

QR-submanifolds of a locally conformal quaternion Kaehler manifold

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Abstract. In this paper, we study QR-submanifolds of a locally conformal quaternion Kaehler manifold. We give the basic formulas for QR-submanifolds of a locally conformal quaternion Kaehler manifold and two examples of QR-submanifolds of a locally conformal quaternion Kaehler manifold. Necessary and sufficient conditions are given for a quaternion distribution on a QR-submanifold to be integrable. Also, a necessary and sufficient condition is given for a distribution $D^\perp$ on a QR-submanifold to be a totally geodesic foliation. Further, a theorem is obtained for a QR-submanifold to be mixed geodesic. Finally, totally umbilical QR-submanifolds are studied and some theorems are given.

1. Introduction

A locally conformal quaternion Kaehler manifold (shortly, l.c.q.K. manifold) is a quaternion Hermitian manifold whose metric is conformal to a quaternion Kaehler metric in some neighborhood of each point.

I. VAISMAN reported on the locally conformal Kaehler structures in [10]. He also gave several results for locally conformal almost Kaehler manifolds to be Kaehler manifolds [10], [11], [12], [13], [14].

A. BEJANCU introduced QR-submanifolds of a quaternion Kaehler manifold [1]. He obtained fundamental results about these submanifolds. It is known that a real hypersurface of a quaternion Kaehler manifold is a

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QR-submanifold [1]. The geometry of these submanifolds has been studied by many authors.

H. PEDERSEN, A. SWANN and Y. S. POON [8] introduced l.c.q.K. manifolds. They showed that a manifold is a quaternion Hermitian–Weyl manifold if and only if it is a l.c.q.K. manifold. L. ORNEA and P. PICCINI [6] showed that the Lee form of a compact l.c.q.K. manifold can be chosen as parallel form without any restrictions. It is known that this property is not guaranteed in the complex case [12], [13]. L. ORNEA and P. PICCINI [6] proved a theorem for a l.c.q.K. manifold to be a quaternion Kaehler manifold.

In this paper, we introduce QR-submanifolds of a l.c.q.K. manifold. We give some necessary and sufficient conditions for a quaternion distribution on a QR-submanifold of a l.c.q.K. manifold to be integrable. We also obtain a necessary and sufficient condition for a distribution $D^\perp$ of a QR-submanifold of a l.c.q.K. manifold to be a totally geodesic foliation. Moreover, we give a characterization for a QR-submanifold to be a mixed geodesic QR-submanifold. Finally, we give some results for totally umbilical QR-submanifolds.

2. Preliminaries

We denote a quaternion Hermitian manifold by $(\bar{M}, g, H)$, where H is a subbundle of $\text{End}(T\bar{M})$ of rank 3 which is spanned by almost complex structures J_1, J_2 and J_3 . We recall that a quaternion Hermitian metric g is said to be a quaternion Kaehler metric if its Levi–Civita connection $\bar{\nabla}$ satisfies $\bar{\nabla}H \subset H$.

A quaternion Hermitian manifold with metric g is a l.c.q.K. manifold if over neighborhoods $\{U_i\}$ covering M , $g|_{U_i} = e^{f_i} g'_i$ with g'_i a quaternion Kaehler metric on U_i . In this case, the Lee form ω is locally defined by $\omega|_{U_i} = df_i$ and satisfies

$$d\Theta = \omega \wedge \Theta, d\omega = 0 \quad (\text{II.1})$$

where $\Theta = \sum_{\alpha=1}^3 \Omega_\alpha \wedge \Omega_\alpha$ is the Kaehler 4-form. We note that property (II.1) is also a sufficient condition for a quaternion Hermitian metric to be a l.c.q.K. metric [6].

The Levi-Civita connections $\bar{D}^i$ of the local Kaehler metrics g'_i glue together on M to a connection $\bar{D}'$ related to the Levi-Civita connection $\bar{\nabla}$ of g by the formula

$$\bar{D}'_X Y = \bar{\nabla}_X Y - \frac{1}{2} \{ \omega(X) Y + \omega(Y) X - g(X, Y) B \} \quad (\text{II.2})$$

for any $X, Y \in \Gamma(TM)$, where $B = \omega^\#$ is the Lee vector field [4].

Let $\bar{M}$ be a l.c.q.K. manifold and M a real submanifold of $\bar{M}$. Then M is called a QR-submanifold if there exists a vector subbundle ν of the normal bundle such that

$$J_a(\nu_x) = \nu_x \quad (\text{II.3})$$

and

$$J_a(\nu_x^\perp) \subset T_M(x) \quad (\text{II.4})$$

for $x \in M$ and $a = 1, 2, 3$, where $\nu^\perp$ is the orthogonal bundle complementary to ν in $TM^\perp$ [1]. Let M be a QR-submanifold of $\bar{M}$. Set $D_{ax} = J_a(\nu_x^\perp)$. We consider $D_{1x} \oplus D_{2x} \oplus D_{3x} = D_x^\perp$. Then the $3s$ -dimensional distribution $D^\perp : x \rightarrow D_x^\perp$ is globally defined on M , where $s = \dim \nu_x^\perp$. Also, we have for each $x \in M$

$$J_a(D_{ax}) = \nu_x^\perp, J_a(D_{bx}) = D_{cx} \quad (\text{II.5})$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. We denote the orthogonal distribution complementary to $D^\perp$ in TM by D . Then D is invariant with respect to the action of J_a , i.e. we have

$$J_a(D_x) = D_x \quad (\text{II.6})$$

for any $x \in M$. D is called a quaternion distribution.

Let $\bar{M}$ be a l.c.q.K. manifold and $\bar{\nabla}$ be the connection of $\bar{M}$. Then the Weyl connection does not preserve the compatible almost complex structures individually but only their 3-dimensional bundle H . Indeed, PEDERSEN, POON and SWANN showed that

$$\bar{D}' J_a = \sum Q_{ab} \otimes J_b \quad (\text{II.7})$$

for $a, b = 1, 2, 3$, and Q_{ab} is a skew-symmetric matrix of local forms [8]. Thus, from (II.1) and (II.2) we have

$$\bar{\nabla}_X J_a Y = J_a \bar{\nabla}_X Y + \frac{1}{2} \{ \theta_o(Y) X - \omega(Y) J_a X - \Omega(X, Y) B + g(X, Y) J_a B \}$$

$$+ Q_{ab}(X)J_bY + Q_{ac}(X)J_cY \quad (\text{II.8})$$

for any $X, Y \in \Gamma(T\bar{M})$, where $\theta_o = \omega o J_a$.

We give the following

Theorem 2.1. [7] *Let $(\bar{M}, \bar{g}, H)$ be a compact quaternion Hermitian Weyl manifold, non-quaternion Kaehler, whose foliation $\bar{D}$ has compact leaves. Then the leaves space $P = \bar{M}/\bar{D}$ is a compact quaternion Kaehler orbifold with positive scalar curvature, the projection is a Riemannian, totally geodesic submersion and a fibre bundle map with fibres as described in Proposition 4.10 of [7], where $\bar{D}$ is locally generated by $B, J_1B = B_1, B_2, B_3$.*

If $\bar{D}$ is a regular foliation, then $P = \bar{M}/\bar{D}$ is a compact quaternion Kaehler manifold.

Let M be a QR-submanifold of a l.c.q.K. manifold $\bar{M}$. Let P denote the projection morphism of TM to the quaternion distribution D and choose a local field of orthonormal frames $\{v_1, \dots, v_s\}$ on the vector subbundle $\nu^\perp$ in $TM^\perp$. Then, on the distribution $D^\perp$, we have the local field of orthonormal frames

$$\{E_{11}, \dots, E_{1s}, E_{21}, \dots, E_{2s}, E_{31}, \dots, E_{3s}\} \quad (\text{II.9})$$

where $E_{ai} = J_a v_i$ and $i = 1, \dots, s$. Thus any vector field Y tangent to M can be written locally as follows

$$Y = PY + \sum_{b=1}^3 \sum_{i=1}^s W_{bi}(Y)E_{bi} \quad (\text{II.10})$$

where the W_{bi} are 1-forms locally defined on M by

$$W_{bi}(Y) = g(Y, E_{bi}). \quad (\text{II.11})$$

Applying J_a to (II.10) and taking account of (II.1) we have

$$J_a Y = J_a P Y + \sum_{i=1}^s \{W_{bi}(Y)E_{ci} - W_{ci}(Y)E_{bi}\} - W_{ai}(Y)v_i. \quad (\text{II.12})$$

We can decompose $J_a Y$ as follows:

$$J_a Y = \phi_a Y + F_a Y, a = 1, 2, 3, \quad (\text{II.13})$$

for $Y \in \Gamma(TM)$, where $\phi_a Y$ and $F_a Y$ are the tangential and normal parts of $J_a Y$, respectively. Similarly, we get

$$J_a V = t_a V + f_a V. \quad (\text{II.14})$$

Example 2.1. Let $\bar{M}$ be a l.c.q.K. manifold. Assume that the foliation $\bar{D}$ is regular. Then $P = \bar{M}/\bar{D}$ is a compact quaternion Kaehler manifold (cf. Theorem 2.1). We denote almost complex structures of $\bar{M}$ and P by J_a and J'_a , respectively. Now we consider the following commutative diagram:

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\pi} & P = \bar{M}/\bar{D} \\ \uparrow i & & \uparrow j \\ N & \xrightarrow{\bar{\pi}} & \bar{N} \end{array}$$

where N and $\bar{N}$ are submanifolds of $\bar{M}$ and P , respectively. We denote the horizontal lift by $*$. Then we have

$$(J'_a X)^* = J_a X^*. \quad (\text{II.15})$$

We note that the projection π is a totally geodesic Riemannian submersion and a fibre bundle map. Hence $\bar{\pi}$ is also a Riemannian submersion. We denote the vertical distribution of the Riemannian submersion π by v , i.e. $\ker \pi_* = v$. Let $\bar{H}$ be the horizontal distribution of π . Then we have $T\bar{M} = \bar{H} \oplus v$. We denote the horizontal distribution of $\bar{\pi}$ by H_0 . We will investigate the relation between normal spaces of N and $\bar{N}$. We denote the Riemannian metrics of $\bar{M}$ and P by g and g' , respectively. Let V^* be the horizontal lift of $V \in \Gamma(T\bar{N}^\perp)$. Then we get

$$g(V^*, X) = g((\pi_*)^* V, X) = g'(\pi_* X, V) = 0,$$

for any $X \in H_0$. Thus, $(T\bar{N}^\perp)^*$ is orthogonal to H_0 . Note that the normal space is always horizontal. Hence $(T\bar{N}^\perp)^*$ is orthogonal to v . Consequently, we have $(T\bar{N}^\perp)^* \subseteq TN^\perp$. Since π is a Riemannian submersion we get

$$(T\bar{N}^\perp)^* = TN^\perp. \quad (\text{II.16})$$

Now, let t_a and f_a be the operators on $\bar{N}$ appearing in (II.14). We denote the operators in N corresponding to t_a and f_a by t'_a and f'_a , respec-

tively. From (II.15) and (II.16) we obtain

$$(t_a V)^* = t'_a V^* \quad (\text{II.17})$$

and

$$(f_a V)^* = f'_a V^*. \quad (\text{II.18})$$

So, from (II.17) and (II.18) we see that N is a QR-submanifold of $\bar{M}$ if and only if $\bar{N}$ is a QR-submanifold of P .

Example 2.2. Let $\bar{M}$ be a l.c.q.K. manifold. We assume that the distribution $\bar{D}$ is regular. Then $P = \bar{M}/\bar{D}$ is a quaternion Kaehler manifold. It is known that a real hypersurface of a quaternion Kaehler manifold is a QR-submanifold [1]. From the previous example, a real hypersurface of a l.c.q.K. manifold is a QR-submanifold. Let M be a real hypersurface of a l.c.q.K. manifold $\bar{M}$. We denote the normal space of M by $TM^\perp$. Set $TM^\perp = Sp\{N\}$. Since $\dim(T_x M^\perp) = 1$ and $g(J_a N, N) = 0$, we obtain $J_a(TM^\perp) \subset TM$. Thus, $\nu_x = \{0\}$ and $\nu_x^\perp = T_x M^\perp$ for $x \in M$.

Let $\bar{M}$ be a l.c.q.K. manifold and M be a QR-submanifold of $\bar{M}$. The formulae of Gauss and Weingarten are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad (\text{II.19})$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla_X^\perp V \quad (\text{II.20})$$

for vector fields X, Y tangent to M and any vector field V normal to M , where ∇ is the induced Riemann connection in M , h is the second fundamental form, A_V is the fundamental tensor field of Weingarten with respect to the normal section V and $\nabla^\perp$ is the normal connection. Moreover, we have the relation

$$g(h(X, Y), V) = g(A_V X, Y). \quad (\text{II.21})$$

3. QR-submanifolds of a l.c.q.K. manifold

Lemma 3.1. *Let $\bar{M}$ be a l.c.q.K. manifold and M be a QR-submanifold of $\bar{M}$. Then we have*

$$h(X, E_{ai}) = W_{ai}(A_{v_i} X)v_i + f_a \nabla_X^\perp v_i \quad (\text{III.1})$$

and

$$g(\nabla_X Y, E_{ai}) = \frac{1}{2}\omega(v_i)g(J_a X, Y) - \frac{1}{2}\theta_o(v_i)g(X, Y) + g(J_a P A_{v_i} X, Y) \quad (\text{III.2})$$

for any $X, Y \in \Gamma(D)$ and $v_i \in \Gamma(D^\perp)$.

PROOF. From (II.8), (II.19) and (II.20), we obtain

$$\begin{aligned} h(X, E_{ai}) &= \bar{\nabla}_X E_{ai} - \nabla_X E_{ai} \\ &= \frac{1}{2} \{ \theta_o(v_i)X - \omega(v_i)J_a X \} - Q_{ab}(X)E_{ci} + Q_{ac}(X)E_{bi} \\ &\quad - \nabla_X E_{ai} - J_a P A_{v_i} X - \sum_{i=1}^s \{ W_{bi}(A_{v_i} X)E_{ci} - W_{ci}(A_{v_i} X)E_{bi} \\ &\quad - W_{ai}(A_{v_i} X)v_i \} + t_a \nabla_X^\perp v_i + f_a \nabla_X^\perp v_i. \end{aligned}$$

Considering the tangential and normal parts of the last equation we get

$$h(X, E_{ai}) = \sum_{i=1}^s W_{ai}(A_{v_i} X)v_i + f_a \nabla_X^\perp v_i,$$

and

$$\begin{aligned} 0 &= \frac{1}{2} \{ \theta_o(v_i)X - \omega(v_i)J_a X \} - Q_{ab}(X)E_{ci} + Q_{ac}(X)E_{bi} - \nabla_X E_{ai} \\ &\quad - J_a P A_{v_i} X - \sum_{i=1}^s \{ W_{bi}(A_{v_i} X)E_{ci} - W_{ci}(A_{v_i} X)E_{bi} \} + t_a \nabla_X^\perp v_i. \end{aligned}$$

The proof of the lemma is complete. $\square$

As a result of the lemma we have the following

Corollary 3.1. *Let $\bar{M}$ be a l.c.q.K. manifold and M be a QR-submanifold of $\bar{M}$. If the Lee vector field is tangent to D and $A_{v_i} X \in \Gamma(D^\perp)$ then D defines a totally geodesic foliation.*

Definition 3.1. A QR-submanifold is called mixed geodesic if $h(X, Y) = 0$ for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$ [2].

Theorem 3.1. *Let $\bar{M}$ be a l.c.q.K. manifold and M be a QR-submanifold of $\bar{M}$. Then M is mixed geodesic if and only if*

$$A_{v_i} X \in \Gamma(D) \quad (\text{III.3})$$

and

$$\nabla_X^\perp v_i \in \Gamma(\nu^\perp) \quad (\text{III.4})$$

for any $X \in \Gamma(D)$.

PROOF. ($\Rightarrow$) Let M be a mixed geodesic QR-submanifold. From (III.1) we get

$$0 = W_{ai}(A_{v_i}X)v_i + f_a \nabla_X^\perp v_i$$

or

$$W_{ai}(A_{v_i}X)v_i = 0, f_a \nabla_X^\perp v_i = 0.$$

Thus we have $A_{v_i}X \in \Gamma(D)$ and $\nabla_X^\perp v_i \in \Gamma(\nu^\perp)$.

($\Leftarrow$) We suppose that (III.3) and (III.4) are satisfied. From (III.1) we have $h(X, E_{ai}) = 0$ for any $X \in \Gamma(D)$. $\square$

Let $\bar{M}$ be a l.c.q.K. manifold and M a QR-submanifold of $\bar{M}$. From (II.8), (II.13), (II.14), (II.19) and (II.20) we get

$$\begin{aligned} h(X, J_aPY) &= f_a h(X, Y) - W_{ai}(\nabla_X Y)v_i + \frac{1}{2}\omega(Y)W_{ai}(X)v_i \\ &\quad - \frac{1}{2}\Omega(X, Y)B^\perp + \frac{1}{2}g(X, Y)B_0^\perp - Q_{ab}(X)\omega_{bi}(Y)v_i \\ &\quad - Q_{ac}(X)W_{ci}(Y)v_i - W_{bi}(Y)h(X, E_{ci}) + W_{ci}(Y)h(X, E_{bi}) \\ &\quad + X(W_{ai}(Y))v_i + W_{ai}(Y)\nabla_X^\perp v_i \end{aligned} \quad (\text{III.5})$$

for any $X, Y \in \Gamma(TM)$, where $B_o = J_a B, B^\perp = \text{Nor} B, B^T = \text{Tan} B$.

Lemma 3.2. *Let $\bar{M}$ be a l.c.q.K. manifold and M be a QR-submanifold of $\bar{M}$. Then we have*

$$\begin{aligned} h(X, J_a Y) &= f_a h(X, Y) - W_{ai}(\nabla_X Y)v_i - \frac{1}{2}\Omega(X, Y)B^\perp \\ &\quad + \frac{1}{2}g(X, Y)B_0^\perp \end{aligned} \quad (\text{III.6})$$

for any $X, Y \in \Gamma(D)$.

PROOF. It can easily be seen from (III.5). $\square$

From Lemma 3.4 we have the following

Corollary 3.2. *Let $\bar{M}$ be a l.c.q.K. manifold and M be a QR-submanifold of $\bar{M}$. If D is integrable and $h(X, J_a Y) = h(J_a X, Y)$ for any $X, Y \in \Gamma(D)$, $a = 1, 2, 3$, then the Lee vector field is tangent to M .*

Definition 3.2. Let M be a QR-submanifold of a l.c.q.K. manifold. Then M is called D -geodesic if $h(X, Y) = 0$ for any $X, Y \in \Gamma(D)$.

Theorem 3.2. *Let $\bar{M}$ be a l.c.q.K. manifold and M be a QR-submanifold of $\bar{M}$. Assume that the Lee vector field is tangent to M . Then the following assertions are equivalent:*

- 1) $h(X, J_a Y) = h(J_a X, Y)$ for any $X, Y \in \Gamma(D)$.
- 2) M is D -geodesic.
- 3) The quaternion distribution is integrable.

PROOF. (1) $\implies$ (2): Since $\bar{M}$ is a quaternion Hermitian manifold, we have $J_c \circ J_b = -J_b \circ J_c = J_a$. Thus we get

$$\begin{aligned} h(X, J_a Y) &= h(J_a X, Y) = h((J_c \circ J_b) X, Y) \\ &= h(J_b X, J_c Y) = h(X, (J_b \circ J_c) Y) = -h(X, J_a Y). \end{aligned}$$

Hence we have $h(X, J_a Y) = 0$.

(2) $\implies$ (3): By using (III.6) we get

$$-W_{ai}(\nabla_X Y)v_i + \frac{1}{2}g(X, Y)B_0^\perp = 0.$$

Thus, interchanging X and Y in the last equation, we have

$$-W_{ai}(\nabla_Y X)v_i + \frac{1}{2}g(Y, X)B_0^\perp = 0.$$

Hence we obtain $[Y, X] \in \Gamma(D)$.

(3) $\implies$ (1): We suppose that D is integrable. From (III.5) we obtain

$$-W_{ai}(\nabla_X Y)v_i - \frac{1}{2}\Omega(X, Y)B^\perp + \frac{1}{2}g(X, Y)B_0^\perp = 0,$$

or

$$-W_{ai}(\nabla_Y X)v_i - \frac{1}{2}\Omega(Y, X)B^\perp + \frac{1}{2}g(Y, X)B_0^\perp = 0,$$

for any $X, Y \in \Gamma(D)$. Hence we get

$$h(X, J_a Y) - h(J_a X, Y) = \Omega(Y, X)B^\perp.$$

Since B is tangent to M we have $h(X, J_a Y) = h(J_a X, Y)$. $\square$

Corollary 3.3. *Let M be a QR-submanifold of a l.c.q.K. manifold $\bar{M}$. Then $D^\perp$ defines a totally geodesic foliation if and only if*

$$A_{v_i} V \in \Gamma(D^\perp)$$

for any $V \in \Gamma(D^\perp)$.

PROOF. From (II.8) we have

$$\begin{aligned} \bar{\nabla}_{E_{bj}} v_i &= -\bar{\nabla}_{E_{bj}} J_a E_{ai} = -(\bar{\nabla}_{E_{bj}} J_a) E_{ai} - J_a \bar{\nabla}_{E_{bj}} E_{ai} \\ &= -\frac{1}{2} \{ \theta_0(E_{ai}) E_{bj} - \omega(E_{ai}) J_a E_{bi} - \Omega(E_{bj}, E_{ai}) B + g(E_{bj}, E_{ai}) J_a B \} \\ &\quad + Q_{ab}(E_{bj}) J_b E_{ai} + Q_{ac}(E_{bj}) J_c E_{ai} - J_a (\nabla_{E_{bj}} E_{ai} + h(E_{bj}, E_{ai})). \end{aligned}$$

Considering (II.20) we have

$$\begin{aligned} P A_{v_i} E_{bj} + \sum_{i=1}^s W_{bi}(A_{v_i} E_{bj}) E_{ci} - W_{ci}(A_{v_i} E_{bj}) E_{bi} + t_a \nabla_{E_{bj}} v_i \\ = -\frac{1}{2} \{ \theta_o(E_{ai}) E_{cj} + \omega(E_{ai}) E_{bi} \} \\ + Q_{ab}(E_{bj}) E_{bi} + Q_{ac}(E_{bj}) E_{ci} + \nabla_{E_{bj}} E_{ai}. \end{aligned} \quad (\text{III.7})$$

If $D^\perp$ defines a totally geodesic foliation then we have $P A_{v_i} E_{bj} = 0$. Hence $A_{v_i} E_{bj} \in \Gamma(D^\perp)$. Conversely, if $A_{v_i} E_{bj} \in \Gamma(D^\perp)$ then $D^\perp$ defines a totally geodesic foliation. $\square$

Lemma 3.3. *Let $\bar{M}$ be a l.c.q.K. manifold and M be a QR-submanifold of $\bar{M}$. Then, we have*

$$A_j E_{ai} = A_i E_{aj} + \frac{1}{2} \omega(v_i) E_{aj} - \frac{1}{2} \omega(v_j) E_{ai} \quad (\text{III.8})$$

for any $v_i, v_j \in \Gamma(D^\perp)$.

PROOF. From (II.8), (II.19) and (II.20) we have

$$\nabla_X E_{ai} + h(X, E_{ai}) = -J_a A_{v_i} X + J_a \nabla_X^\perp v_i$$

$$\begin{aligned}
& + \frac{1}{2} \{ \theta_o(v_i)X - \omega(v_i) J_a X - g(X, E_{ai})B \} \\
& + Q_{ab}(X)E_{bi} + Q_{ac}(X)E_{ci}
\end{aligned} \tag{III.9}$$

or

$$\begin{aligned}
g(h(X, E_{ai}), v_j) &= -g(J_a A_{v_i} X, v_j) + g(J_a \nabla_X^\perp v_i, v_j) \\
&+ \frac{1}{2} \omega(v_i) g(X, J_a v_j) - \frac{1}{2} \omega(v_j) g(E_{ai}, X) \\
g(A_j E_{ai}, X) &= g(A_{v_i} X, E_{aj}) + \frac{1}{2} \omega(v_i) g(X, E_{aj}) - \frac{1}{2} \omega(v_j) g(E_{ai}, X)
\end{aligned}$$

for any $X \in \Gamma(TM)$. Hence we get

$$A_j E_{ai} = A_i E_{aj} + \frac{1}{2} \omega(v_i) E_{aj} - \frac{1}{2} \omega(v_j) E_{ai}. \quad \square$$

Lemma 3.4. *Let M be a QR-submanifold of a l.c.q.K. manifold $\bar{M}$. Then, we have*

$$B_{aij}(X) = -\frac{1}{2} \delta_{ij} \omega(X) + g(A_j E_{ai}, J_a X) \tag{III.10}$$

for any $X \in \Gamma(D)$, where $B_{aij}(X) = g(\nabla_{E_{ai}} E_{aj}, X)$.

PROOF. From (III.9) we get

$$\begin{aligned}
g(\nabla_{E_{ai}} E_{aj}, X) &= -g(J_a A_{v_j} E_{ai}, X) - \frac{1}{2} g(E_{ai}, E_{aj}) g(B, X) \\
&= g(A_{v_j} E_{ai}, J_a X) - \frac{1}{2} \delta_{ij} \omega(X). \quad \square
\end{aligned}$$

Lemma 3.5. *Let $\bar{M}$ be a l.c.q.K. manifold and M be a QR-submanifold of $\bar{M}$. Then we have*

$$g(\nabla_{E_{ai}} E_{bj}, X) = -B_{aji}(J_c X) - \frac{1}{2} \delta_{ij} g(B, J_c X) \tag{III.11}$$

for any $X \in \Gamma(D)$.

PROOF. From (II.20) we obtain

$$g(\nabla_{E_{ai}} E_{bj}, X) = g(\bar{\nabla}_{E_{ai}} E_{bj}, X) = g(J_c \bar{\nabla}_{E_{ai}} E_{bj}, J_c X)$$

$$= g(\bar{\nabla}_{E_{ai}} J_c E_{bj} - (\bar{\nabla}_{E_{ai}} J_c) E_{bj}, J_c X).$$

Since $J_c E_{bj} = -E_{aj}$, we get

$$g(\nabla_{E_{ai}} E_{bj}, X) = -g(\bar{\nabla}_{E_{ai}} E_{aj}, J_c X) - g((\bar{\nabla}_{E_{ai}} J_c) E_{bj}, J_c X).$$

By using (II.8) we get

$$g(\nabla_{E_{ai}} E_{bj}, X) = -B_{aij}(J_c X) - \frac{1}{2} \delta_{ij} g(B, J_c X). \quad \square$$

From Lemma 3.4, Lemma 3.5 and Corollary 3.3, we have the following corollaries:

Corollary 3.4. *Let M be a QR-submanifold of a l.c.q.K. manifold $\bar{M}$. Then $D^\perp$ defines a totally geodesic foliation if and only if B is normal to D and $B_{aij}(X) = 0, X \in \Gamma(D)$.*

Corollary 3.5. *Let M be a QR-submanifold of a l.c.q.K. manifold $\bar{M}$. If the distribution $D^\perp$ is integrable and $B_{aij}(X) = 0, X \in \Gamma(D)$ for all $i, j = 1, \dots, s$, then B is normal to D .*

PROOF. From (III.8) and (III.10) we get

$$\begin{aligned} B_{aij}(X) &= -\frac{1}{2} \delta_{ij} \omega(X) + g(A_i E_{aj} + \frac{1}{2} \omega(v_i) E_{aj} - \frac{1}{2} \omega(v_j) E_{ai}, J_a X) \\ &= -\frac{1}{2} \delta_{ij} \omega(X) + g(A_j E_{ai}, J_a X) = B_{aji}(X). \end{aligned} \quad (III.12)$$

On the other hand we have

$$g(\nabla_{E_{bj}} E_{ai}, X) = -B_{bji}(J_c X) - \frac{1}{2} \delta_{ij} g(B, J_c X). \quad (III.13)$$

Thus, from (III.11) and (III.13) we get

$$g([E_{ai}, E_{bj}], X) = -B_{aij}(J_c X) - B_{bji}(J_c X) + \delta_{ij} g(B, J_c X). \quad (III.14)$$

From (III.14) and (III.12) the proof is results. $\square$

The rest of this section is devoted to the study of totally umbilical QR-submanifolds of a l.c.q.K. manifold.

We recall that any submanifold is called totally umbilical in a Riemann manifold if

$$h(X, Y) = g(X, Y)H \quad (III.15)$$

for any $X, Y \in \Gamma(TM)$, where H is the mean curvature vector.

Corollary 3.6. *Let $\bar{M}$ be a l.c.q.K. manifold and M be a totally umbilical QR-submanifold of $\bar{M}$. Then $D^\perp$ defines a totally geodesic foliation if and only if B is normal to the quaternion distribution.*

PROOF. From (III.10) and (III.15) we have

$$\begin{aligned} B_{aij}(X) &= -\frac{1}{2}\delta_{ij}\omega(X) + g(A_j E_{ai}, J_a X) = -\frac{1}{2}\delta_{ij}g(X, B) \\ &\quad + g(h(E_{ai}, J_a X), v_j) = -\frac{1}{2}\delta_{ij}g(X, B), \end{aligned} \quad (\text{III.16})$$

for any $X \in \Gamma(D)$. By using (III.11) we get

$$g(\nabla_{E_{ai}} E_{bj}, X) = -B_{aij}(J_c X) - \frac{1}{2}\delta_{ij}g(B, J_c X). \quad (\text{III.17})$$

Thus from (III.16) and (III.17) we have the assertion of the corollary. $\square$

Theorem 3.3. *Let $\bar{M}$ be a l.c.q.K. manifold and M be a totally umbilical QR-submanifold of $\bar{M}$. Assume that the Lee vector field is tangent to M . If $\dim \nu_x^\perp > 1$ for $x \in M$, then the QR-submanifold is totally geodesic.*

PROOF. From (III.8) we have

$$A_j E_{ai} = A_i E_{aj} + \frac{1}{2}\omega(v_i) E_{aj} - \frac{1}{2}\omega(v_j) E_{ai}$$

for $X, Y \in \Gamma(D_{ax})$, hence

$$A_{J_a X} Y = A_{J_a Y} X + \frac{1}{2}\omega(v_i) X - \frac{1}{2}\omega(v_j) Y,$$

where $J_a v_i = Y, J_a v_j = X$. Since $t_a H \in \Gamma(D_{ax})$ at each $x \in M$, we have

$$A_{J_a X} t_a H = A_{J_a t_a H} X + \frac{1}{2}\omega(t_a H) X - \frac{1}{2}\omega(v_j) t_a H.$$

Now we derive

$$\begin{aligned} g(A_{J_a X} t_a H, X) &= g(A_{J_a t_a H} X, X) + \frac{1}{2}\omega(J_a t_a H) g(X, X) \\ &\quad - \frac{1}{2}\omega(v_j) g(t_a H, X) \end{aligned}$$

$$g(h(t_a H, X), J_a X) = g(h(X, X), J_a t_a H) + \frac{1}{2} \omega(J_a t_a H) g(X, X) - \frac{1}{2} \omega(v_j) g(t_a H, X).$$

Since M is totally umbilical, we have

$$\begin{aligned} g(t_a H, X) g(H, J_a X) &= g(X, X) g(H, J_a t_a H) + \frac{1}{2} \omega(J_a t_a H) g(X, X) \\ &\quad - \frac{1}{2} \omega(v_j) g(t_a H, X) \\ &= -g(X, X) g(t_a H, t_a H) + \frac{1}{2} \omega(J_a t_a H) g(X, X) - \frac{1}{2} \omega(v_j) g(t_a H, X). \end{aligned}$$

By the hypothesis of the theorem, we can choose $X \in \Gamma(TM)$ such that $X \neq 0$ and X is orthogonal to $B_a H$. Since the Lee vector field is tangent to M , we obtain

$$0 = -g(X, X) g(t_a H, t_a H),$$

that is

$$t_a H = 0. \quad (\text{III.18})$$

On the other hand, by using (II.12), (II.14), (II.19) and (II.20) in (II.8) and taking the tangential parts we obtain

$$\begin{aligned} \nabla_Y t_a V - A_{f_a V} Y &= -J_a P A_V Y - W_{bk} (A_V Y) E_{ck} + W_{ck} (A_V Y) E_{bk} \\ &\quad + t_a \nabla_Y^\perp V + \frac{1}{2} \theta_o(V) Y - \frac{1}{2} \omega(V) \phi_a Y \\ &\quad - \frac{1}{2} \Omega(Y, V) B^T + Q_{ab}(Y) t_b V + Q_{ac}(Y) t_c V \end{aligned}$$

or

$$\begin{aligned} P \nabla_Y t_a V - P A_{f_a V} Y &= -J_a P A_V Y + \frac{1}{2} \theta_o(V) P Y - \frac{1}{2} \omega(V) P \phi_a Y \\ &\quad - \frac{1}{2} \Omega(Y, V) P B^T \end{aligned} \quad (\text{III.19})$$

for any $Y \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$. From (III.18) and (III.19) we get

$$-P A_{f_a H} Y = -J_a P A_H Y + \frac{1}{2} \theta_o(H) P Y - \frac{1}{2} \Omega(Y, H) P B^T.$$

For $Z \in \Gamma(D)$, we have

$$\begin{aligned} g(PA_{f_a H}Y, Z) &= -g(A_{f_a H}Y, Z) = -g(h(Y, Z), J_a H) \\ &= -g(Y, Z)g(H, J_a H) = 0. \end{aligned}$$

Hence we obtain

$$\begin{aligned} g(J_a PA_H Y, Z) + \frac{1}{2}\theta_o(H)g(PY, Z) - \frac{1}{2}\Omega(Y, H)g(PB^T, Z) &= 0 \\ g(PA_H Y, J_a Z) + \frac{1}{2}\theta_o(H)g(PY, Z) - \frac{1}{2}g(Y, J_a H)g(PB^T, Z) &= 0 \\ g(A_H Y, J_a Z) + \frac{1}{2}\theta_o(H)g(PY, Z) - \frac{1}{2}g(Y, t_a H)g(PB^T, Z) &= 0 \\ g(h(Y, J_a Z), H) + \frac{1}{2}\theta_o(H)g(PY, Z) - \frac{1}{2}g(Y, t_a H)g(PB^T, Z) &= 0. \end{aligned}$$

Since the Lee vector field is tangent to M and $t_a H = 0$, we get

$$\begin{aligned} g(h(Y, J_a Z), H) &= 0 \\ g(Y, J_a Z)g(H, H) &= 0. \end{aligned}$$

Thus, we obtain $H = 0$ for $Y = J_a Z$. $\square$

Let $\bar{M}$ be a compact l.c.q.K. manifold. Then we can choose the metric g such that

i) The fixed metric g makes ω parallel:

$$\bar{\nabla}\omega = 0, \quad (\text{III.20})$$

ii)

$$\|\omega\| = 1 \quad (\text{III.21})$$

[6]. From now on we will denote a compact l.c.q.K. manifold by $\bar{M}$.

Lemma 3.6. *Let K_0 be the curvature tensor field of the Weyl connection $\bar{D}'$ of the l.c.q.K. manifold $\bar{M}$ and $\bar{R}$ the curvature tensor field of the Levi-Civita connection ∇ of the l.c.q.K. manifold $\bar{M}$. Then we have*

$$\begin{aligned} K_0(X, Y)Z &= \bar{R}(X, Y)Z + \frac{1}{4}\{\omega(Z)\omega(Y)X - \omega(Z)\omega(X)Y\} \\ &\quad + \frac{1}{4}\{-\omega(Y)g(X, Z) + \omega(X)g(Y, Z)\}B - \frac{1}{4}(X \wedge Y)Z, \end{aligned} \quad (\text{III.22})$$

for any $X, Y, Z \in \Gamma(T\bar{M})$.

PROOF. From (II.2) we have

$$\begin{aligned} K_0(X, Y)Z &= \bar{R}(X, Y)Z - \frac{1}{2}\{L_0(X, Z)Y - L_0(Y, Z)X \\ &\quad + g(X, Z)L_0(Y, \cdot)^\# - g(Y, Z)L_0(X, \cdot)^\#\} - \frac{1}{4}\|\omega\|^2(X \wedge Y)Z, \end{aligned}$$

for any $X, Y, Z \in \Gamma(T\bar{M})$, where $L_0 = \bar{\nabla}\omega + \frac{1}{2}\omega \otimes \omega$. Since $\bar{\nabla}\omega = 0$ we have

$$\bar{\nabla}B = 0. \quad (\text{III.23})$$

Thus, from (III.20), (III.21) and (III.23) we have the assertion of the lemma. $\square$

Let $\bar{D}'$ be the Weyl connection of $\bar{M}$. Then we have

$$K_0(X, Y)J_1Z - J_1K_0(X, Y)Z = \alpha(X, Y)J_2Z - \beta(X, Y)J_3Z, \quad (\text{III.24})$$

where

$$\alpha = dQ_{12} + Q_{32} \wedge Q_{13}$$

and

$$\beta = dQ_{13} + Q_{23} \wedge Q_{12}.$$

Theorem 3.4. *There exist no proper totally umbilical QR-submanifolds in negatively curved L.c.q.K. manifolds with $B^T = 0$.*

PROOF. Considering the definition of a QR-submanifold, from (III.22) and (III.24) we have

$$\begin{aligned} &-\bar{R}(X, E_{ai}, J_a X, v_i) - \bar{R}(X, E_{ai}, X, E_{ai}) \\ &= -\frac{1}{4}\omega(X)\omega(X) - \frac{1}{4}\omega(E_{ai})\omega(E_{ai}) + \frac{1}{4} \end{aligned}$$

for any orthonormal vector field $X \in \Gamma(D)$ and $E_{ai} \in \Gamma(D^\perp)$. Thus, if B is normal to M we get

$$-\bar{R}(X, E_{ai}, J_a X, v_i) + \bar{R}(X, E_{ai}, E_{ai}, X) = \frac{1}{4}. \quad (\text{III.25})$$

Now suppose that M is a proper totally umbilical QR-submanifold of $\bar{M}$ with $K_{\bar{M}} < 0$.

Then from the equation of Codazzi we have

$$g(\bar{R}(X, Y)Z, W) = g(Y, Z)g\left(\nabla_X^\perp H, W\right) - g(X, Z)g\left(\nabla_Y^\perp H, W\right),$$

for any X, Y, Z tangent to M and V normal to M . Thus, if we take $X \in \Gamma(D), Z = J_1 X, Y = E_{1i}$ and $W = v_i$ we obtain

$$\bar{R}(X, E_{ai}, J_a X, v_i) = 0. \quad (\text{III.26})$$

Using (III.25) and (III.26) we get $K_{\bar{M}}(X, E_{1i}) = \frac{1}{4}$ which is a contradiction. $\square$

From the Gauss equation for totally umbilical submanifolds we have

$$K_M(X, Y) = K_{\bar{M}}(X, Y) + \|H\|^2,$$

for any X, Y tangent to M [3]. Now we take $X \in \Gamma(D)$ and $E_{ai} = Y$ in this equation and taking account of $K_{\bar{M}}(X, E_{ai}) = 0$ we obtain

$$K_M(X, E_{ai}) = \|H\|^2 + \frac{1}{4}. \quad (\text{III.27})$$

Thus, from (III.27) and (III.26) we have the following.

Corollary 3.7. *There exist no proper totally umbilical negatively curved QR-submanifolds of a l.c.q.K. manifold with $B^T = 0$.*

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