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Existence of solutions for nonlinear second order systems on a measure chain

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Abstract. Under suitable conditions on positive functions $f(t,v)$ and $g(t,u)$, we prove that the nonlinear second order systems on a measure chain

$$
\text{(BVPS)}\qquad \begin{cases} (\mathrm{E}_1) & u^{\Delta\Delta}(t) + f(t, v(\sigma(t))) = 0, \quad 0 < t < 1,\\ (\mathrm{E}_2) & v^{\Delta\Delta}(t) + g(t, u(\sigma(t))) = 0, \quad 0 < t < 1,\\ (\mathrm{BC}_1) & \begin{cases} \alpha_1 u(0) - \beta_1 u^{\Delta}(0) = 0,\\ \gamma_1 u(\sigma(1)) + \delta_1 u^{\Delta}(\sigma(1)) = 0, \end{cases}\\ (\mathrm{BC}_2) & \begin{cases} \alpha_2 v(0) - \beta_2 v^{\Delta}(0) = 0,\\ \gamma_2 v(\sigma(1)) + \delta_2 v^{\Delta}(\sigma(1)) = 0, \end{cases} \end{cases}
$$

has at least one positive solution.

1. Introduction

In 1990, S. Hilger [8] introduced the theorey of measure chain in order to unify continuous and discrete calculus. Recently, the developement of theory of measure chain has received a lot of attention, see [1]–[7], [9], [11], [12].

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The existence of solutions of the boundary value problem

$$\text{(BVP)}\qquad \begin{cases} \text{(E)} & u^{\Delta\Delta}(t) + f(t, u(\sigma(t))) = 0, \quad 0 < t < 1, \\ \text{(BC)} & \begin{cases} \alpha u(0) - \beta u^{\Delta}(0) = 0, \\ \gamma u(\sigma(1)) + \delta u^{\Delta}(\sigma(1)) = 0 \end{cases} \end{cases}$$

on a measure chain has been studied by many authors, see, for examples, Chyan and Henderson [4], Erbe and Peterson [7], Hong and Yeh [9] and W. C. Lian, C. C. Chou, C. T. Liu and F. H. Wong [12].

In this article, we shall consider the existence of positive solutions of the following boundary value problem systems

$$\text{(BVPS)}\qquad \begin{cases} \text{(E}_1\text{)} & u^{\Delta\Delta}(t) + f(t, v(\sigma(t))) = 0, \quad 0 < t < 1, \\ \text{(E}_2\text{)} & v^{\Delta\Delta}(t) + g(t, u(\sigma(t))) = 0, \quad 0 < t < 1, \\ \text{(BC}_1\text{)} & \begin{cases} \alpha_1 u(0) - \beta_1 u^{\Delta}(0) = 0, \\ \gamma_1 u(\sigma(1)) + \delta_1 u^{\Delta}(\sigma(1)) = 0, \end{cases} \\ \text{(BC}_2\text{)} & \begin{cases} \alpha_2 v(0) - \beta_2 v^{\Delta}(0) = 0, \\ \gamma_2 v(\sigma(1)) + \delta_2 v^{\Delta}(\sigma(1)) = 0, \end{cases} \end{cases}$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i$ are nonnegative real numbers and $r_i := \gamma_i\beta_i + \alpha_i\delta_i + \alpha_i\gamma_i\sigma(1) > 0$, $i = 1, 2$ and $f, g \in C_{rd}([0, \sigma(1)] \times [0, \infty), (0, \infty))$.

2. Main results

In order to abbreviate our discussion, throughout this paper, we suppose that the following assumptions hold:

(C$_1$) $\xi := \min\left\{t \in T \mid t \geq \frac{\sigma(1)}{4}\right\}$ and $\omega := \max\left\{t \in T \mid t \leq \frac{3\sigma(1)}{4}\right\}$ both exist and satisfy

$$\frac{\sigma(1)}{4} \leq \xi < \omega \leq \frac{3\sigma(1)}{4}.$$

(C$_2$) $G_i(t, s)$ is the Green's function of the differential equation

$$-u^{\Delta\Delta}(t) = 0 \quad \text{in } (0, 1)$$

satisfying the boundary value condition (BC_i);

(C_3) $M_i = \min\{d_i, l_i\}$, where

$$d_i := \min\left\{\frac{\gamma_i\sigma(1)+4\delta_i}{4(\gamma_i\sigma(1)+\delta_i)}, \frac{\alpha_i\sigma(1)+4\beta_i}{4(\alpha_i\sigma(1)+\beta_i)}\right\} \in (0,1)$$

and

$$l_i = \min_{s\in[0,\sigma(1)]} \frac{G_i(\sigma(\omega),s)}{G_i(\sigma(s),s)}.$$

(C_4) $f, g \in C([0,\sigma(1)]\times[0,\infty);(0,\infty))$.

In order to prove our main result (Theorem 2.1 below), we shall need the following two useful lemmas:

Lemma 2A (Erbe and Peterson [7]). *Let M_i be defined as in* (C_3). *For the Green's function $G_i(t,s)$ $(i=1,2)$ the following results hold:*

$$\begin{cases} (\mathrm{R}_1) \quad \dfrac{G_i(t,s)}{G_i(\sigma(s),s)} \le 1 \quad \text{for } t\in[0,\sigma(1)] \text{ and } s\in[0,\sigma(1)], \\ (\mathrm{R}_2) \quad \dfrac{G_i(t,s)}{G_i(\sigma(s),s)} \ge M_i \quad \text{for } t\in[\xi,\omega] \text{ and } s\in[0,\sigma(1)]. \end{cases}$$

Lemma 2B (Krasnoselskii [10]). *Let $P \subseteq E$ be a cone in a Banach space E. Assume that Ω_1, Ω_2 are open subsets of E with $0\in\Omega_1, \overline{\Omega_1}\subset\Omega_2$. If*

$$\Phi : P\cap(\overline{\Omega_2}\backslash\Omega_1) \longrightarrow P$$

is a completely continuous operator such that either

(i) $\|\Phi u\| \le \|u\|,\ u\in P\cap\partial\Omega_1$ *and* $\|\Phi u\|\ge\|u\|,\ u\in P\cap\partial\Omega_2$*; or*

(ii) $\|\Phi u\| \ge \|u\|,\ u\in P\cap\partial\Omega_1$ *and* $\|\Phi u\|\le\|u\|,\ u\in P\cap\partial\Omega_2$,

then Φ has a fixed point in $P\cap(\overline{\Omega_2}\backslash\Omega_1)$.

Denote the Banach space $E = \{\mathcal{U} = (u,v)\in (C[0,\sigma(1)])^2\}$ with norm

$$\|\mathcal{U}\| = \|(u,v)\| := \max\{\|u\|_\infty, \|v\|_\infty\}.$$

Here $\|u\|_\infty := \sup\limits_{0\le t\le\sigma(1)} |u(t)|$. Define a set $P\subset E$ by

$$P := \Big\{(u,v) \mid (u,v)\ge(0,0), \\ \big(\min_{\xi\le t\le\sigma(\omega)]} u(t), \min_{\xi\le t\le\sigma(\omega)} v(t)\big) \ge (M_1\|u\|_\infty, M_2||v||_\infty)\Big\},$$

where $(a,b) \geq (c,d)$ means that $a \geq c$ and $b \geq d$. Here $a,b,c,d \in \Re$. It is clear that P is a cone in E.

Now, we can state and prove our main result.

Theorem 2.1 (Main result)**.** *Assume that there exist four positive constants η_1, η_2, λ_1 and λ_2 such that for $(v,u) \in [0,\lambda_1] \times [0,\lambda_2]$,*

$$\Big(\int_0^{\sigma(1)} G_1(t,s)f(s,v)\Delta s, \int_0^{\sigma(1)} G_2(t,s)g(s,u)\Delta s\Big) \leq (\lambda_1,\lambda_2) \qquad (1)$$

and for $(v,u) \in [M_1\eta_1,\eta_1] \times [M_2\eta_2,\eta_2]$,

$$\Big(\int_\xi^{\omega} G_1(\theta,s)f(s,v)\Delta s, \int_\xi^{\omega} G_2(\theta,s)g(s,u)\Delta s\Big) \geq (\eta_1,\eta_2), \qquad (2)$$

where $\theta \in (\xi,\omega)$. Then (1) *has at least one positive solution* (u,v) *between λ and η, where $\lambda := \max\{\lambda_1,\lambda_2\}$ and $\eta := \min\{\eta_1,\eta_2\}$.*

Proof. Without loss of generality, we may assume that $\lambda < \eta$. It is clear that (1) has a solution $\mathcal{U} := (u,v) = (u(t),v(t))$ if and only if $\mathcal{U}$ is the solution of the operator equation

$$\begin{aligned}\Phi\mathcal{U}(t) &:= (\Phi u(t), \Phi v(t))\\ &:= \Big(\int_0^{\sigma(1)} G_1(t,s)f(s,v(\sigma(s)))\Delta s, \int_0^{\sigma(1)} G_2(t,s)g(s,u(\sigma(s)))\Delta s\Big)\\ &= \mathcal{U}(t) \text{ for } t \in [0,\sigma(1)] \text{ and } \mathcal{U} \in E.\end{aligned}$$

It follows from the definition of P and Lemma 2A that

$$\begin{aligned}&\min_{t\in[\xi,\omega]}(\Phi\mathcal{U})(t)\\ &= \Big(\min_{t\in[\xi,\omega]}\int_0^{\sigma(1)} G_1(t,s)f(s,v(\sigma(s)))\Delta s, \min_{t\in[\xi,\omega]}\int_0^{\sigma(1)} G_2(t,s)g(s,u(\sigma(s)))\Delta s\Big)\\ &\geq \Big(M_1\int_0^{\sigma(1)} G_1(\sigma(s),s)f(s,v(\sigma(s)))\Delta s,\ M_2\int_0^{\sigma(1)} G_2(\sigma(s),s)g(s,u(\sigma(s)))\Delta s\Big)\\ &\geq \Big(M_1\int_0^{\sigma(1)} G_1(t,s)f(s,v(\sigma(s)))\Delta s,\ M_2\int_0^{\sigma(1)} G_2(t,s)g(s,u(\sigma(s)))\Delta s\Big)\end{aligned}$$

and

$$\begin{aligned}
&(\Phi\mathcal{U})(\sigma(\omega))\\
&= \Big(\int_0^{\sigma(1)} G_1(\sigma(\omega),s)f(s,v(\sigma(s)))\Delta s,\ \int_0^{\sigma(1)} G_2(\sigma(\omega),s)g(s,u(\sigma(s)))\Delta s\Big)\\
&\geq \Big(l_1 \int_0^{\sigma(1)} G_1(\sigma(s),s)f(s,v(\sigma(s)))\Delta s,\ l_2\int_0^{\sigma(1)} G_2(\sigma(s),s)g(s,u(\sigma(s)))\Delta s\Big)\\
&\geq \Big(M_1 \int_0^{\sigma(1)} G_1(\sigma(s),s)f(s,v(\sigma(s)))\Delta s,\ M_2\int_0^{\sigma(1)} G_2(\sigma(s),s)g(s,u(\sigma(s)))\Delta s\Big)\\
&\geq \Big(M_1 \int_0^{\sigma(1)} G_1(t,s)f(s,v(\sigma(s)))\Delta s,\ M_2\int_0^{\sigma(1)} G_2(t,s)g(s,u(\sigma(s)))\Delta s\Big).
\end{aligned}$$

Hence

$$\min_{t\in[\xi,\sigma(\omega)]} \Phi\mathcal{U}(t) \geq \Big(M_1\|\Phi u\|_\infty, M_2\|\Phi v\|_\infty\Big),$$

which implies $\Phi P\subset P$. Furthermore, it is easy to check that $\Phi : P \to P$ is completely continuous. In order to complete the proof, we separate the rest of the proof into the following two steps:

Step (I) Let $\Omega_1 := \{\mathcal{U} \in P \mid \|\mathcal{U}\| < \lambda\}$. It follows from (1), Lemmas 2A–2B and the fact $\mathcal{U}\in P$ that for $\mathcal{U}\in\partial\Omega_1$,

$$\begin{aligned}
\Phi\mathcal{U}(t) &= \Big(\int_0^{\sigma(1)} G_1(t,s)f(s,v(\sigma(s)))\Delta s, \int_0^{\sigma(1)} G_2(t,s)g(s,u(\sigma(s)))\Delta s\Big)\\
&\leq \Big(\int_0^{\sigma(1)} G_1(\sigma(s),s)f(s,v(\sigma(s)))\Delta s,\ \int_0^{\sigma(1)} G_2(\sigma(s),s)g(s,u(\sigma(s)))\Delta s\Big)\\
&\leq (\lambda_1,\lambda_2) \leq (\lambda,\lambda) = (||\mathcal{U}||, ||\mathcal{U}||).
\end{aligned}$$

Hence,

$$\|\Phi\mathcal{U}\| \leq \|\mathcal{U}\| \quad \text{for } \mathcal{U}\in\partial\Omega_1.$$

Step (II) Let $\Omega_2 := \{\mathcal{U} \in P \mid \|\mathcal{U}\| < \eta\}$. It follows from the definitions of $\|\mathcal{U}\|$, P and Lemma 2B that for $\mathcal{U} \in \partial\Omega_2$,

$$\mathcal{U} = (u(t), v(t)) \leq (\|\mathcal{U}\|, \|\mathcal{U}\|) = (\eta, \eta) \text{ for } t \in [0, \sigma(1)],$$

and for $t \in [\xi, \sigma(\omega)]$,

$$\begin{aligned}(u(t), v(t)) &\geq (\min_{t\in[\xi,\sigma(\omega)]} u(t), \min_{t\in[\xi,\sigma(\omega)]} v(t)) \\ &\geq (M_1\|u\|_\infty, M_2\|v\|_\infty) \\ &= (M_1\eta_1, M_2\eta_2) \\ &\geq (M_1\eta, M_2\eta),\end{aligned}$$

which implies

$$(M_1\eta, M_2\eta) \leq (u(t), v(t)) \leq (\eta, \eta).$$

Hence, by (2),

$$\begin{aligned}(\Phi\mathcal{U})(\theta) &= \left(\int_0^{\sigma(1)} G_1(\theta, s) f(s, v(\sigma(s)))\Delta s, \int_0^{\sigma(1)} G_2(\theta, s) g(s, u(\sigma(s)))\Delta s\right) \\ &\geq \left(\int_\xi^{\omega} G_1(\theta, s) f(s, v(\sigma(s)))\Delta s, \int_\xi^{\omega} G_2(\theta, s) g(s, u(\sigma(s)))\Delta s\right) \\ &\geq (\eta_1, \eta_2) \\ &\geq (\eta, \eta) = (\|\mathcal{U}\|, \|\mathcal{U}\|).\end{aligned}$$

Thus,

$$\|\Phi\mathcal{U}\| \geq \|\mathcal{U}\| \quad \text{for } \mathcal{U} \in \partial\Omega_2.$$

Hence, by the first part of Lemma 2B, we complete the proof. □

Let

$$\max h_0 := \lim_{w\to 0^+} \max_{t\in[0,\sigma(1)]} \frac{h(t, w)}{w},$$

$$\min h_0 := \lim_{w\to 0^+} \min_{t\in[\xi,\sigma(\omega)]} \frac{h(t, w)}{w},$$

$$\max h_\infty := \lim_{w\to\infty} \max_{t\in[0,\sigma(1)]} \frac{h(t,w)}{w},$$

$$\min h_\infty := \lim_{w\to\infty} \min_{t\in[\xi,\sigma(\omega)]} \frac{h(t,w)}{w}.$$

Then we have the following:

Remark 2.2. Let α, β, γ and δ be nonnegative constants, $r := \gamma\beta + \alpha\delta + \alpha\gamma\sigma(1) > 0$,

$$M := \min\left\{\frac{\gamma\sigma(1)+4\delta}{4(\gamma\sigma(1)+\delta)}, \frac{\alpha\sigma(1)+4\beta}{4(\alpha\sigma(1)+\beta)}\right\},$$

and $G(t,s)$ the Green's function of the differential equation

$$u^{\Delta\Delta}(t) = 0 \quad \text{on} \quad (0,1)$$

satisfying the boundary value conditions

$$\begin{cases} \alpha w(0) - \beta w^{\Delta}(0) = 0, \\ \gamma w(\sigma(1)) + w^{\Delta}(\sigma(1)) = 0. \end{cases}$$

Let

$$\left(\int_0^{\sigma(1)} G(\sigma(s),s)\Delta s\right)^{-1} := A \quad \text{and} \quad \left(\int_\xi^{\omega} G(\theta,s)\Delta s\right)^{-1} := B.$$

$$\left(\int_0^{\sigma(1)} G_i(\sigma(s),s)\Delta s\right)^{-1} := A_i \quad \text{and} \quad \left(\int_\xi^{\omega} G_i(\theta,s)\Delta s\right)^{-1} := B_i,\ (i=1,2).$$

Then, we have the following results.

(I) Suppose that $\max h_0 := C_1 \in [0, A)$. Taking $\epsilon = A - C_1 > 0$, there exists $\lambda_1 > 0$ (λ_1 can be chosen arbitrarily small) such that

$$\max_{t\in[0,\sigma(1)]} \frac{h(t,w)}{w} \le \epsilon + C_1 = A \quad \text{on } [0,\lambda_1].$$

Hence,

$$h(t,w) \le Aw \le A\lambda_1 \text{ on } \quad [0,\sigma(1)]\times[0,\lambda_1].$$

If we replace h by f and g, and replace A by A_1 and A_2, respectively. Then, the hypothesis (1) of Theorem 2.1 is satisfied.

(II) Suppose that $\min h_\infty := C_2 \in (\frac{B}{M}, \infty]$. Taking $\epsilon = C_2 - \frac{B}{M} > 0$, there exists $\eta_1 > 0$ (η_1 can be chosen arbitrarily large) such that

$$\min_{t\in[\xi,\sigma(\omega)]} \frac{h(t,w)}{w} \geq -\epsilon + C_2 = \frac{B}{M} \quad \text{on } [M\eta_1, \infty).$$

Hence,

$$h(t,w) \geq \frac{B}{M} w \geq \frac{B}{M} M\eta_1 = B\eta_1$$

on $[\xi, \sigma(\omega)] \times [M\eta_1, \eta_1]$. If we replace h by f and g, and replace B by B_1 and B_2, respectively. Then, the hypothesis (2) of Theorem 2.1 is satisfied.

(III) Suppose that $\min h_0 := C_3 \in (\frac{B}{M}, \infty]$. Taking $\epsilon = C_3 - \frac{B}{M} > 0$, there exists $\eta_2 > 0$ (η_2 can be chosen arbitrarily small) such that

$$\min_{t\in[\xi,\sigma(\omega)]} \frac{h(t,w)}{w} \geq -\epsilon + C_3 = \frac{B}{M} \quad \text{on } [0, \eta_2].$$

Hence,

$$h(t,w) \geq \frac{B}{M} w \geq \frac{B}{M} M\eta_2 = B\eta_2$$

on $[\xi, \sigma(\omega)] \times [0, \eta_2]$. If we replace h by f and g, and replace B by B_1 and B_2, respectively. Then, the hypothesis (2) of Theorem 2.1 is satisfied.

(IV) Suppose that $\max h_\infty := C_4 \in [0, A)$. Taking $\epsilon = A - C_4 > 0$, there exist $\theta > 0$ (θ can be chosen arbitrarily large) such that

$$\max_{t\in[0,\sigma(1)]} \frac{h(t,w)}{w} \leq \epsilon + C_4 = A \quad \text{on } [\theta, \infty). \tag{3}$$

Hence, we have the following two cases:

Case (i): Assume that $\max_{t\in[0,\sigma(1)]} h(t,w)$ is bounded, say,

$$h(t,w) \leq L \quad \text{on } [0, \sigma(1)] \times [0, \infty).$$

Taking $\lambda_2 = \frac{L}{A}$ (since L can be chosen arbitrarily large, λ_2 can be chosen arbitrarily large, too),

$$h(t,w) \leq L = A\lambda_2 \quad \text{on } [0, \sigma(1)] \times [0, \lambda_1] \subseteq [0, \sigma(1)] \times [0, \infty).$$

Case (ii): Assume that $\max_{t\in[0,\sigma(1)]} h(t,w)$ is unbounded, hence, there exists a $\lambda_2 \geq \theta$ (λ_2 can be chosen arbitrarily large) and $t_0 \in [0,\sigma(1)]$ such that

$$h(t,w) \leq h(t_0,\lambda_2) \quad \text{on } [0,\sigma(1)] \times [0,\lambda_2].$$

It follows from $\lambda_2 \geq \theta$ and (3) that

$$h(t,w) \leq h(t_0,\lambda_2) \leq A\lambda_2 \quad \text{on } [0,\sigma(1)] \times [0,\lambda_2].$$

By cases (i) and (ii), if we replace h by f and g, and replace A by A_1 and A_2, respectively. Then, the hypothesis (1) of Theorem 2.1 is satisfied.

By Remark 2.2, we have the following three corollaries.

Corollary 2.3. *Let*

$$A_i := \left(\int\limits_0^{\sigma(1)} G_i(\sigma(s),s)\Delta s \right)^{-1} \text{ and } \quad B_i := \left(\int\limits_\xi^{\omega} G_i(\theta,s)\Delta s \right)^{-1}, \quad (i=1,2).$$

Then, (1) *has at least one positive solution if one of the following conditions holds:*

(H$_1$) $\max f_0 = D_1 \in [0,A_1)$, $\max g_0 = E_1 \in [0,A_2)$, $\min f_\infty = D_2 \in (\frac{B_1}{M_1},\infty]$ *and* $\min g_\infty = E_2 \in (\frac{B_2}{M_2},\infty]$*;*

(H$_2$) $\min f_0 = D_3 \in (\frac{B_1}{M_1},\infty]$, $\min g_0 = E_3 \in (\frac{B_2}{M_2},\infty]$, $\max f_\infty = D_4 \in [0,A_1)$ *and* $\max g_\infty = E_4 \in [0,A_2)$*;*

(H$_3$) $\max f_0 = D_1 \in [0,A_1)$, $\min g_0 = E_3 \in (\frac{B_2}{M_2},\infty]$, $\min f_\infty = D_2 \in (\frac{B_1}{M_1},\infty]$, *and* $\max g_\infty = E_4 \in [0,A_2)$*;*

(H$_4$) $\min f_0 = D_3 \in (\frac{B_1}{M_1},\infty]$, $\max g_0 = E_1 \in [0,A_2)$, $\max f_\infty = D_4 \in [0,A_1)$ *and* $\min g_\infty = E_2 \in (\frac{B_2}{M_2},\infty]$.

Proof. It follows from Remark 2.2 and Theorem 2.1 that the desired result holds, immediately. □

Corollary 2.4. *Let A_i and B_i be defined as in Corollary 2.3. Then,* (1) *has at least two positive solutions $\mathcal{U}_1$ and $\mathcal{U}_2$ such that*

$$0 < \|\mathcal{U}_1\| < \lambda^* < \|\mathcal{U}_2\|,$$

if the following hypotheses hold:

(H$_5$) $\min f_\infty = C_1$, $\min f_0 = C_2 \in (\frac{B_1}{M_1}, \infty]$ *and* $\min g_\infty = D_1$, $\min g_0 = D_2 \in (\frac{B_2}{M_2}, \infty]$;

(H$_6$) *there exists a real number* $\lambda^* = \max\{\lambda_1^*, \lambda_2^*\} > 0$ *such that*

$$\begin{cases} f(t,v) \le A_1\lambda_1^* & \text{on } [0,\lambda_1^*], \\ g(t,u) \le A_2\lambda_2^* & \text{on } [0,\lambda_2^*]. \end{cases}$$

Proof. It follows from Remark 2.2 that there exist four real numbers $\eta_{1,1}$, $\eta_{1,2}$, $\eta_{2,1}$ and $\eta_{1,2}$ satisfying

$$0 < \eta_{1,1} < \lambda^* < \eta_{1,2}, \quad 0 < \eta_{2,1} < \lambda^* < \eta_{2,2},$$

$$\begin{cases} f(t,v) \ge B_1\eta_{1,1} & \text{on } [\xi,\sigma(\omega)] \times [M_1\eta_{1,1}, \eta_{1,1}], \\ g(t,u) \ge B_2\eta_{1,2} & \text{on } [\xi,\sigma(\omega)] \times [M_2\eta_{1,2}, \eta_{1,2}] \end{cases}$$

and

$$\begin{cases} f(t,v) \ge B_1\eta_{2,1} & \text{on } [\xi,\sigma(\omega)] \times [M_1\eta_{2,1}, \eta_{2,1}], \\ g(t,u) \ge B_2\eta_{2,2} & \text{on } [\xi,\sigma(\omega)] \times [M_2\eta_{2,2}, \eta_{2,2}]. \end{cases}$$

Hence, by Theorem 2.1, we see that (1) has two positive solutions $\mathcal{U}_1$ and $\mathcal{U}_2$ such that

$$\eta_2 < \|\mathcal{U}_1\| < \lambda^* < \|\mathcal{U}_2\| < \eta_1;$$

where $\eta_1 := \min\{\eta_{1,2}, \eta_{2,2}\}$ and $\eta_2 := \min\{\eta_{1,1}, \eta_{2,1}\}$. Thus, we complete the proof. □

Corollary 2.5. *Let A_i and B_i be defined as in Corollary 2.3. Then, (1) has at least two positive solutions $\mathcal{U}_1$ and $\mathcal{U}_2$ such that*

$$0 < \|\mathcal{U}_1\| < \eta^* < \|\mathcal{U}_2\|,$$

if the following hypotheses hold:

(H$_7$) $\max f_0 = D_1$, $\max f_\infty = D_4 \in [0, A_1)$ *and* $\max g_0 = E_1$, $\min g_\infty = E_4 \in [0, A_2)$;

(H$_8$) *there exists a real number* $\eta^* := \min\{\eta_1^*, \eta_2^*\} > 0$ *such that*

$$\begin{cases} f(t,v) \ge B_1\eta_1^* & \text{on } [\xi,\sigma(\omega)] \times [M_1\eta_1^*, \eta_1^*], \\ g(t,u) \le B_2\eta_2^* & \text{on } [\xi,\sigma(\omega)] \times [M_2\eta_2^*, \eta_2^*]. \end{cases}$$

PROOF. It follows from Remark 2.2, there exist four real numbers $\lambda_{1,1}$, $\lambda_{1,2}$, $\lambda_{2,1}$ and $\lambda_{2,2}$ satisfying

$$0 < \lambda_{1,1} < \eta^* < \lambda_{1,2}, \quad 0 < \lambda_{2,1} < \eta^* < \lambda_{2,2}$$

$$\begin{cases} f(t,v) \le A_1\lambda_{1,1} & \text{on } [0,\sigma(1)] \times [0,\lambda_{1,1}], \\ g(t,u) \le A_2\lambda_{2,1} & \text{on } [0,\sigma(1)] \times [0,\lambda_{2,1}]. \end{cases}$$

and

$$\begin{cases} f(t,v) \le A_1\lambda_{1,2} & \text{on } [0,\sigma(1)] \times [0,\lambda_{1,2}], \\ g(t,u) \le A_2\lambda_{2,2} & \text{on } \quad [0,\sigma(1)] \times [0,\lambda_{2,2}]. \end{cases}$$

Hence, by Theorem 2.1 that (1) has two positive solutions $\mathcal{U}_1$ and $\mathcal{U}_2$ such that

$$\lambda_1 < \|\mathcal{U}_1\| < \eta^* < \|\mathcal{U}_2\| < \lambda_2,$$

where $\lambda_1 = \max\{\lambda_{1,1}, \lambda_{2,1}\}$ and $\lambda_2 = \max\{\lambda_{1,2}, \lambda_{2,2}\}$. Thus, we complete the proof. □

Remark 2.6. Consider the following fourth order boundary value problem

$$\text{(BVP.1)} \qquad \begin{cases} (\mathrm{E}_3) & u^{\Delta\Delta\Delta\Delta}(t) + g(t,u(\sigma(t))) = 0, \quad 0 < t < 1, \\ (\mathrm{BC}_3) & \begin{cases} \alpha_1 u(0) - \beta_1 u^{\Delta}(0) = 0, \\ \gamma_1 u(\sigma(1)) + \delta_1 u^{\Delta}(\sigma(1)) = 0, \end{cases} \\ (\mathrm{BC}_4) & \begin{cases} -\alpha_2 u^{\Delta\Delta}(0) + \beta_2 v^{\Delta\Delta\Delta}(0) = 0, \\ -\gamma_2 u^{\Delta\Delta}(\sigma(1)) - \delta_2 u^{\Delta\Delta\Delta}(\sigma(1)) = 0, \end{cases} \end{cases}$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i$ are nonnegative real numbers, $r_i := \gamma_i\beta_i + \alpha_i\delta_i + \alpha_i\gamma_i\sigma(1) > 0$, $i = 1,2$ and $g \in C_{rd}([0,\sigma(1)] \times [0,\infty), [0,\infty))$. If we let

$-u^{\Delta\Delta}(t) = v(t)$, then (2.6) can be transformed into

$$
\text{(BVP.2)} \qquad \begin{cases} (\mathrm{E}_4) & u^{\Delta\Delta}(t) + v(t) = 0, \quad 0 < t < 1, \\ (\mathrm{E}_5) & v^{\Delta\Delta}(t) + g(t, u(\sigma(t))) = 0, \quad 0 < t < 1, \\ (\mathrm{BC}_1) & \begin{cases} \alpha_1 u(0) - \beta_1 u^{\Delta}(0) = 0, \\ \gamma_1 u(\sigma(1)) + \delta_1 u^{\Delta}(\sigma(1)) = 0, \end{cases} \\ (\mathrm{BC}_2) & \begin{cases} \alpha_2 v(0) - \beta_2 v^{\Delta}(0) = 0, \\ \gamma_2 u(\sigma(1)) + \delta_2 u^{\Delta}(\sigma(1)) = 0. \end{cases} \end{cases}
$$

Thus, we can apply the above-mentioned results to study the existence of solutions of (2.6).

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