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On the range of elementary operators

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Abstract. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of operators on $\mathcal{H}$ into itself. For a two-sided ideal $\mathcal{J} \subset \mathcal{B}(\mathcal{H})$ with unitarily invariant norm and an elementary operator $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ we consider the question when $\overline{\operatorname{ran}(\mathcal{E}|_{\mathcal{J}})}^{\mathcal{J}} = \overline{\operatorname{ran}\mathcal{E} \cap \mathcal{J}}^{\mathcal{J}}$. We prove that this holds when: (i) $\mathcal{E}(X) = AXB$ and $\mathcal{J}$ is separable: (ii) $\mathcal{E}(X) = AXB + CXD$, where A and C, respectively B and D are commuting normal operators and $\mathcal{J} = \mathcal{C}_p$ with $p > 1$; (iii) $\mathcal{E}(X) = \sum A_i X B_i$, where $\boldsymbol{A} = (A_1, \dots, A_n)$ and $\boldsymbol{B} = (B_1, \dots, B_n)$ are n-tuples of mutually commuting normal operators and $\mathcal{J} = \mathcal{C}_2$. Finally, as an application of (iii) we prove some results about double operator integrals.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on an infinite dimensional complex separable Hilbert space $\mathcal{H}$. For a compact operator X let $s_1(X) \geq s_2(X) \geq \dots$ denote the singular values of X, i.e., the eigenvalues of $|X| = \sqrt{X^*X}$, arranged in a non-increasing order, with their multiplicities counted. A unitarily invariant norm is any norm $|||\cdot|||$ defined on some two-sided ideal $\mathcal{J} \subseteq \mathcal{B}(\mathcal{H})$ which satisfies the following two conditions. For unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ the equality

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$\|UXV\| = \|X\|$ holds, and $\|X\| = s_1(X)$ for all rank one operators. All ideals considered in this note are two-sided with unitarily invariant norms; as usual we call them norm ideals. It is a well-known fact that there is a one-to-one correspondence between symmetric gauge functions defined on sequences of real numbers and unitarily invariant norms defined on ideals of operators. More precisely, if $\|\cdot\|$ is a unitarily invariant norm, then there is a unique symmetric gauge function ϕ such that

$$\|X\| = \phi(\{s_j(X)\})$$

for all $X \in \mathcal{J}$.

Especially well-known among unitarily invariant norms are the von Neumann–Schatten p-norms defined as

$$\|X\|_p = \left(\sum_j s_j^p(X)\right)^{1/p}$$

if $1 \le p < \infty$, and $\|X\|_\infty = s_1(X)$. The associated norm ideals, denoted by $\mathcal{C}_p$, are the von Neumann–Schatten p-classes. For a complete account on the theory of norm ideals, the reader is referred to [6].

Let $\boldsymbol{A} = (A_1, \dots, A_n)$ and $\boldsymbol{B} = (B_1, \dots, B_n)$ be n-tuples of operators and define the elementary operator $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by $\mathcal{E}(X) = \sum_{i=1}^n A_i X B_i$. If $\mathcal{J}$ is any norm ideal, then $\mathcal{E}(\mathcal{J}) \subseteq \mathcal{J}$. However it can also happen that $\mathcal{E}(X) \in \mathcal{J}$ for some $X \in \mathcal{B}(\mathcal{H}) \backslash \mathcal{J}$; hence $\operatorname{ran}(\mathcal{E}|_{\mathcal{J}}) \subseteq \operatorname{ran}\mathcal{E} \cap \mathcal{J}$ and then we also have $\overline{\operatorname{ran}(\mathcal{E}|_{\mathcal{J}})}^{\mathcal{J}} \subseteq \overline{\operatorname{ran}\mathcal{E} \cap \mathcal{J}}^{\mathcal{J}}$, where $\overline{(\cdot)}^{\mathcal{J}}$ denotes closure in the norm of the ideal $\mathcal{J}$. So we may well ask if the reverse inclusion is possible. Let us say that elementary operator $\mathcal{E}$ satisfies *property (R)* with respect to the norm ideal $\mathcal{J}$, if

$$\overline{\operatorname{ran}(\mathcal{E}|_{\mathcal{J}})}^{\mathcal{J}} = \overline{\operatorname{ran}\mathcal{E} \cap \mathcal{J}}^{\mathcal{J}},$$

or equivalently,

$$\text{if } \mathcal{E}(X) \in \mathcal{J}, \text{ then } \mathcal{E}(X) = \lim_n \mathcal{E}(X_n), \text{ and } X_n \in \mathcal{J}.$$

In this note we prove that multiplications, i.e., $\mathcal{E}(X) = AXB$, satisfy property (R) with respect to each separable norm ideal $\mathcal{J}$. When $\mathcal{E}$ is an elementary operator of length two this is not true anymore. Namely, this is a consequence of the fact that there exists a trace class (even rank one) non-zero trace commutator $AX - XA$, where A is normal, see [14]. However, let A and C, respectively B and D be normal commuting operators and let $\mathcal{E}(X) = AXB + CXD$. We show then that $\mathcal{E}$ satisfies property (R) with respect to the von Neumann–Schatten classes $\mathcal{C}_p$ with $p > 1$. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are n-tuples of commuting normal operators and $\mathcal{E}(X) = \sum_{i=1}^{n} A_i X B_i$, then $\mathcal{E}$ satisfies property (R) with respect to the Hilbert–Schmidt class $\mathcal{C}_2$. Applying this last fact, we prove some results about double operator integrals, see [12].

2. Main results

Lemma 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and suppose that $\mathcal{E}(X) = AXA^*$. Then $\mathcal{E}$ satisfies property (R) with respect to each separable norm ideal $\mathcal{J}$.*

Proof. We divide the proof into three steps.

First step. Suppose that $A \geq 0$ and injective; let E be its spectral measure and denote $E_n = E(\{z \in \mathbb{C} : |z| > 1/n\})$. Then with respect to the decomposition $\mathcal{H} = \operatorname{ran} E_n \oplus \operatorname{ran}^{\perp} E_n$ one obtains

$$A = \begin{bmatrix} A_1^{(n)} & 0 \\ 0 & A_2^{(n)} \end{bmatrix}, \qquad X = \begin{bmatrix} X_{11}^{(n)} & X_{12}^{(n)} \\ X_{21}^{(n)} & X_{22}^{(n)} \end{bmatrix},$$

where $A_1^{(n)}$ is invertible. From $AXA \in \mathcal{J}$, using the fact that $\mathcal{J}$ is an ideal, we get that

$$\begin{bmatrix} A_1^{(n)} X_{11}^{(n)} A_1^{(n)} & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{J}.$$

Since $A_1^{(n)}$ is invertible, it follows that

$$X_n = \begin{bmatrix} X_{11}^{(n)} & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{J}.$$

Notice that $X_n = E_n X E_n$ and we have

$$AXA - AX_nA = AXA - AE_nXE_nA = AXA - E_nAXAE_n.$$

Since $E_n \underset{n}{\to} 1$ strongly (A is injective) [6, Theorem 6.3] (at this point we need separability) tells us that the sequence $E_n AXAE_n \underset{n}{\to} AXA$ in the norm of the ideal $\mathcal{J}$, and in this case the desired conclusion follows.

Second step. Now suppose that $A \geq 0$ but not necessarily injective. With respect to the decomposition $\mathcal{H} = \ker^{\perp} A \oplus \ker A$ one obtains

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix},$$

where A_1 is injective. Then

$$AXA = \begin{bmatrix} A_1 X_{11} A_1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{J}$$

and we can apply the previous step to finish the proof.

Third step. Finally, suppose that $A \in \mathcal{B}(\mathcal{H})$ is arbitrary and let $A = U|A|$ be its polar decomposition. Then since $\mathcal{J}$ is an ideal, we have $U^*(AXA^*)U = |A|X|A| \in \mathcal{J}$. As we already know we can write $|A|X|A| = \lim_n |A|X_n|A|$ with operators $X_n \in \mathcal{J}$. From the estimate

$$\begin{aligned} \|\!|AXA^* - AX_nA^*|\!\| &= \|\!|U(|A|X|A| - |A|X_n|A|)U^*|\!\| \\ &\leq \|\!||A|X|A| - |A|X_n|A||\!\| \end{aligned}$$

the lemma follows. □

A familiar device of considering 2×2 operator matrices enables us to give the following theorem.

Theorem 2.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and suppose that $\mathcal{E}(X) = AXB$. Then $\mathcal{E}$ satisfies property (R) with respect to each separable norm ideal $\mathcal{J}$.*

Proof. On $\mathcal{H} \oplus \mathcal{H}$ put

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix}, \qquad \tilde{X} = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}.$$

Then

$$\tilde{A}\tilde{X}\tilde{A}^* = \begin{bmatrix} 0 & AXB \\ 0 & 0 \end{bmatrix} \in \mathcal{J}$$

and by the previous lemma we can find a sequence

$$\tilde{X}_n = \begin{bmatrix} X_{11}^{(n)} & X_{12}^{(n)} \\ X_{21}^{(n)} & X_{22}^{(n)} \end{bmatrix} \in \mathcal{J}$$

such that $\tilde{A}\tilde{X}_n\tilde{A}^* \underset{n}{\rightarrow} \tilde{A}\tilde{X}\tilde{A}^*$. Hence $AX_{12}^{(n)}B \underset{n}{\rightarrow} AXB$ and operators $X_{12}^{(n)} \in \mathcal{J}$. □

Before going on to prove the next theorem we state some known results. Recall that if $\mathcal{U}$ and $\mathcal{V}$ are subspaces of a Banach space $\mathcal{X}$ with norm $\|\cdot\|$, $\mathcal{U}$ is said to be orthogonal to $\mathcal{V}$ if $\|u+v\| \geq \|v\|$ for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$. The range-kernel orthogonality of elementary operators has been considered by a number of authors, see [1], [2], [3], [9], [10] for generalized derivations $X \mapsto AX - XB$, and [7], [16], [17] for some more general results. Let $\mathcal{E}(X) = AXB + CXD$, where A, C, respectively B, D are commuting normal operators.

Theorem 2.3 ([17, Theorem 3.4])**.** *Let $p > 1$, $S \in \mathcal{C}_p$ and suppose that $\ker A \cap \ker C = \ker B^* \cap \ker D^* = \{0\}$. Then $S \in \ker \mathcal{E}$ if and only if $\|\mathcal{E}(X) + S\|_p \geq \|S\|_p$ for all $X \in \mathcal{C}_p$.*

Theorem 2.4 ([7])**.** *Let $p \geq 1$, $S \in \mathcal{C}_p \cap \ker \mathcal{E}$ and $\mathcal{E}(X) \in \mathcal{C}_p$ for some $X \in \mathcal{B}(\mathcal{H})$. If $\ker A \cap \ker C = \ker B^* \cap \ker D^* = \{0\}$, then $\|\mathcal{E}(X) + S\|_p \geq \|S\|_p$.*

Let $\mathcal{X}$ be a Banach space, $\mathcal{U}$ a subspace and define the orthogonal complement of $\mathcal{U}$ as

$$\mathcal{U}_\perp = \{x \in \mathcal{X} : \|u + x\| \geq \|x\| \text{ for all } u \in \mathcal{U}\}.$$

Applying the preceding two theorems we obtain

Proposition 2.5. *Let $p > 1$. Then*

$$\operatorname{ran}(\mathcal{E}|_{\mathcal{C}_p})_\perp = (\operatorname{ran}\mathcal{E} \cap \mathcal{C}_p)_\perp = \ker(\mathcal{E}|_{\mathcal{C}_p}).$$

Next simple proposition deals with orthogonal complements.

Proposition 2.6. *Let $\mathcal{U}$ and $\mathcal{V}$ be closed subspaces of a Banach space $\mathcal{X}$ and suppose that $\mathcal{U}_\perp$ and $\mathcal{V}_\perp$ are also subspaces. If $\mathcal{U} \subseteq \mathcal{V}$, $\mathcal{U}_\perp = \mathcal{V}_\perp$ and $\mathcal{U} \oplus \mathcal{U}_\perp = \mathcal{V} \oplus \mathcal{V}_\perp = \mathcal{X}$, then $\mathcal{U} = \mathcal{V}$.*

Proof. Take $v \in \mathcal{V}$. Then $v = v + 0 = u + u_\perp$ and from this it follows that $v - u = u_\perp$. But $v - u \in \mathcal{V}$ and $u_\perp \in \mathcal{U}_\perp = \mathcal{V}_\perp$, hence $v - u = 0$. So we have proved that $v = u \in \mathcal{U}$ and this proves the reverse inclusion $\mathcal{V} \subseteq \mathcal{U}$. $\square$

Lemma 2.7. *Let $\mathcal{E}(X) = AXB + CXD$ with A, C, respectively B, D commuting normal operators and let $p > 1$. If $\ker A \cap \ker C = \ker B^* \cap \ker D^* = \{0\}$, then*

$$\overline{\operatorname{ran}(\mathcal{E}|_{\mathcal{C}_p})}^{\mathcal{C}_p} = \overline{\operatorname{ran}\mathcal{E} \cap \mathcal{C}_p}^{\mathcal{C}_p}.$$

In other words this means that $\mathcal{E}$ satisfies property (R) with respect to the von Neumann–Schatten ideals $\mathcal{C}_p$ with $p > 1$.

Proof. [17, Corollary 3.7] tells us that

$$\overline{\operatorname{ran}(\mathcal{E}|_{\mathcal{C}_p})}^{\mathcal{C}_p} \oplus \ker(\mathcal{E}|_{\mathcal{C}_p}) = \mathcal{C}_p.$$

From Proposition 2.5 we know that $\operatorname{ran}(\mathcal{E}|_{\mathcal{C}_p})_\perp = (\operatorname{ran}\mathcal{E} \cap \mathcal{C}_p)_\perp$. Since furthermore $\overline{\operatorname{ran}(\mathcal{E}|_{\mathcal{C}_p})}^{\mathcal{C}_p} \subseteq \overline{\operatorname{ran}\mathcal{E} \cap \mathcal{C}_p}^{\mathcal{C}_p}$ we can apply Proposition 2.6 to complete the proof. $\square$

In the next theorem we shall show that the condition $\ker A \cap \ker C = \ker B^* \cap \ker D^* = \{0\}$ is superfluous.

Theorem 2.8. *Let $\mathcal{E}(X) = AXB + CXD$ with A, C, respectively B, D commuting normal operators. Then $\mathcal{E}$ satisfies property (R) with respect to the von Neumann–Schatten ideals $\mathcal{C}_p$ with $p > 1$.*

Proof. With respect to the decompositions $\mathcal{H}_1 = \mathcal{H} = \ker^\perp A \oplus \ker A$ and $\mathcal{H}_2 = \mathcal{H} = \ker^\perp B \oplus \ker B$ we can write operators $A, C : \mathcal{H}_1 \to \mathcal{H}_1$, $B, D : \mathcal{H}_2 \to \mathcal{H}_2$ and $X : \mathcal{H}_2 \to \mathcal{H}_1$ as

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \ X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

Then

$$\mathcal{E}(X) = \begin{bmatrix} A_1X_{11}B_1 + C_1X_{11}D_1 & C_1X_{12}D_2 \\ C_2X_{21}D_1 & C_2X_{22}D_2 \end{bmatrix}.$$

Since $\ker A_1 \cap \ker C_1 = \ker B_1^* \cap \ker D_1^* = \{0\}$ we can apply Lemma 2.7 for the elementary operator in the upper left corner. For the other entries we use Theorem 2.2 and the proof is finished. $\square$

Let $\boldsymbol{A} = (A_1, \dots, A_n)$ and $\boldsymbol{B} = (B_1, \dots, B_n)$ be n-tuples of mutually commuting normal operators and let $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be the elementary operator $\mathcal{E}(X) = \sum_{i=1}^n A_i X B_i$. We can prove property (R) for the elementary operator $\mathcal{E}$ in the case of Hilbert–Schmidt operators. It is a consequence of the following theorem:

Theorem 2.9 ([16, Theorem 3.1])**.** *Let $\mathcal{E}(S) = 0$ for some $S \in \mathcal{C}_2$. If $\mathcal{E}(X) \in \mathcal{C}_2$ for some $X \in \mathcal{B}(\mathcal{H})$, then $\|\mathcal{E}(X) + S\|_2^2 = \|\mathcal{E}(X)\|_2^2 + \|S\|_2^2$.*

Namely, we reason as follows. Since $\mathcal{E}|_{\mathcal{C}_2}$ is normal operator, $\operatorname{ran}^\perp(\mathcal{E}|_{\mathcal{C}_2}) = \ker(\mathcal{E}|_{\mathcal{C}_2})$. Hence for $S \in \operatorname{ran}^\perp(\mathcal{E}|_{\mathcal{C}_2})$ it follows that $S \in \ker(\mathcal{E}|_{\mathcal{C}_2})$. But then Theorem 2.9 implies that $S \in (\operatorname{ran} \mathcal{E} \cap \mathcal{C}_2)^\perp$. This means that $\operatorname{ran}^\perp(\mathcal{E}|_{\mathcal{C}_2}) \subseteq (\operatorname{ran} \mathcal{E} \cap \mathcal{C}_2)^\perp$. Since the reverse inclusion is trivial, we have thus proved that $\operatorname{ran}^\perp(\mathcal{E}|_{\mathcal{C}_2}) = (\operatorname{ran} \mathcal{E} \cap \mathcal{C}_2)^\perp$. In particular we have $\overline{\operatorname{ran}(\mathcal{E}|_{\mathcal{C}_2})}^{\mathcal{C}_2} = \overline{\operatorname{ran} \mathcal{E} \cap \mathcal{C}_2}^{\mathcal{C}_2}$. We can summarize these results in

Theorem 2.10. *Let $\boldsymbol{A} = (A_1, \dots, A_n)$ and $\boldsymbol{B} = (B_1, \dots, B_n)$ be n-tuples of mutually commuting normal operators and let $\mathcal{E}$ be the elementary operator $\mathcal{E}(X) = \sum_{i=1}^n A_i X B_i$. Then $\mathcal{E}$ satisfies property (R) with respect to the Hilbert–Schmidt ideal.*

3. An application

Let A and B be normal bounded operators and denote by E and F respectively, their spectral measures. Furthermore let f be bounded Borel measurable function defined on $\sigma(A) \times \sigma(B)$. Then let $f(A, B) : \mathcal{C}_2 \to \mathcal{C}_2$ denote bounded operator on $\mathcal{C}_2$ defined by

$$f(A, B)X = \int_{\sigma(A)} \int_{\sigma(B)} f(z, w) E(dz) X F(dw).$$

It is known that this functional calculus could not, in general, be defined on the whole $\mathcal{B}(\mathcal{H})$, see [5], but merely on the Hilbert–Schmidt class. For a nice account on double operator integrals see [11], [12]. For a more prospective insight we list the main properties of this functional calculus:

(i) $f(A,B)X = AX$ for $f(z,w) = z$, $f(A,B)X = XB$ for $f(z,w) = w$.

(ii) $(\alpha f + \beta g)(A,B)X = \alpha f(A,B)X + \beta g(A,B)X$.

(iii) $(fg)(A,B)X = f(A,B)(g(A,B)X)$.

(iv) $f(A,B)^* = \overline{f}(A,B)$.

(v) $\operatorname{tr}(f(A,B)X(g(A,B)Y)^*) = \operatorname{tr}(((f\overline{g})(A,B)X)Y^*)$.

As pointed out in [12], the following integral representation formula is an important special case of (iii): if f is a Lipschitz function on $\sigma(A) \cup \sigma(B)$ and $X \in \mathcal{C}_2$, then

$$f(A)X - Xf(B) = \int_{\sigma(A)} \int_{\sigma(B)} \frac{f(z) - f(w)}{z - w} E(dz)(AX - XB)F(dw).$$

As one notes, the right side of this formula is well defined if merely $AX - XB \in \mathcal{C}_2$. Thus a natural question is, whether we have equality also in this case. In fact, the main result in [12] states that this formula remains valid also in this case. Our intention is to present a proof based on our results.

Theorem 3.1. *Let A and B be normal bounded operators such that $AX - XB \in \mathcal{C}_2$ for some $X \in \mathcal{B}(\mathcal{H})$. Then for every Lipschitz function f defined on $\sigma(A) \cup \sigma(B)$ we have*

$$f(A)X - Xf(B) = \tilde{f}(A,B)(AX - XB),$$

where

$$\tilde{f}(z,w) = \begin{cases} (f(z) - f(w))(z-w)^{-1} & \text{if } z \neq w, \\ 0 & \text{if } z = w. \end{cases}$$

Proof. Since $AX - XB \in \mathcal{C}_2$, it follows from Theorem 2.10 that we can find operators $X_n \in \mathcal{C}_2$ such that

$$AX_n - X_nB \underset{n}{\rightarrow} AX - XB$$

in $\mathcal{C}_2$ norm. Then we have

$$\begin{aligned} \tilde{f}(A,B)(AX_n - X_nB) &= f(A)X_n - X_nf(B) \\ &\underset{n}{\rightarrow} \tilde{f}(A,B)(AX - XB). \end{aligned} \tag{1}$$

If L is a Lipschitz constant of the function f, then applying [8, Corollary 1] we get

$$\begin{aligned}\|(f(A)X_n - X_n f(B)) - (f(A)X - Xf(B))\|_2 \\ &= \|f(A)(X_n - X) - (X_n - X)f(B)\|_2 \\ &\leq L\|A(X_n - X) - (X_n - X)B\|_2.\end{aligned}$$

Hence

$$f(A)X_n - X_n f(B) \underset{n}{\rightarrow} f(A)X - Xf(B),$$

and this together with (1) completes the proof. □

Remember that we have denoted by $\boldsymbol{A} = (A_1, \ldots, A_n)$ and $\boldsymbol{B} = (B_1, \ldots, B_n)$ n- tuples of mutually commuting normal operators and define, besides $\mathcal{E}$, also the elementary operator $\mathcal{E}^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by $\mathcal{E}^*(X) = \sum_{i=1}^n A_i^* X B_i^*$. Recall, see [4], that left (Harte), respectively right (Harte) spectrum of the n-tuple $\boldsymbol{A}$ is given by

$$\sigma_l(\boldsymbol{A}) = \Big\{\boldsymbol{\lambda} \in \mathbb{C}^n : \sum_{i=1}^n X_i(A_i - \lambda_i) = 1 \text{ can not be solved for } X_i \in \mathcal{B}(\mathcal{H})\Big\},$$
$$\sigma_r(\boldsymbol{A}) = \Big\{\boldsymbol{\lambda} \in \mathbb{C}^n : \sum_{i=1}^n (A_i - \lambda_i)X_i = 1 \text{ can not be solved for } X_i \in \mathcal{B}(\mathcal{H})\Big\},$$

and that joint or Harte spectrum of $\boldsymbol{A}$ is defined by

$$\sigma_H(\boldsymbol{A}) = \sigma_l(\boldsymbol{A}) \cup \sigma_r(\boldsymbol{A}).$$

If $C^*(\boldsymbol{A}) \subseteq \mathcal{B}(\mathcal{H})$ is a commutative C^*-algebra generated with identity and operators $A_1, \ldots, A_n$, then it is a well-known fact that its spectrum (the space of all multiplicative functionals) is homeomorphic to the joint spectrum $\sigma_H(\boldsymbol{A})$. Furthermore, [13, Theorem 12.22] gives us spectral measure $\boldsymbol{E}$ defined on all Borel subsets of $\sigma_H(\boldsymbol{A})$ such that

$$f(\boldsymbol{A}) = \int_{\sigma_H(\boldsymbol{A})} f(\boldsymbol{z})\boldsymbol{E}(d\boldsymbol{z})$$

for every bounded Borel measurable function on $\sigma_H(\boldsymbol{A})$. Let us again turn our attention to double operator integrals. Namely, we are in analogous position as before with two single operators A and B. We have two n-tuples $\boldsymbol{A} = (A_1, \ldots, A_n)$ and $\boldsymbol{B} = (B_1, \ldots, B_n)$ of mutually commuting normal operators with their spectral measures $\boldsymbol{E}$, respectively $\boldsymbol{F}$. Then for every bounded Borel measurable function $f(\boldsymbol{z}, \boldsymbol{w})$ defined on $\sigma_H(\boldsymbol{A}) \times \sigma_H(\boldsymbol{B})$ and for every $X \in \mathcal{C}_2$ operator

$$f(\boldsymbol{A}, \boldsymbol{B})X = \int_{\sigma_H(\boldsymbol{A})} \int_{\sigma_H(\boldsymbol{B})} f(\boldsymbol{z}, \boldsymbol{w}) \boldsymbol{E}(d\boldsymbol{z}) X \boldsymbol{F}(d\boldsymbol{w})$$

is again in $\mathcal{C}_2$. Hence for the function $f(\boldsymbol{z}, \boldsymbol{w}) = \overline{\boldsymbol{z}} \cdot \overline{\boldsymbol{w}} = \overline{z_1}\,\overline{w_1} + \ldots + \overline{z_n}\,\overline{w_n}$ and $X \in \mathcal{C}_2$ we have

$$\mathcal{E}^*(X) = \int_{\sigma_H(\boldsymbol{A})} \int_{\sigma_H(\boldsymbol{B})} \frac{\overline{\boldsymbol{z}} \cdot \overline{\boldsymbol{w}}}{\boldsymbol{z} \cdot \boldsymbol{w}} \boldsymbol{E}(d\boldsymbol{z}) \mathcal{E}(X) \boldsymbol{F}(d\boldsymbol{w}).$$

As before, we notice that for the right side of this formula we do not need $X \in \mathcal{C}_2$, but merely $\mathcal{E}(X) \in \mathcal{C}_2$. However if this formula were true merely under the assumption $\mathcal{E}(X) \in \mathcal{C}_2$, then $\mathcal{E}(X) = 0$ would imply $\mathcal{E}^*(X) = 0$; a contradiction with Shulman's result [15, Corollary 3]. Nevertheless with stronger assumption that both $\mathcal{E}(X)$ and $\mathcal{E}^*(X)$ are in $\mathcal{C}_2$ we have

Theorem 3.2. *Suppose that $\mathcal{E}(X),\ \mathcal{E}^*(X) \in \mathcal{C}_2$ for some $X \in \mathcal{B}(\mathcal{H})$. Then*

$$\mathcal{E}^*(X) = \tilde{f}(\boldsymbol{A}, \boldsymbol{B}) \mathcal{E}(X),$$

where

$$\tilde{f}(\boldsymbol{z}, \boldsymbol{w}) = \begin{cases} \dfrac{\overline{\boldsymbol{z}} \cdot \overline{\boldsymbol{w}}}{\boldsymbol{z} \cdot \boldsymbol{w}} & \text{if } \boldsymbol{z} \cdot \boldsymbol{w} \neq 0, \\ 0 & \text{if } \boldsymbol{z} \cdot \boldsymbol{w} = 0. \end{cases}$$

Proof. We reason as in Theorem 3.1. Since $\mathcal{E}(X) \in \mathcal{C}_2$ we can find operators $X_n \in \mathcal{C}_2$ such that

$$\mathcal{E}(X_n) \underset{n}{\to} \mathcal{E}(X)$$

in $\mathcal{C}_2$ norm. Thus

$$\tilde{f}(\boldsymbol{A}, \boldsymbol{B}) \mathcal{E}(X_n) = \mathcal{E}^*(X_n) \underset{n}{\to} \tilde{f}(\boldsymbol{A}, \boldsymbol{B}) \mathcal{E}(X).$$

But a result of Weiss, see [18], says that whenever both $\mathcal{E}(X)$and $\mathcal{E}^*(X)$

belong to $\mathcal{C}_2$, then $\|\mathcal{E}^*(X)\|_2 = \|\mathcal{E}(X)\|_2$. Hence

$$\begin{aligned}\|\mathcal{E}^*(X) - \mathcal{E}^*(X_n)\|_2 &= \|\mathcal{E}^*(X - X_n)\|_2 \\ &= \|\mathcal{E}(X - X_n)\|_2 = \|\mathcal{E}(X) - \mathcal{E}(X_n)\|_2 \underset{n}{\to} 0,\end{aligned}$$

and this completes the proof. □

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