

## Some remarks on the Lorenz five component model

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*Dedicated to Professor L. Tamásy on his 80th birthday*

**Abstract.** Some old aspects from the theory of the Lorenz five component model are discussed and some of its new properties are pointed out.

### 1. Introduction

The study of balanced dynamics is a central subject in geophysical fluid dynamics. It can be realized using three different approaches. The most recent one is due to LORENZ [6] and is called the five component model. Its dynamics is usually described by the following set of differential equations:

$$\begin{cases} \dot{x}_1 = -x_2x_3 + bx_2x_5 \\ \dot{x}_2 = x_1x_3 - bx_1x_5 \\ \dot{x}_3 = -x_1x_2 \\ \dot{x}_4 = -\frac{x_5}{\varepsilon} \\ \dot{x}_5 = \frac{x_4}{\varepsilon} + bx_1x_2, \end{cases} \quad (1.1)$$

where  $b, \varepsilon \in \mathbb{R}$ .

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The goal of our paper is to present some new properties of this dynamics from the Poisson geometry point of view.

## 2. Stability problem – Periodic solutions

Let us start with the following result dues to BOKHOVE [2]. More exactly we have:

**Proposition 2.1** ([2]). *The dynamics (1.1) has the following Hamilton–Poisson realization:*

$$(\mathbb{R}^5, \Pi_{\varepsilon, b}, H),$$

where

$$\Pi_{\varepsilon, b} = \begin{bmatrix} 0 & 0 & -x_2 & 0 & bx_2 \\ 0 & 0 & x_1 & 0 & -bx_1 \\ x_2 & -x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\varepsilon} \\ -bx_2 & bx_1 & 0 & \frac{1}{\varepsilon} & 0 \end{bmatrix}$$

and

$$H(x_1, x_2, x_3, x_4, x_5) = \frac{1}{2}(x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + x_5^2).$$

**Proposition 2.2.** *There exists only one functionally independent Casimir of our configuration  $(\mathbb{R}^5, \Pi_{\varepsilon, b})$ .*

PROOF. For the proof we shall use the technique of BERMEJO and FAIREN [1].

Let

$$(\Pi_{\varepsilon, b})_{2m} = \begin{bmatrix} 0 & x_1 & 0 & -bx_1 \\ -x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\varepsilon} \\ bx_1 & 0 & \frac{1}{\varepsilon} & 0 \end{bmatrix}.$$

Then

$$\det(\Pi_{\varepsilon, b})_{2m} = \frac{x_1^2}{\varepsilon} \neq 0$$

iff

$$x_1 \neq 0.$$

It follows that

$$(\Pi_{\varepsilon,b})_{2m}^{-1} = \begin{bmatrix} 0 & -\frac{1}{x_1} & 0 & 0 \\ \frac{1}{x_1} & 0 & -b\varepsilon & 0 \\ 0 & b\varepsilon & 0 & \varepsilon \\ 0 & 0 & -\varepsilon & 0 \end{bmatrix}.$$

Hence

$$\begin{aligned} (\Pi_{\varepsilon,b})_{n-2m} &= [0, -x_2, 0, bx_2]; \\ \Gamma &= [(\Pi_{\varepsilon,b})_{n-2m} \cdot (\Pi_{\varepsilon,b})_{2m}^{-1}]^t \\ &= \left[ -\frac{x_2}{x_1}, 0, 0, 0 \right]^t. \end{aligned}$$

Therefore

$$dx_1 = -\frac{x_2}{x_1} dx_2$$

or equivalent

$$C(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{1}{2}(x_1^2 + x_2^2)$$

is the Casimir of our configuration  $(\mathbb{R}^5, \Pi_{\varepsilon,b})$ . □

It is not hard to see that the equilibrium states of our system are:

$$e_1^M = (M, 0, 0, 0, 0), \quad M \in \mathbb{R};$$

$$e_2^M = (0, M, 0, 0, 0), \quad M \in \mathbb{R};$$

$$e_3^M = (0, 0, M, 0, 0), \quad M \in \mathbb{R}.$$

We shall now discuss their nonlinear stability. Recall that an equilibrium state  $x_e$  is nonlinear stable if the trajectories close to  $x_e$  stay close to  $x_e$  for each  $t \in \mathbb{R}$ . In other words, at least one neighborhood of  $x_e$  must be flow invariant.

**Proposition 2.3.** *The equilibrium state  $e_1^M$ ,  $M \in \mathbb{R}$ ,  $M \neq 0$  is nonlinear stable.*

PROOF. We shall make the proof using the energy-Casimir method, see [5], [7] or [9].

Let  $H_\varphi$  be the energy-Casimir function given by

$$H_\varphi(x_1, x_2, x_3, x_4, x_5) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2) + \varphi\left(\frac{1}{2}(x_1^2 + x_2^2)\right),$$

where  $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ .

Then we have

$$\begin{aligned}\delta H_\varphi &= x_1 \delta x_1 + 2x_2 \delta x_2 + x_3 \delta x_3 + x_4 \delta x_4 + x_5 \delta x_5 \\ &\quad + \varphi'(x_1 \delta x_1 + x_2 \delta x_2),\end{aligned}$$

where

$$\varphi' = \frac{\partial \varphi}{\partial \left(\frac{1}{2}(x_1^2 + x_2^2)\right)}.$$

Hence

$$\delta H_\varphi(M, 0, 0, 0, 0) = 0$$

if and only if

$$\varphi'\left(\frac{1}{2}M^2\right) = -1. \quad (2.1)$$

The second variation of  $H_\varphi$  is given by

$$\begin{aligned}\delta^2 H_\varphi &= (\delta x_1)^2 + 2(\delta x_2)^2 + (\delta x_3)^2 + (\delta x_4)^2 + (\delta x_5)^2 \\ &\quad + \varphi''(x_1 \delta x_1 + x_2 \delta x_2)^2 + \varphi'[(\delta x_1)^2 + (\delta x_2)^2].\end{aligned}$$

At the equilibrium of interest we have via (2.1)

$$\begin{aligned}\delta^2 H_\varphi(M, 0, 0, 0, 0) &= (\delta x_2)^2 + (\delta x_3)^2 + (\delta x_4)^2 + (\delta x_5)^2 \\ &\quad + \varphi''\left(\frac{1}{2}M^2\right) M^2 (\delta x_1)^2.\end{aligned}$$

If we can choose  $\varphi$  such that

$$\varphi''\left(\frac{1}{2}M^2\right) > 0 \quad (2.2)$$

then the second variation of  $H_\varphi$  at the equilibrium of interest is positive definite and so we can conclude that the equilibrium state  $(M, 0, 0, 0, 0)$ ,  $M \in \mathbb{R}$ ,  $M \neq 0$  is nonlinear stable.

For instance such a  $\varphi$  is given by

$$\varphi(x) = \left(x - \frac{1}{2}M^2\right)^2 - x. \quad \square$$

Using now the linear part of our system (1.1) at the equilibrium of interest  $e_2^M$  [resp.  $e_3^M$ ] we have immediately:

**Proposition 2.4.** *The equilibrium states  $e_2^M$ ,  $e_3^M$ ,  $M \in \mathbb{R}$ ,  $M \neq 0$ , have the following behaviour:*

- i)  $e_2^M = (0, M, 0, 0, 0)$ ,  $M \in \mathbb{R}$ ,  $M \neq 0$  is unstable.
- ii)  $e_3^M = (0, 0, M, 0, 0)$ ,  $M \in \mathbb{R}$ ,  $M \neq 0$  is spectrally stable.

If we take now the function  $H \in C^\infty(\mathbb{R}^5, \mathbb{R})$  given by

$$H(x_1, x_2, x_3, x_4, x_5) = \frac{1}{2}(x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + x_5^2)$$

as a Lyapunov function, then via Lyapunov theorem we have

**Proposition 2.5.** *The equilibrium state  $e^0 = (0, 0, 0, 0, 0)$  is nonlinear stable.*

*Remark 2.1.* It is an open problem to decide the nonlinear stability or instability of the equilibrium states  $e_3^M = (0, 0, M, 0, 0)$ ,  $M \in \mathbb{R}$ ,  $M \neq 0$ .

We shall discuss now the existence of the periodic solutions for the dynamics (1.1).

Let  $K_\varphi$  be a first integral of the dynamics (1.1) given by

$$K_\varphi(x_1, x_2, x_3, x_4, x_5) = \frac{1}{2}(x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + x_5^2) + \varphi\left(\frac{1}{2}(x_1^2 + x_2^2)\right) - \frac{1}{2}M^2 - \varphi\left(\frac{1}{2}M^2\right)$$

where  $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$  satisfies to the conditions (2.1), (2.2). Then we have

- i)  $K_\varphi \in C^\infty(\mathbb{R}^5, \mathbb{R})$ ;
- ii)  $K_\varphi(M, 0, 0, 0, 0) = 0$ ;
- iii)  $dK_\varphi(M, 0, 0, 0, 0) = 0$ ;
- iv)  $d^2K_\varphi(M, 0, 0, 0, 0)$  is positive definite.

Then via the MOSER theorem [8] we have

**Proposition 2.6.** *For each  $\varepsilon > 0$  sufficiently small any integral surface*

$$K_\varphi(x_1, x_2, x_3, x_4, x_5) = \varepsilon^2$$

*contains at least one periodic solution for the dynamics (1.1) whose periods are close to those of the corresponding linear system.*

Let  $H$  be the Hamiltonian (or the energy of the dynamics (1.1)), i.e.

$$H(x_1, x_2, x_3, x_4, x_5) = \frac{1}{2}(x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + x_5^2).$$

Then we have

- i)  $H \in C^\infty(\mathbb{R}^5, \mathbb{R})$ ;
- ii)  $H(0, 0, 0, 0, 0) = 0$ ;
- iii)  $DH(0, 0, 0, 0, 0) = 0$ ;
- iv)  $D^2H(0, 0, 0, 0, 0)$  is positive definite.

Then via the MOSER [8] we have

**Proposition 2.7.** *For each  $\varepsilon > 0$  sufficiently small any integral surface*

$$H(x_1, x_2, x_3, x_4, x_5) = \varepsilon^2$$

*contains at least one periodic solution for the dynamics (1.1) whose periods are close to those of the corresponding linear system.*

### 3. The reduced phase space and the reduced dynamics

It is clear that the function  $C \in C^\infty(\mathbb{R}^5, \mathbb{R})$  given by

$$C(x_1, x_2, x_3, x_4, x_5) = \frac{1}{2}(x_1^2 + x_2^2)$$

is constant of motion (1.1). Let us make the change of variables:

$$\begin{cases} x_1 = \sqrt{2C} \cos \varphi \\ x_2 = \sqrt{2C} \sin \varphi. \end{cases}$$

Then the dynamics (1.1) takes the following form:

$$\begin{cases} \frac{d\varphi}{dt} = x_3 - bx_5 \\ \frac{dx_3}{dt} = -C \sin 2\varphi \\ \frac{dx_4}{dt} = -\frac{x_5}{\varepsilon} \\ \frac{dx_5}{dt} = \frac{x_4}{\varepsilon} + bC \sin 2\varphi \end{cases} \quad (3.1)$$

and it is usually called the reduced dynamics.

**Proposition 3.1** ([3]). *The reduced dynamics (3.1) has the following Hamilton–Poisson realization:*

$$(M_C, \Pi_{\text{red}}, H_{\text{red}}),$$

where

$$M_C = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1^2 + x_2^2 = 2C\};$$

$$\Pi_{\text{red}} = \begin{bmatrix} 0 & 1 & 0 & -b \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\varepsilon} \\ b & 0 & \frac{1}{\varepsilon} & 0 \end{bmatrix};$$

$$H_{\text{red}}(\varphi, x_3, x_4, x_5) = -\frac{C}{2} \cos 2\varphi + \frac{1}{2}(x_3^2 + x_4^2 + x_5^2);$$

$$C \in \mathbb{R}, \quad C > 0.$$

Since

$$\det(\Pi_{\text{red}}) = -\frac{1}{\varepsilon^2} \neq 0,$$

we have immediately:

**Proposition 3.2.** *The reduced dynamics (3.1) has the following Hamiltonian realization:*

$$(M_C, \omega_C, H_{\text{red}}),$$

where

$$\omega_C = -d\varphi \wedge dx_3 - \varepsilon b dx_3 \wedge dx_4 + \varepsilon dx_4 \wedge dx_5.$$

*Remark 3.1.* It is clear from the above considerations that  $(M_C, \omega_C)$ ,  $C \in \mathbb{R}$ ,  $C > 0$ , is in fact the symplectic foliation of the Poisson manifold  $(\mathbb{R}^5, \Pi)$ .

It is easy to see that the equilibrium states of our system (3.1) are

$$e_k = \left( \frac{k\pi}{2}, 0, 0, 0 \right), \quad k \in \mathbb{Z}.$$

**Proposition 3.3.** *The equilibrium states*

$$e_{2l} = (l\pi, 0, 0, 0), \quad l \in \mathbb{Z}$$

*are nonlinear stable.*

PROOF. Let  $L \in C^\infty(M_C, \mathbb{R})$  be a smooth function given by

$$L(\varphi, x_3, x_4, x_5) = -\frac{C}{2} \cos 2\varphi + \frac{C}{2} + \frac{1}{2}(x_3^2 + x_4^2 + x_5^2).$$

Then we have

- i)  $L(l\pi, 0, 0, 0) = 0$ ;
- ii)  $L(\varphi, x_3, x_4, x_5) > 0$ ,  $(\forall) (\varphi, x_3, x_4, x_5) \in M_C$ ,  
 $(\varphi, x_3, x_4, x_5) \neq (l\pi, 0, 0, 0)$ ;
- iii)  $\dot{L} = 0$ , along the trajectories of the system (3.1).

Hence  $L$  is a Lyapunov function and then via the Lyapunov theorem the equilibrium states

$$e_{2l} = (l\pi, 0, 0, 0), \quad l \in \mathbb{Z}$$

are nonlinear stable. □

**Proposition 3.4.** *The equilibrium states*

$$e_{2l+1} = \left(l\pi + \frac{\pi}{2}, 0, 0, 0\right), \quad l \in \mathbb{Z}$$

*are unstable.*

PROOF. Indeed, let  $A$  be the matrix of the linear part of our dynamics (3.1),

$$A = \begin{bmatrix} 0 & 1 & 0 & -b \\ -2C \cos 2\varphi & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\varepsilon} \\ 2bC \cos 2\varphi & 0 & \frac{1}{\varepsilon} & 0 \end{bmatrix}.$$

Then we have

$$A(e_{2l+1}) = \begin{bmatrix} 0 & 1 & 0 & -b \\ 2C & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\varepsilon} \\ 2bC & 0 & \frac{1}{\varepsilon} & 0 \end{bmatrix};$$

$$p_{A(e_{2l+1})}(x) = x^4 + x^2 \left( \frac{1}{\varepsilon^2} - 2C + 2b^2C \right) - 2C \frac{1}{\varepsilon^2}.$$

It is easy to see now that the characteristic equation

$$p_{A(e_{2l+1})}(x) = 0$$

has a positive root and so the equilibrium states

$$e_{2l+1} = \left( l\pi + \frac{\pi}{2}, 0, 0, 0 \right), \quad l \in \mathbb{Z}$$

are unstable. □

Let  $L$  be a real valued function given by

$$L(\varphi, x_3, x_4, x_5) = -\frac{C}{2} \cos 2\varphi + \frac{C}{2} + \frac{1}{2}(x_3^2 + x_4^2 + x_5^2).$$

Then we have

- i)  $L \in C^\infty(M_C, \mathbb{R})$ ;
- ii)  $L(\pi l, 0, 0, 0) = 0$ ;
- iii)  $dL(\pi l, 0, 0, 0) = 0$ ;
- iv)  $d^2L(\pi l, 0, 0, 0)$  is positive definite.

Then via WEINSTEIN theorem [10] we have:

**Proposition 3.5.** *For each sufficiently small  $\varepsilon$  any integral surface*

$$L(\varphi, x_3, x_4, x_5) = \varepsilon^2$$

*contains at least two periodic solutions of the dynamics (3.1) whose periods are close to those of the linearized system.*

#### 4. Geometric prequantization of the reduced dynamics (3.1)

We have seen in the previous section that the dynamics (3.1) has the following Hamiltonian realization:

$$(M_C, \omega_C, H_{\text{red}})$$

where

$$M_C = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1^2 + x_2^2 = 2C\};$$

$$\omega_C = -d\varphi \wedge dx_3 - \varepsilon b dx_3 \wedge dx_4 + \varepsilon dx_4 \wedge dx_5;$$

$$H_{\text{red}}(\varphi, x_3, x_4, x_5) = -\frac{C}{2} \cos 2\varphi + \frac{1}{2}(x_3^2 + x_4^2 + x_5^2);$$

$$C \in \mathbb{R}, \quad C > 0.$$

It is easy to see that

$$\omega_C = d\theta_C,$$

where

$$\theta_C = -\varphi dx_3 - \varepsilon b x_3 dx_4 + \varepsilon x_4 dx_5$$

and so  $(M_C, \omega_C)$  is a quantizable manifold. Moreover, the Hilbert representation space is given by

$$\mathcal{H} = L^2(M_C, \mathbb{C}),$$

and the prequantum operator  $\delta_f$  has the following expression for each  $f \in C^\infty(M_C, \mathbb{R})$ :

$$\delta_f = -i\hbar \left[ X_f - \frac{i}{\hbar} \theta_C(X_f) \right] + f,$$

where  $\hbar$  is the Planck constant divided by  $2\pi$ .

Therefore we have

**Proposition 4.1.** *The pair  $(\mathcal{H}, \delta)$  gives rise to a prequantization of the reduced dynamics (3.1).*

Using now the same arguments as in [4] with obvious modifications we can prove:

**Proposition 4.2.** *Let  $O(L^2(M_C, \mathbb{C}))$  be the space of self adjoint operators on the Hilbert space  $L^2(M_C, \mathbb{C})$ . Then the map*

$$f \in C^\infty(M_C, \mathbb{R}) \mapsto \delta_f \in O(L^2(M_C, \mathbb{C}))$$

*gives rise to an irreducible representation of  $C^\infty(M_C, \mathbb{R})$  on  $O(L^2(M_C, \mathbb{C}))$ .*

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