu_j} \right| < \frac{\pi}{m+1} \quad (j = 1, \ldots, m).$$

If $m = 2n + 1$ is odd, then $-1 = e^{iu_{n+1}}$ is always a zero, and all zeros of P_{2n+1} are single.
If $m = 2n$ is even, if the "coefficient condition" holds with equality and if

$$\operatorname{sgn} A_{2n} = \operatorname{sgn}(-1)^{k+1}(A_k - A_{2n}) = \operatorname{sgn}(-1)^{n+1}\frac{A_n - A_{2n}}{2} \quad (k = 1, 2, \ldots, n-1),$$

then $u_n = u_{n+1} = \pi$, the number $-1 = e^{iu_n} = e^{iu_{n+1}}$ is a double zero of P_{2n}. Otherwise all zeros of P_{2n} are single.

Mathematics Subject Classification: Primary 30C15; Secondary 12D10, 42C05.
Key words and phrases: reciprocal, semi-reciprocal polynomials, Chebyshev transform, zeros on the unit circle.
Research partially supported by Hungarian NFSR grant No. TO 29525.

1. Introduction

The Coxeter transformation was introduced in the representation theory of finite dimensional algebras (see [2]). The characteristic polynomial of the Coxeter transformation of an oriented graph whose underlying graph is a wild star is a Salem polynomial (see [3], [4]).

Allowing circles in the underlying graph, the spectral properties of the Coxeter transformations get much more complicated. These properties are related to polynomials of the form

$$l(z^m + z^{m-1} + \cdots + z + 1) + \left(z^k + z^{m-k}\right) \quad (z \in \mathbb{C})$$

where m, k are fixed non-negative integers with $m \geq 2$, $1 \leq k \leq \left[\frac{m}{2}\right]$ and l is a fixed real number.

The zeros of the first expression $l(z^m + z^{m-1} + \cdots + z + 1)$ are

$$\epsilon_j = e^{i\frac{j}{m+1}2\pi} \quad (j = 1, 2, \ldots, m)$$

the $(m+1)$st roots of unity except 1, *they are on the unit circle*. It is surprising that adding $z^k + z^{m-k}$ to the first expression the polynomial obtained inherits this property. Moreover, *not just all zeros remain on the unit circle but they move away from ϵ_j just a little* even if we add a linear combination $\sum_{k=1}^{\left[\frac{m}{2}\right]} a_k(z^k + z^{m-k})$ to the expression $l(z^m + z^{m-1} + \cdots + z + 1)$, *provided that $|l|$ is large enough*. This leads to the main result of the paper: giving a sufficient condition for reciprocal polynomials to have all of their zeros on the unit circle and also giving the location of the zeros.

Our basic tool is a transformation of semi-reciprocal polynomials called the Chebyshev transformation. Although this transformation seems to be well known we could not find a suitable reference. In Section 2, based on [1], we summarize the properties of the Chebyshev transformation. In Section 3 we formulate our results and prove them. In Section 4 we discuss the necessity of our sufficient condition.

2. The Chebyshev transformation

A polynomial p of the form $p(z) = \sum_{j=0}^{m} a_j z^j$ $(z \in \mathbb{C})$ where $a_j \in \mathbb{C}$ are given numbers with $a_m \neq 0$, $a_j = a_{m-j}$ $\left(j = 0, \ldots, \left[\frac{m}{2}\right]\right)$ is called a *reciprocal polynomial of degree m*.

We need a more general class of reciprocal polynomials (of even degree).

Definition 1. A polynomial p of the form

$$p(z) = \sum_{j=0}^{2n} a_j z^j \quad (z \in \mathbb{C}) \tag{1}$$

where $n \in \mathbb{N}, a_0, \dots, a_{2n} \in \mathbb{R}$ and

$$a_j = a_{2n-j} \ (j = 0, \dots, n-1) \tag{2}$$

is called a *real semi-reciprocal polynomial* of degree at most $2n$. If $a_{2n} \neq 0$ we call p a *real reciprocal polynomial* of degree $2n$.

Denote by $\mathcal{R}_{2n}$ the set of all real semi-reciprocal polynomials of degree at most $2n$.

If $p \in \mathcal{R}_{2n}$, $p \neq o$ ($o =$ the zero polynomial), then there is an integer k, $0 \leq k \leq n$, such that

$$\begin{gathered} a_{2n} = a_{2n-1} = \cdots = a_{n+k+1} = 0 = a_{n-k-1} = \cdots = a_0 \\ \text{but} \quad a_{n+k} = a_{n-k} \neq 0. \end{gathered} \tag{3}$$

Hence

$$p(z) = \sum_{j=0}^{2n} a_j z^j = z^n \left[a_{n+k} \left(z^k + \frac{1}{z^k} \right) + \cdots + a_{n+1} \left(z + \frac{1}{z} \right) + a_n \right]. \tag{4}$$

Let T_j be the jth Chebyshev polynomial of the first kind, defined by

$$T_j(\cos x) = \cos jx \quad (j = 0, 1, \dots).$$

With $z + \frac{1}{z} = x$ we have $z^j + \frac{1}{z^j} = C_j(x)$ $(j = 1, 2, \dots)$ (see e.g. [6], p. 224) where

$$C_j(x) := 2T_j\left(\frac{x}{2}\right) \quad (x \in \mathbb{C},\ j = 1, 2, \dots)$$

are the normalized Chebyshev polynomials of the first kind. For us it will be now more convenient to define C_0 by

$$C_0(x) := T_0(x) \quad (x \in \mathbb{C}).$$

Hence, from (4)

$$p(z) = z^n \sum_{j=0}^{k} a_{n+j} C_j(x) = a_{n+k}\, z^n \prod_{j=1}^{k} (x - \alpha_j) \tag{5}$$

where $\alpha_j \in \mathbb{C}$ $(j = 1, \ldots, k)$ are the zeros of the polynomial $\sum_{j=0}^{k} a_{n+j} C_j(x)$. Equation (5) remains true in the case when $k = 0$, i.e. $p(z) = a_n z^n$ if we agree that

$$\prod_{j=1}^{0} b_j := 1. \tag{6}$$

Going back to the variable z we get that

$$p(z) = a_{n+k}\, z^{n-k} \prod_{j=1}^{k} z \left(z + \frac{1}{z} - \alpha_j \right) = a_{n+k}\, z^{n-k} \prod_{j=1}^{k} (z^2 - \alpha_j\, z + 1).$$

With this we have justified

Proposition 1. *Every non-zero polynomial $p \in \mathcal{R}_{2n}$ has the decomposition*

$$p(z) = a_{n+k}\, z^{n-k} \prod_{j=1}^{k} (z^2 - \alpha_j\, z + 1) \tag{7}$$

where $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$, $a_{n+k} \neq 0$ for some k with $0 \leq k \leq n$ and the convention (6) *is adopted. If $p \in \mathcal{R}_{2n}$ is a reciprocal polynomial of degree $2n$, then* (7) *holds with $k = n$.*

Definition 2. The Chebyshev transform of a non-zero polynomial $p \in \mathcal{R}_{2n}$ having the decomposition (7) is defined by

$$\mathcal{T}p(x) = a_{n+k} \prod_{j=1}^{k} (x - \alpha_j) \tag{8}$$

(with (6) adopted) while for the zero polynomial $p = o$ let

$$\mathcal{T}o(x) = 0. \tag{9}$$

It is clear that $\mathcal{T}$ maps $\mathcal{R}_{2n}$ into the set $\mathcal{P}_n$ of all polynomials of degree $\leq n$ with real coefficients.

Proposition 2. *The Chebyshev transform $\mathcal{T}$ is an isomorphism of the (real) vector space $\mathcal{R}_{2n}$ onto $\mathcal{P}_n$.*

PROOF. (i) *$\mathcal{T}$ preserves the addition and the multiplication by a real constant.* Using (5) and (3) (to include also the zero coefficients into the sum) we can write $\mathcal{T}p$ into the form

$$\mathcal{T}p(x) = a_{n+k}\prod_{j=1}^{k}(x-\alpha_j) = \sum_{j=0}^{k} a_{n+j}C_j(x) = \sum_{j=0}^{n} a_{n+j}C_j(x)$$

and the last form of $\mathcal{T}p$ is valid also for the zero polynomial. Taking now another $q \in \mathcal{R}_{2n}$ with $q(z) = \sum_{j=0}^{2n} b_j z^j$ ($b_j = b_{2n-j}$ for $j = 0, \dots, n-1$) and constants $\alpha, \beta \in \mathbb{R}$ we have

$$(\alpha p + \beta q)(z) = \sum_{j=0}^{2n}(\alpha a_j + \beta b_j)z^j$$

thus

$$\begin{aligned}\mathcal{T}(\alpha p + \beta q)(x) &= \sum_{j=0}^{n}(\alpha a_{n+j} + \beta b_{n+j})C_j(x)\\ &= \alpha\sum_{j=0}^{n} a_{n+j}C_j(x) + \beta\sum_{j=0}^{n} b_{n+j}C_j(x) = \alpha\left(\mathcal{T}p(x)\right) + \beta\left(\mathcal{T}q(x)\right).\end{aligned}$$

(ii) *$\mathcal{T}$ maps onto $\mathcal{P}_n$.* Every polynomial $\tilde{r} \in \mathcal{P}_n$ can uniquely be written as a (real) linear combination of $C_0, C_1, \dots, C_n$ in the form $\tilde{r}(x) = \sum_{j=0}^{n} A_{n+j}C_j(x)$ ($A_{n+j} \in \mathbb{R}$). With $r(z) := \sum_{j=0}^{2n} A_j z^j$ where $A_j := A_{2n-j}$ for $j = 0, \dots, n-1$ we have $r \in \mathcal{R}_{2n}$ and $\mathcal{T}r = \tilde{r}$ proving our claim.

(iii) *$\mathcal{T}$ is one-to-one.* Namely, if $\mathcal{T}p = \mathcal{T}q$ for $p, q \in \mathcal{R}_{2n}$, then $\mathcal{T}p - \mathcal{T}q = \mathcal{T}(p-q) = o$ hence, by (8), (9) $p - q = o$, $p = q$. □

Lemma 1. (i) *Let p be a real reciprocal polynomial of degree $2n$. Then all zeros of p are on the unit circle if and only if all zeros of its Chebyshev transform $\mathcal{T}p$ are in the closed interval $[-2, 2]$.*

(ii) *Moreover, if all zeros α_j of $\mathcal{T}p$ are in $[-2, 2]$, written as $\alpha_j = 2\cos u_j$ with $u_j \in [0, \pi]$ ($j = 1, 2, \dots, n$), then all zeros of p are given by*

$$e^{\pm i u_j} \quad (j = 1, 2, \dots, n).$$

The multiplicity of $\alpha_j \neq \pm 2$ is the same as the multiplicities of e^{iu_j} and e^{-iu_j} $(j = 1, 2, \dots, n)$ while in the case of $\alpha_j = \pm 2$ the multiplicities of the corresponding zeros $e^{iu_j} = \pm 1$ of p are doubled.

PROOF. (i) *Necessity.* Suppose that all zeros of p are on the unit circle. They can be arranged in conjugate pairs $(\beta_1, \bar{\beta}_1), (\beta_2, \bar{\beta}_2) \dots (\beta_n, \bar{\beta}_n)$. By assumption $|\beta_j|^2 = \beta_j \bar{\beta}_j = 1$, $\bar{\beta}_j = \frac{1}{\beta_j}$ $(j = 1, \dots n)$, hence

$$p(z) = a_{2n} \prod_{j=1}^{n} (z - \beta_j)(z - \bar{\beta}_j) = a_{2n} \prod_{j=1}^{n} \left(z^2 - (\beta_j + \bar{\beta}_j) z + 1 \right)$$

and

$$\mathcal{T}p(x) = a_{2n} \prod_{j=1}^{n} \left(x - (\beta_j + \bar{\beta}_j) \right).$$

It is clear that $|\beta_j + \bar{\beta}_j| = |2\mathrm{Re}\,(\beta_j)| \leq 2|\beta_j| = 2$.

(i) *Sufficiency.* Assume that the Chebyshev transform has the form

$$\mathcal{T}p(x) = a_{2n} \prod_{j=1}^{n} (x - \alpha_j)$$

where $a_{2n} \neq 0$ and $\alpha_j \in [-2, 2]$ $(j = 1, \dots, n)$. Then

$$p(z) = a_{2n} \prod_{j=1}^{n} \left(z^2 - \alpha_j z + 1 \right).$$

Since $\alpha_j \in [-2, 2]$ we have $z^2 - \alpha_j z + 1 = (z - \beta_j)(z - \bar{\beta}_j)$ with $\beta_j \bar{\beta}_j = 1 = |\beta_j|^2$ proving that all zeros $\beta_1, \bar{\beta}_1, \beta_2, \bar{\beta}_2 \dots \beta_n, \bar{\beta}_n$ of p are on the unit circle.

(ii) We have $\alpha_j = 2 \cos u_j = \beta_j + \bar{\beta}_j$. Writing β_j as e^{iq_j} (here we may suppose that $0 \leq q_j \leq \pi$) we obtain that $2 \cos u_j = e^{iq_j} + e^{-iq_j} = 2 \cos q_j$ hence $u_j = q_j$ $(j = 1, 2, \dots, n)$. The statement concerning the multiplicities is obvious. □

3. Results and proofs

Theorem 1. *All zeros of the (real reciprocal) polynomial*

$$h_m(z) = l(z^m + z^{m-1} + \dots + z + 1) + \sum_{k=1}^{[\frac{m}{2}]} a_k \left(z^{m-k} + z^k \right) \quad (z \in \mathbb{C}) \tag{10}$$

of degree m where $l, a_1, \dots, a_{[\frac{m}{2}]} \in \mathbb{R}$, $l \neq 0$, $m \in \mathbb{N}$, $m \geq 2$, are on the unit circle if

$$|l| \geq 2 \sum_{k=1}^{[\frac{m}{2}]} |a_k|. \tag{11}$$

Moreover, if (11) is satisfied, then for even $m = 2n$ all zeros of h_m can be given as

$$e^{iu_j}, \quad e^{-iu_j} \quad (j = 1, 2, \dots, n)$$

where

$$\frac{j - \frac{1}{2}}{m+1} 2\pi < u_j < \frac{j + \frac{1}{2}}{m+1} 2\pi \quad (j = 1, 2, \dots, n-1)$$

$$\frac{n - \frac{1}{2}}{m+1} 2\pi < u_n \leq \pi.$$

In the last inequality $u_n \leq \pi$, we have equality if and only if

$$|l| = 2 \sum_{k=1}^{[\frac{m}{2}]} |a_k| \quad \text{and} \quad \operatorname{sgn} l = \operatorname{sgn}(-1)^{k+1} \operatorname{sgn} a_k \tag{12}$$

$$\text{for all } k = 1, 2, \dots, n.$$

If (12) holds, then $-1 = e^{i\pi} = e^{-i\pi}$ is a double zero of h_m and all other zeros are single.

For odd $m = 2n + 1$ all zeros of h_m are single, they can be given as

$$-1, \ e^{iu_j}, \ e^{-iu_j} \quad (j = 1, 2, \dots, n)$$

where

$$\frac{j - \frac{1}{2}}{m+1} 2\pi < u_j < \frac{j + \frac{1}{2}}{m+1} 2\pi \quad (j = 1, 2, \dots, n).$$

Remark 1. The statement concerning the location of the zeros of h_m can also be formulated as follows.

If (11) is satisfied, then all the zeros e^{iu_j} $(j = 1, 2, \dots, m)$ of h_m can be arranged such that

$$\left| \epsilon_j - e^{iu_j} \right| < \frac{\pi}{m+1} \quad (j = 1, \dots, m)$$

where, as in the introduction, ϵ_j are the $(m+1)$st roots of unity, except 1.

Namely, for even $m = 2n$, let u_j $(j = 1, 2, \dots, n)$ be the same as in Theorem 1 and $u_{n+j} := 2\pi - u_{n+1-j}$ $(j = 1, 2, \dots, n)$. If (12) does not hold, then all zeros of h_m are single. If (12) holds, then $u_n = u_{n+1} = \pi$ and $-1 = e^{iu_n} = e^{iu_{n+1}}$ is a double zero and all other zeros are single.

For odd $m = 2n + 1$ let u_j $(j = 1, 2, \dots, n)$ be the same as in Theorem 1, $u_{n+1} := \pi$ and $u_{n+1+j} := 2\pi - u_{n+1-j}$ $(j = 1, 2, \dots, n)$. The number $-1 = e^{iu_{n+1}}$ is always a zero and all zeros are single.

Proof. The basic idea of our proof is the following. Assume that (11) holds and let

$$x_j = 2\cos\frac{j+\frac{1}{2}}{m+1}\,2\pi \quad \left(j = 0, \dots, \left[\tfrac{m}{2}\right]\right).$$

If $m = 2n$ is an even number, we show that $\operatorname{sgn} \mathcal{T}h_{2n}(x_j) = \operatorname{sgn}(-1)^j \operatorname{sgn} l$ $(j = 0, 1, \dots, n-1)$ and $\mathcal{T}h_{2n}(x_n) = 0$ if (12) holds, otherwise $\operatorname{sgn} \mathcal{T}h_{2n}(x_j) = \operatorname{sgn}(-1)^j \operatorname{sgn} l$ $(j = 0, \dots, n)$.

If $m = 2n + 1$ is odd, then $h_{2n+1}(z) = (z+1)\bar{h}_{2n}(z)$ with a suitable reciprocal polynomial $\bar{h}_{2n}$ from $\mathcal{R}_{2n}$. We show that $\operatorname{sgn} \mathcal{T}\bar{h}_{2n}(x_j) = \operatorname{sgn} l \operatorname{sgn}(-1)^j$ $(j = 0, 1, \dots, n)$.

Applying Lemma 1 completes the proof.

Case 1: $m = 2n$. With the notation $v_j(z) = z^j + z^{j-1} + \dots + 1 = \frac{z^{j+1}-1}{z-1}$, $e_j(z) = z^j$, $w_j(z) = z^j + 1$ $(j = 0, 1, \dots)$ we have

$$h_{2n}(z) = lv_{2n}(z) + \sum_{k=1}^{n} a_k e_k(z) \cdot w_{2n-2k}(z),$$

$$\mathcal{T}h_{2n}(x) = l\mathcal{T}v_{2n}(x) + \sum_{k=1}^{n} a_k \mathcal{T}(e_k \cdot w_{2n-2k})(x).$$

The zeros of v_{2n} are the $(2n+1)$st roots of unity, except 1: $e^{\frac{2j\pi i}{2n+1}}$ $(j = 1, 2, \dots, 2n)$. They can be arranged into conjugate pairs: $\left(e^{\frac{2j\pi i}{2n+1}}, e^{\frac{2(2n+1-j)\pi i}{2n+1}}\right) = \left(e^{\frac{2j\pi i}{2n+1}}, e^{-\frac{2j\pi i}{2n+1}}\right)$ $(j = 1, \dots, n)$, thus

$$v_{2n}(z) = \prod_{j=1}^{2n}\left(z - e^{\frac{2j\pi i}{2n+1}}\right) = \prod_{j=1}^{n}\left(z - e^{\frac{2j\pi i}{2n+1}}\right)\left(z - e^{-\frac{2j\pi i}{2n+1}}\right)$$

$$= \prod_{j=1}^{n} \left(z^2 - 2\cos\frac{2j\pi}{2n+1} z + 1 \right),$$

$$\mathcal{T} v_{2n}(x) = \prod_{j=1}^{n} \left(x - 2\cos\frac{2j\pi}{2n+1} \right).$$

Similarly, for each $0 \le k \le n$ the zeros of w_{2n-2k} are the $(2n-2k)$ st roots of $-1 : e^{\frac{(2j-1)\pi i}{2n-2k}}$ $(j = 1, \dots, 2n-2k)$. They can be arranged into conjugate pairs

$$\left(e^{\frac{(2j-1)\pi i}{2n-2k}}, e^{\frac{(2(2n-2k+1-j)-1)\pi i}{2n-2k}} \right) = \left(e^{\frac{(2j-1)\pi i}{2n-2k}}, e^{-\frac{(2j-1)\pi i}{2n-2k}} \right) \quad (j = 1, \dots, n-k).$$

Therefore

$$w_{2n-2k}(z) = \prod_{j=1}^{2n-2k} \left(z - e^{\frac{(2j-1)\pi i}{2n-2k}} \right) = \prod_{j=1}^{n-k} \left(z^2 - 2\cos\frac{(2j-1)\pi}{2n-2k} z + 1 \right),$$

$$\mathcal{T}(e_k w_{2n-2k})(x) = \prod_{j=1}^{n-k} \left(x - 2\cos\frac{(2j-1)\pi}{2n-2k} \right).$$

Denote by U_n the nth Chebyshev polynomial of the second kind (see for example in [6]), defined by

$$U_n(\cos x) = \frac{\sin(n+1)x}{\sin x} \quad (n = 0, 1, \dots).$$

We claim that

$$\mathcal{T} v_{2n}(x) = U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right), \tag{13}$$

$$\mathcal{T}(e_k \cdot w_{2n-2k})(x) = 2T_{n-k}\left(\frac{x}{2}\right). \tag{14}$$

To justify the first identity we note that

$$\begin{aligned} U_n(\cos y) + U_{n-1}(\cos y) &= \frac{\sin(n+1)y + \sin ny}{\sin y} \\ &= 2\frac{\sin\frac{(2n+1)y}{2}\cos\frac{y}{2}}{\sin y} = \frac{\sin\frac{(2n+1)y}{2}}{\sin\frac{y}{2}}. \end{aligned} \tag{15}$$

The right hand side is zero if and only if $y = \frac{2j\pi}{2n+1}$ $(j \in \mathbb{Z} \setminus \{0\})$ hence all zeros of $U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right)$ are $2\cos\frac{2j\pi}{2n+1}$ $(j = 1, \dots, n)$. Since both sides of (13) are monics which have the same zeros, they are identical.

The zeros of T_p can be calculated easily from their definition, for $p \in \mathbb{N}$ they are

$$\cos\frac{(2j-1)\pi}{2p} \quad (j = 1, \dots, p).$$

Thus for $k < n$ the zeros of the monic $2T_{n-k}\left(\frac{x}{2}\right)$ are $2\cos\frac{(2j-1)\pi}{2n-2k}$ $(j = 1, \dots, n-k)$. They are the same as the zeros of $\mathcal{T}(e_k \cdot w_{2n-2k})$, hence (14) holds. It also holds for $k = n$ since then both sides of (14) are equal to 2.

Next we evaluate $\mathcal{T}h_{2n}$ at the points

$$x_j = 2\cos\frac{j+\frac{1}{2}}{m+1}\,2\pi \quad (j = 0, \dots, n)$$

of the interval $[-2, 2]$. Since $x_j = 2\cos y_j$ with $y_j = \frac{j+\frac{1}{2}}{2n+1}\,2\pi$ we have by (13), (14)

$$\begin{aligned}\mathcal{T}h_{2n}(x_j) &= l\left(U_n\left(\frac{x_j}{2}\right) + U_{n-1}\left(\frac{x_j}{2}\right)\right) + \sum_{k=1}^{n} 2a_k T_{n-k}\left(\frac{x_j}{2}\right)\\ &= 2\left[\frac{\frac{l}{2}\,\sin\frac{2n+1}{2}y_j}{\sin\frac{1}{2}y_j} + \sum_{k=1}^{n} a_k\cos(n-k)y_j\right]\\ &= 2\left[\frac{\frac{l}{2}(-1)^j}{\sin\frac{y_j}{2}} + \sum_{k=1}^{n} a_k\cos(n-k)y_j\right].\end{aligned}$$

If $j = 0, 1, \dots, n-1$, then $0 < \sin\frac{y_j}{2} < 1$, $\sum_{k=1}^{n}|a_k\cos(n-k)y_j| \le \sum_{k=1}^{n}|a_k|$ and by (11) the sign of the expression in the bracket is $(-1)^j \operatorname{sgn} l$.

If $j = n$, then $y_n = \pi$ and the expression in the bracket is

$$\frac{l}{2}(-1)^n + \sum_{k=1}^{n} a_k(-1)^{n-k} = (-1)^n\left(\frac{l}{2} + \sum_{k=1}^{n} a_k(-1)^k\right).$$

Its sign is $(-1)^n \operatorname{sgn} l$ if in (11) strict inequality holds or if in (11) we have equality and at least for one k $(1 \le k \le n)$ we have $\operatorname{sgn} l = \operatorname{sgn}(-1)^k \operatorname{sgn} a_k$. If we have equality in (11) and $\operatorname{sgn} l = \operatorname{sgn}(-1)^{k+1} \operatorname{sgn} a_k$ for all $k = 1, \dots, n$, then the expression in the bracket is zero.

Thus either $\operatorname{sgn} \mathcal{T}h_{2n}(x_j) = \operatorname{sgn}(-1)^j \operatorname{sgn} l$ $(j = 0, \dots, n)$ or $\operatorname{sgn} \mathcal{T}h_{2n}(x_j) = \operatorname{sgn}(-1)^j \operatorname{sgn} l$ $(j = 0, 1, \dots, n-1)$ and $\mathcal{T}h_{2n}(x_n) = 0$. In both cases $\mathcal{T}h_{2n}$ has n distinct zeros in the interval $[-2, 2]$. Writing these in the form $2\cos u_j$ with $0 \le u_1 \le u_2 \le \cdots \le u_n \le \pi$ and applying Lemma 1 we can complete the proof in the first case.

Case 2: $m = 2n+1$. We have $h_{2n+1}(z) = (z+1)\bar{h}_{2n}(z)$ with

$$\bar{h}_{2n}(z) = l\bar{v}_{2n}(z) + \sum_{k=1}^{n} a_k z^k \bar{w}_{2n-2k}(z)$$

where

$$\bar{v}_{2n}(z) = z^{2n} + z^{2n-2} + \cdots + z^2 + 1 = v_n(z^2),$$

$$\bar{w}_{2n-2k}(z) = \frac{w_{2n+1-2k}(z)}{z+1} = \frac{z^{2n+1-2k}+1}{z+1}.$$

Using the factorization of v_n we get

$$\bar{v}_{2n}(z) = v_n(z^2) = \prod_{j=1}^{n} \left(z^2 - e^{\frac{2j\pi i}{n+1}}\right) = \prod_{j=1}^{n} \left(z - e^{\frac{j\pi i}{n+1}}\right)\left(z - e^{\frac{j\pi i}{2n+1} - \pi i}\right).$$

Arranging the zeros of $\bar{v}_{2n}$ into conjugate pairs $\left(e^{\frac{j\pi i}{n+1}}, e^{-\frac{j\pi i}{n+1}}\right)$ $(j = 1, \dots, n)$ we have

$$\bar{v}_{2n}(z) = \prod_{j=1}^{n} \left(z - e^{\frac{j\pi i}{n+1}}\right)\left(z - e^{-\frac{j\pi i}{n+1}}\right) = \prod_{j=1}^{n} \left(z^2 - 2\cos\frac{2j\pi}{2n+1} z + 1\right)$$

therefore

$$\mathcal{T}\bar{v}_{2n}(x) = \prod_{j=1}^{n} \left(x - 2\cos\frac{j\pi}{n+1}\right).$$

We can easily calculate the zeros of $\bar{w}_{2n-2k}$ (we omit this elementary calculation) and obtain the factorization

$$\begin{aligned}\bar{w}_{2n-2k}(z) &= \prod_{j=1}^{n-k} \left(z - e^{\frac{(2j-1)\pi i}{2n-2k+1}}\right)\left(z - e^{-\frac{(2j-1)\pi i}{2n-2k+1}}\right) \\ &= \prod_{j=1}^{n-k} \left(z^2 - 2\cos\frac{(2j-1)\pi}{2n-2k+1} z + 1\right)\end{aligned}$$

therefore

$$\mathcal{T}(e_k \cdot \bar{w}_{2n-2k})(x) = \prod_{j=1}^{n-k} \left(x - 2\cos\frac{(2j-1)\pi}{2n-2k+1} \right).$$

Next we show that

$$\mathcal{T}\bar{v}_{2n}(x) = U_n\left(\frac{x}{2}\right), \tag{16}$$

$$\mathcal{T}(e_k \cdot \bar{w}_{2n-2k})(x) = U_{n-k}\left(\frac{x}{2}\right) - U_{n-k-1}\left(\frac{x}{2}\right). \tag{17}$$

where we have to adopt the convention

$$U_{-1}(x) = 0 \quad (x \in \mathbb{C}). \tag{18}$$

The first identity follows from the fact that the zeros of both sides are the same.

To justify the second we note that

$$\begin{aligned} U_{n-k}(\cos y) - U_{n-k-1}(\cos y) &= \frac{\sin(n-k+1)y - \sin(n-k)y}{\sin y} \\ &= \frac{2\cos\frac{(2n-2k+1)y}{2}\sin\frac{y}{2}}{\sin y} = \frac{\cos\frac{(2n-2k+1)y}{2}}{\cos\frac{y}{2}} \end{aligned}$$

for all $k = 0, \ldots, n$ provided that the convention (18) is adopted.

If $k = n$, then both sides of (17) are equal to 1 thus (17) holds. For $k < n$ the right hand side of (17) is zero if and only if $y = \frac{(2j-1)\pi}{2n-2k+1}$ $(j \in \mathbb{Z})$ hence all zeros of $U_{n-k}\left(\frac{x}{2}\right) - U_{n-k-1}\left(\frac{x}{2}\right)$ are $2\cos\frac{(2j-1)\pi}{2n-2k+1}$ $(j = 1, \ldots, n-k)$, they are the same as the zeros of $\mathcal{T}(e_k \cdot \bar{w}_{2n-2k})$ proving (17).

By the linearity of the Chebyshev transform and by (16), (17) we have

$$\begin{aligned} \mathcal{T}\bar{h}_{2n}(x) &= l\mathcal{T}\bar{v}_{2n}(x) + \sum_{k=1}^{n} a_k \mathcal{T}(e_k \cdot \bar{w}_{2n-2k})(x) \\ &= lU_n\left(\frac{x}{2}\right) + \sum_{k=1}^{n} a_k \left[U_{n-k}\left(\frac{x}{2}\right) - U_{n-k-1}\left(\frac{x}{2}\right) \right]. \end{aligned}$$

Next we evaluate $\mathcal{T}\bar{h}_{2n}$ at the points

$$\bar{x}_j = x_j = 2\cos\frac{j+\frac{1}{2}}{2n+2}\,2\pi \quad (j=0,\dots,n)$$

of the interval $[-2,2]$. Since $\bar{x}_j = 2\cos\bar{y}_j$ with $\bar{y}_j = \frac{j+\frac{1}{2}}{2n+2}\,2\pi$ we have

$$\begin{aligned}\mathcal{T}\bar{h}_{2n}(\bar{x}_j) &= 2\left[\frac{l}{2}\frac{\sin(n+1)\bar{y}_j}{\sin\bar{y}_j} + \frac{\sum_{k=1}^{n} a_k\cos\frac{2n-2k+1}{2}\bar{y}_j}{2\cos\frac{\bar{y}_j}{2}}\right]\\ &= 2\left[\frac{l}{2}\frac{(-1)^j}{\sin\bar{y}_j} + \sum_{k=1}^{n} a_k\frac{\cos\frac{2n-2k+1}{2}\bar{y}_j}{2\cos\frac{\bar{y}_j}{2}}\right]\\ &= 2\frac{\frac{l}{2}(-1)^j + \sum_{k=1}^{n} a_k\sin\frac{\bar{y}_j}{2}\cos\frac{2n-2k+1}{2}\bar{y}_j}{\sin\bar{y}_j}.\end{aligned}$$

Since $\bar{y}_j \in\,]0,\pi[$ we have $\sin\bar{y}_j > 0$, $0 < \sin\frac{\bar{y}_j}{2} < 1$, $|\cos\frac{2n-2k+1}{2}\bar{y}_j| \le 1$ for all $k=1,\dots,n$ therefore the sign of the expression in the bracket is $\operatorname{sgn} l\,\operatorname{sgn}(-1)^j$. Thus $\operatorname{sgn}\big(\mathcal{T}\bar{h}_{2n}(x_j)\big) = \operatorname{sgn} l\,\operatorname{sgn}(-1)^j$ $(j=0,1,\dots,n)$ proving that $\mathcal{T}\bar{h}_{2n}$ has n different zeros in $[-2,2]$. Writing these zeros in the form $2\cos u_j$ with $0 \le u_1 \le u_2 \le \cdots \le u_n \le \pi$ and applying Lemma 1 the proof is completed in the second case as well. □

We can formulate Theorem 1 in a more symmetric way. This formulation explains, in a certain way, the appearance of the factor 2 in (11).

Theorem 2. *All zeros of the reciprocal polynomial*

$$P_m(z) = \sum_{k=0}^{m} A_k z^k \quad (z\in\mathbb{C}) \tag{19}$$

of degree $m \ge 2$ with real coefficients $A_k \in \mathbb{R}$ (i.e. $A_m \ne 0$ and $A_k = A_{m-k}$ for all $k = 0,\dots,\left[\frac{m}{2}\right]$) are on the unit circle, provided that

$$|A_m| \ge \sum_{k=1}^{m-1} |A_k - A_m|. \tag{20}$$

If (20) holds, then all zeros e^{iu_j} $(j=1,2,\dots,m)$ of P_m can be arranged such that

$$\left|\epsilon_j - e^{iu_j}\right| < \frac{\pi}{m+1} \quad (j=1,\dots,m).$$

If $m = 2n + 1$ is odd, then $-1 = e^{iu_{n+1}}$ is always a zero and all zeros of P_m are single.

If $m = 2n$ is even

$$(21)\qquad \begin{cases} |A_{2n}| = \sum\limits_{k=1}^{2n-1} |A_k - A_{2n}| \quad \textit{and} \\ \operatorname{sgn} A_{2n} = \operatorname{sgn}(-1)^{k+1}(A_k - A_{2n}) = \operatorname{sgn}(-1)^{n+1}\dfrac{A_n - A_{2n}}{2} \\ \qquad (k = 1, 2, \ldots, n-1) \end{cases}$$

holds, then $u_n = u_{n+1} = \pi$, the number $-1 = e^{iu_n} = e^{iu_{n+1}}$ is a double zero of P_m and all other zeros are single. Otherwise (i.e. if $m = 2n$, (21) does not hold) all zeros of P_m are single.

Proof. Comparing the coefficients of z^j in h_m and P_m we see that for even $m = 2n$

$$A_{2n} = A_0 = l,\ A_{2n-1} = A_1 = l + a_1, \ldots,\ A_{n+1} = A_{n-1} = l + a_{n-1},$$
$$A_n = l + 2a_n$$

thus $l = A_{2n}$, $a_k = A_{2n-k} - A_{2n} = A_k - A_{2n}$ for $k = 1, 2, \ldots, n-1$ and $2a_n = A_n - A_{2n}$. Therefore the condition (11)

$$|l| \geq 2\sum_{k=1}^{n} |a_k|$$

can be written as

$$|A_{2n}| \geq 2\sum_{k=1}^{n-1} |A_k - A_{2n}| + |A_n - A_{2n}| = \sum_{k=1}^{2n-1} |A_k - A_{2n}|$$

which is the same as (20).

For odd $m = 2n + 1$ the comparison of the coefficients gives that

$$A_{2n+1} = A_0 = l,\ A_{2n} = A_1 = l + a_1, \ldots,\ A_{n+1} = A_n = l + a_n$$

thus $l = A_{2n+1}$, $a_k = A_{2n+1-k} - A_{2n+1} = A_k - A_{2n+1}$ for $k = 1, 2, \ldots, n$

and (11) can be written as

$$|A_{2n+1}| \geq 2\sum_{k=1}^{n}|A_k - A_{2n+1}| = \sum_{k=1}^{n}\left(|A_k - A_{2n+1}| + |A_{2n+1-k} - A_{2n+1}|\right)$$
$$= \sum_{k=1}^{2n}|A_k - A_{2n+1}|$$

proving (20). The statement concerning the location of the zeros follows from Remark 1. □

4. Necessary and sufficient conditions

If the degree m of P_m is small we can easily obtain necessary and sufficient conditions for all zeros of P_m to be on the unit circle.

If $m = 2$, then $P_2(z) = A_2z^2 + A_1z + A_2 = z\left(A_2(z + \frac{1}{z}) + A_1\right)$ hence $\mathcal{T}P_2(x) = A_2x + A_1$. The only zero of $\mathcal{T}P_2$ is in $[-2, 2]$ if and only if

$$|A_2| \geq \frac{1}{2}|A_1|. \tag{22}$$

This is the criteria for P_2 to have all zeros on the unit circle.

If $m = 3$, then $P_3(z) = A_3z^3 + A_2z^2 + A_2z + A_3 = (z + 1)(A_3z^2 + (A_2 - A_3)z + A_3)$. By (22) the zeros of P_3 are on the unit circle if and only if

$$|A_3| \geq \frac{1}{2}|A_2 - A_3|. \tag{23}$$

If $m = 4$, then $P_4(z) = A_4z^4 + A_3z^3 + A_2z^2 + A_3z + A_4 = z^2(A_4(z^2 + \frac{1}{z^2}) + A_3(z + \frac{1}{z}) + A_2)$ hence with $x = z + \frac{1}{z}$ we get that $\mathcal{T}P_4(x) = A_4(x^2 - 2) + A_3x + A_2$. By Lemma 1 all zeros of P_4 are on the unit circle if and only if the discriminant of $\mathcal{T}P_4$ is non-negative:

$$A_3^2 - 4A_4(A_2 - 2A_4) \geq 0 \tag{24}$$

and

$$-2 \leq x_1, \quad x_2 \leq 2 \tag{25}$$

hold where $x_1 \le x_2$ are the real zeros of $\mathcal{T}P_4$. A simple calculation shows that (24) and (25) are equivalent to

$$2\sqrt{\max\{A_2A_4 - 2A_4^2, 0\}} \le |A_3| \le \min\left\{4|A_4|, |A_4| + \frac{1}{2}A_2 \operatorname{sgn} A_4\right\}. \tag{26}$$

This is the criterion for P_4 to have all of its zeros on the unit circle.

For $m = 2$ (22) holds if and only if

$$A_1 \in [-2|A_2|, 2|A_2|]$$

while (20) gives only the smaller interval

$$A_1 \in [A_2 - |A_2|, A_2 + |A_2|]\,.$$

This shows that (20) for $m = 2$ is not necessary. The situation is similar for $m = 3$.

For $m = 4$ the necessary and sufficient condition (26) is non-linear in the coefficients, while our sufficient condition (20) is linear for all $m \ge 2$. In some special cases we get necessary and sufficient conditions.

Corollary 1. *All zeros of the polynomial*

$$l(z^m + z^{m-1} + \cdots + z + 1) + \left(z^k + z^{m-k}\right) \quad (z \in \mathbb{C})$$

where m, k are fixed non-negative integers with $m \ge 2$, $1 \le k \le \left[\frac{m}{2}\right]$ and l is a fixed positive number, are on the unit circle for all $m \ge 2$ and for all $k = 1, 2, \ldots, \left[\frac{m}{2}\right]$ if and only if $l \ge 2$.

Namely, taking $m = 2$, $k = 1$ by (22) all zeros of the resulting polynomial $lz^2 + (l + 2)z + l$ are on the unit circle if and only if $l \notin \left(-\frac{2}{3}, 2\right)$ therefore $l \ge 2$. On the other hand if $l \ge 2$, then by Theorem 1 all zeros of the polynomial $l(z^m + z^{m-1} + \cdots + z + 1) + \left(z^k + z^{m-k}\right)$ are on the unit circle.

Remark 2. A preliminary version of some parts of this paper was reported in [5].

A. Schinzel [7] generalized Theorem 2 to the case of self-inversive polynomials over $\mathbb{C}$, i.e. polynomials $P_m(z) = \sum_{k=0}^{n} A_k z^k$ for which $A_k \in \mathbb{C}$,

$A_m \neq 0$, $\epsilon \bar{A}_k = A_{m-k}$ for all $k = 0, \ldots, m$ with a fixed $\epsilon \in \mathbb{C}$, $|\epsilon| = 1$. He proved that all zeros of P_m are on the unit circle, provided that

$$|A_m| \geq \inf \sum_{k=0}^{m} |cA_k - d^{m-j} A_m|,$$

where the infimum is taken over all $c, d \in \mathbb{C}$ and $|d| = 1$.

References

[1] J. W. Cannon and P. Wagreich, Geowth function of surface groups, *Math. Ann.* **293** (1992), 239–257.

[2] V. Dlab and C. M. Ringel, Indecomposable representations of graphs and algebras, *Memoirs of the Amer. Math. Soc.* **173** (1976), 1–57.

[3] J. F. McKee, P. Rowlinson and C. J. Smyth, Salem numbers and Pisot numbers from stars, Number theory in progress, Proc. Internat. Conf. Banach Internat. Math. Center, vol 1: Diophantine problems and polynomials (K. Győry, *et al.*, eds.), *de Gruyter*, *Berlin*, 1999.

[4] P. Lakatos, Salem numbers, PV numbers and spectral radii of Coxeter transformations, *C. R. Math. Acad. Sci. Soc. R. Can.* **23** no. 3 (2001), 71–77.

[5] P. Lakatos, On polynomials having zeros on the unit circle, *C. R. Math. Acad. Sci. Soc. R. Can.* **24** no. 2 (2002), 91–96.

[6] T. J. Rivlin, Chebyshev polynomials, *A Wiley-Interscience Publication*, 1990.

[7] A. Schinzel, Self-inversive polynomials with all zeros on the unit circle (*to appear*).

PIROSKA LAKATOS
INSTITUTE OF MATHEMATICS AND INFORMATICS
DEBRECEN UNIVERSITY
4010 DEBRECEN, P.O. BOX 12
HUNGARY

E-mail: lapi@math.klte.hu

(Received March 7, 2002; revised August 27, 2002)