

On the differences between polynomial values and perfect powers

By I. PINK (Debrecen)

Dedicated to the memory of Professor Péter Kiss

Abstract. Let $F(X) \in \mathbb{Z}[X]$ be a polynomial of degree $n \geq 2$, and let x , b , y , m be non-zero integers with $m \geq 2$, $|y| \geq 2$ and $F(x) \neq by^m$. Under some natural assumptions on F , we give explicit lower bounds for $|F(x) - by^m|$, depending only on n , m , b , $H(F)$ and n , b , $F(x)$, $H(F)$, respectively. These results generalize Theorems 1 and 2 of BUGEAUD and HAJDU [8]. To prove our results, we slightly improve and make completely explicit the upper bound obtained in [3] for the unknown exponent m in the superelliptic equation (1).

1. Introduction

Let a , b , x , y , n , m be non-zero integers with $n, m \geq 2$, $|y| \geq 2$ and $ax^n \neq by^m$. The first effective lower bound for $|ax^n - by^m|$ which is independent of x and y was proved by TURK [16], in case of $a = b = 1$. A result of similar strength valid for arbitrary a and b , however not completely explicit, can also be deduced from the work of SHOREY [14]. Later,

Mathematics Subject Classification: 11D41, 11D61.

Key words and phrases: polynomial values, perfect powers, Baker's method.

Research supported in part by the Netherlands Organization for Scientific Research (NWO), by grant F034981 of the Hungarian National Foundation for Scientific Research and by the FKFP grant 3272-13/066/2001.

BUGEAUD [5] considerably sharpened Turk's estimate for $|x^n - y^m|$. Recently, thanks to some refined arguments, BUGEAUD and HAJDU [8] improved and extended Bugeaud's result to arbitrary a and b . The purpose of this paper is to generalize the results of BUGEAUD and HAJDU [8] to differences of the form $|F(x) - by^m|$, where $F(X) \in \mathbb{Z}[X]$ is a polynomial of degree $n \geq 2$.

Under certain assumptions on F , we derive explicit lower bounds for $|F(x) - by^m|$ (cf. Theorem 2) from our Theorem 1 which provides an explicit upper bound for the exponent m in the equation

$$f(x) = by^m \text{ in } x, y, m \in \mathbb{Z}, \quad \text{with } |y| \geq 2, \quad m \geq 1, \quad (1)$$

in terms of b and the height of $f \in \mathbb{Z}[X]$. The first results proving that m is bounded were given by TIJDEMAN [15] and SCHINZEL and TIJDEMAN [13]. Later, several effective but not completely explicit upper bounds were obtained for m ; see [2], [4], [3] and the references given there. Our Theorem 1 slightly improves and makes explicit in each parameter the previously best known bound (cf. [3]) on m . In our proof we will follow the approach of BRINDZA, EVERTSE and GYÖRY [4]. They gave an estimate for m from above in terms of the discriminant of f .

2. New results

Throughout the paper, we use the following notation. For every positive real number s , we put $\log_* s = \max\{1, \log s\}$. Let

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n = a_0 \prod_{i=1}^n (x - \alpha_i), \quad a_0 \neq 0,$$

be a polynomial with integer coefficients. We write

$$H(f) = \max_{0 \leq i \leq n} |a_i| \quad \text{and} \quad M(f) = |a_0| \prod_{i=1}^n \max(1, |\alpha_i|)$$

for the “classical” height and the Mahler-height of f , respectively.

Theorem 1. *Let $f(X) \in \mathbb{Z}[X]$ be a polynomial of degree $n \geq 2$ and b a non-zero integer. If f has at least two distinct roots, then equation (1) with $x, y, m \in \mathbb{Z}$ and $|y| \geq 2$, $m \geq 1$ implies*

$$m \leq 2^{24n+56} n^{7n+17} M(f)^{3n-3} (\log_* M(f))^{3n} (\log_* |b|)^{\frac{5}{2}}.$$

As was mentioned above, our Theorem 1 slightly improves and makes completely explicit the previously best known result of this type, established in [3]. In the special case $f(x) = ax^n + c$, a similar result was proved in [8]. Our Theorem 1 is also related to Theorem 5 of BRINDZA, EVERTSE and GYÖRY [4], where it is assumed that $b = 1$ and f is irreducible and monic, but the bound given for m depends only on n and the discriminant of f .

In the proof of Theorem 1 we will follow the approach of [4]. We obtain the following result as a consequence of Theorem 1.

Theorem 2. *Let $F(X) \in \mathbb{Z}[X]$ be a polynomial of degree $n \geq 2$, and let b, x, y, m be integers with $b \neq 0$, $m \geq 1$, $|y| \geq 2$. Suppose that $F(x) \neq by^m$, and if $F(X)$ is of the special form $F(X) = t_1(X - t_2)^n + t_3$ with $t_1, t_2, t_3 \in \mathbb{Z}$, then also assume that $F(x) \neq by^m + t_3$. Then we have*

$$|F(x) - by^m| \geq m^{\frac{1}{3n}} 2^{-8 - \frac{56}{3n}} n^{-\frac{23}{6} - \frac{17}{3n}} \left(H(F) \log_*^{\frac{5}{6n}} |b| \right)^{-1}. \quad (2)$$

We note that to give a lower bound for $|F(x) - by^m|$, we need to use the classical height instead of the Mahler-height. The reason is that for every $k \in \mathbb{Z}$, plainly $H(F - k) \leq H(F) + |k|$, but $M(f)$ does not have a similar nice property. However, the use of the classical height already in Theorem 1 would result in a worse estimate for $|F(x) - by^m|$.

As in [8], by combining Theorem 2 with an estimate for the size of the solutions of superelliptic equations, we derive a lower bound for $|F(x) - by^m|$ in terms of $|F(x)|$.

Theorem 3. *Let $F(X) \in \mathbb{Z}[X]$ be a polynomial of degree $n \geq 2$, and let b, x, y, m be integers with $b \neq 0$, $m \geq 3$, $|y| \geq 2$. Suppose that $F(x) \neq by^m$, and if $F(X)$ is of the special form $F(X) = t_1(X - t_2)^n + t_3$*

with $t_1, t_2, t_3 \in \mathbb{Z}$, then also assume that $F(x) \neq by^m + t_3$. Then

$$|F(x) - by^m| \geq c_1 n^{-\frac{23}{6}} H(F)^{-1} (\log_* |b|)^{-\frac{4}{3n+1}} (\log_* \log_* |F(x)|)^{\frac{1}{3n+1}}, \quad (3)$$

where c_1 denotes an effectively computable absolute constant.

Theorem 2 generalizes the estimate

$$|ax^n - by^m| \geq m^{2/5n} (20n)^{-2-11/n} \left(|a| \log_*^{\frac{1}{n}} |b| \right)^{-1}$$

of BUGEAUD and HAJDU [8]. Similarly, our Theorem 3 is an extension of Theorem 2 of [8]. Observe that our bound in (2) in the special case $F(x) = ax^n$ yields an estimate of similar strength as in [8], up to the exponent of m . This difference comes from the fact that $\Delta(ax^n + k) \leq c_2 |k|^n$, while in general we only have $\Delta(F(x) + k) \leq c_3 |k|^{2n}$. Here $\Delta(g(x))$ denotes the discriminant of $g(x) \in \mathbb{Z}[x]$ and c_2, c_3 are constants depending on a, n and F , respectively.

3. Some lemmas

For a non-zero algebraic number α of degree l over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a \prod_{i=1}^l (X - \alpha_i)$, let

$$h(\alpha) = \frac{1}{l} \left(\log |a| + \sum_{i=1}^l \log \max(1, |\alpha_i|) \right)$$

denote the absolute logarithmic height of α . Let $\mathbb{K}$ be a number field of degree $d_{\mathbb{K}}$, with unit rank $r_{\mathbb{K}}$ and regulator $R_{\mathbb{K}}$. In the course of our proof, we use an independent system of units in $\mathbb{K}$ with small height, provided by the following lemma.

Lemma 1. *There exists an independent system $\varepsilon_1, \dots, \varepsilon_{r_{\mathbb{K}}}$ of units in $\mathbb{K}$ satisfying*

$$\prod_{i=1}^{r_{\mathbb{K}}} h(\varepsilon_i) \leq d_{\mathbb{K}}^{-r_{\mathbb{K}}} r_{\mathbb{K}}! R_{\mathbb{K}} \quad (4)$$

and

$$h(\varepsilon_i) \leq r_{\mathbb{K}}! d_{\mathbb{K}}^{-1} (9(\log 3d_{\mathbb{K}})^3/8)^{r_{\mathbb{K}}-1} R_{\mathbb{K}}, \quad i = 1, \dots, r_{\mathbb{K}}. \quad (5)$$

Moreover, for every non-zero algebraic integer $\alpha \in \mathbb{K}$, there exists a unit ε in the multiplicative subgroup generated by $\varepsilon_1, \dots, \varepsilon_{r_{\mathbb{K}}}$ such that

$$h(\varepsilon\alpha) \leq (\log N_{\mathbb{K}/\mathbb{Q}}(\alpha))/(2d_{\mathbb{K}}) + (r_{\mathbb{K}} + 1)^{r_{\mathbb{K}}+1} \log^{3r_{\mathbb{K}}+3}(3d_{\mathbb{K}})R_{\mathbb{K}}. \quad (6)$$

PROOF. This is a reformulation of Lemme 1 and Lemme 2 of [6]. $\square$

Our proof ultimately depends on Baker's estimate for linear forms in logarithms. We use the following version due to MATVEEV [12], which is a sharpening of an estimate given by BAKER and WÜSTHOLZ [1].

Lemma 2. *Let $\mathbb{K}$ be an algebraic number field of degree D over $\mathbb{Q}$. Let $\alpha_1, \dots, \alpha_n \in \mathbb{K}^*$ with absolute logarithmic heights $h(\alpha_j)$ ($1 \leq j \leq n$), and $\log \alpha_1, \dots, \log \alpha_n$ arbitrary fixed non-zero values of the logarithms. Suppose that*

$$A_j \geq \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\} \quad (1 \leq j \leq n).$$

Consider the linear form

$$\Lambda = b_1 \log \alpha_1 + \dots + b_n \log \alpha_n,$$

with $b_1, \dots, b_n \in \mathbb{Z}$ and put $B = \max\{|b_1|, \dots, |b_n|\}$. If $\Lambda \neq 0$, then

$$\log |\Lambda| > -C(n) \log(eD) \log(eB) D^2 \Omega,$$

where $\Omega = A_1 \cdots A_n$ and $C(n) = 2^{6n+20}$.

PROOF. This is a reformulation of Corollary 2.3 of Matveev [12]. $\square$

We deduce Theorem 3 from Theorem 2 by using an explicit upper bound for the size of the solutions of superelliptic equations.

Lemma 3. *Let a and m be non-zero integers with $m \geq 3$ and $Q(X) = \prod_{i=1}^r (X - \alpha_i)^{e_i} \in \mathbb{Z}[X]$ be a monic polynomial of degree $n \geq 2$ with distinct roots $\alpha_1, \dots, \alpha_r$. Put $\Delta(Q) = \prod_{i \neq j} (\alpha_i - \alpha_j)$ and let $m_i = m / \gcd(m, e_i)$ for $i = 1, \dots, r$. Suppose that for some i, j with $1 \leq i \neq j \leq r$, we have $\gcd(m_i, m_j) \geq 3$. Then all the solutions $(x, y) \in \mathbb{Z}^2$ of*

$$Q(x) = ay^m \quad (7)$$

satisfy

$$|x| \leq H(Q)^{m+1} \exp \left\{ (c_4 n m)^{c_5 n^2 m} |\Delta(Q)|^{5nm} |a|^{n^2 m} (\log_* |a \Delta(Q)|)^{2n^2 m} \right\},$$

where c_4 and c_5 are effectively computable absolute constants.

PROOF. This easily follows from the Proposition of BUGEAUD [7]. $\square$

4. Proof of the theorems

We follow the method of the proofs in [4] and [3], but with explicit constants.

PROOF of Theorem 1. We have two cases to distinguish. First we assume that f has an irreducible factor $P \in \mathbb{Z}[X]$ of degree ≥ 2 . Let δ be a root of P , moreover, let $R_{\mathbb{K}}$, $h_{\mathbb{K}}$, $D_{\mathbb{K}}$ and $r_{\mathbb{K}}$ be the regulator, class number, discriminant and unit rank of the field $\mathbb{K} = \mathbb{Q}(\delta)$, respectively. Combining the inequality

$$d_{\mathbb{K}} \leq \frac{2}{\log 3} \log |D_{\mathbb{K}}|,$$

due to GYÖRY [9] with a result of LENSTRA [10], we have

$$h_{\mathbb{K}} R_{\mathbb{K}} \leq \frac{1}{(d_{\mathbb{K}} - 1)!} |D_{\mathbb{K}}|^{\frac{1}{2}} \log^{d_{\mathbb{K}} - 1} |D_{\mathbb{K}}|. \quad (8)$$

By an estimate of MAHLER [11] on the discriminant of P , we get

$$|D_{\mathbb{K}}| \leq d_{\mathbb{K}}^{d_{\mathbb{K}}} M(P)^{2d_{\mathbb{K}} - 2}.$$

Since $P|f$ implies $M(P) \leq M(f)$, this yields

$$|D_{\mathbb{K}}| \leq d_{\mathbb{K}}^{d_{\mathbb{K}}} M(f)^{2d_{\mathbb{K}} - 2}.$$

Combining the last inequality with (8) we obtain

$$h_{\mathbb{K}} R_{\mathbb{K}} \leq \frac{1}{\sqrt{2\pi}} d_{\mathbb{K}}^{d_{\mathbb{K}} - 1} e^{2d_{\mathbb{K}} - 1} M(f)^{d_{\mathbb{K}} - 1} (\log_* M(f))^{d_{\mathbb{K}} - 1}. \quad (9)$$

Let a_0 denote the leading coefficient of f , and let $\beta_1, \dots, \beta_n$ be the zeros of $g(x) = a_0^{n-1} f(\frac{x}{a_0})$. Set

$$\Delta(g) = \prod_{\beta_i \neq \beta_j} (\beta_i - \beta_j),$$

and write g in the form $g(x) = P_1^{k_1}(x) P_2(x)$ where $P_1(x) = a_0^{d_{\mathbb{K}}} P(\frac{x}{a_0})$ and P_2 are relatively prime polynomials in $\mathbb{Z}[X]$. Let $\beta_1, \dots, \beta_{d_{\mathbb{K}}}$ be the

zeros of P_1 with $\beta_1 = \delta$ and (x, y) be an arbitrary, however, fixed solution to (1). The greatest common divisor of the principal ideals $\langle a_0x - \beta_1 \rangle$ and $\langle g(a_0x)(a_0x - \beta_1)^{-k_1} \rangle$ divides $\Delta^n(g)$. Therefore there are integral ideals A, B, C in $\mathbb{K}$ such that

$$A\langle a_0x - \beta_1 \rangle = BC^w \quad (10)$$

where

$$w = \frac{m}{\gcd(m, k_1)}.$$

Further,

$$\max\{N_{\mathbb{K}/\mathbb{Q}}(A), N_{\mathbb{K}/\mathbb{Q}}(B)\} \leq |a_0 \cdot b \cdot \Delta(g)|^{n^2}.$$

Hence using Lemma 1, (6) and (9), by a simple calculation we obtain that the ideals $A^{h_{\mathbb{K}}}$ and $B^{h_{\mathbb{K}}}$ have generators α and β , respectively, with

$$\max\{h(\alpha), h(\beta)\} \leq c_6. \quad (11)$$

Here

$$\begin{aligned} c_6 &= 0.12n^3(n-1)d_{\mathbb{K}}^{d_{\mathbb{K}}-1}e^{2d_{\mathbb{K}}-1}(r_{\mathbb{K}}+1)^{r_{\mathbb{K}}+1} \\ &\quad \times (\log 3d_{\mathbb{K}})^{3r_{\mathbb{K}}+3}M(f)^{d_{\mathbb{K}}-1}(\log_* M(f))^{d_{\mathbb{K}}} \log_* |b|. \end{aligned}$$

Equation (10) can be rewritten as

$$\alpha(a_0x - \beta_1)^{h_{\mathbb{K}}} = \varepsilon\beta\gamma^w, \quad (12)$$

where γ is a generator of $C^{h_{\mathbb{K}}}$ and ε is a unit in $\mathbb{K}$. Let $\varepsilon_1, \dots, \varepsilon_{r_{\mathbb{K}}}$ be an independent system of units with the properties specified in Lemma 1. Then we can express ε as $\varepsilon = \varepsilon' \varepsilon_1^{l_1} \dots \varepsilon_{r_{\mathbb{K}}}^{l_{r_{\mathbb{K}}}}$, where ε' is a unit with

$$h(\varepsilon') \leq (r_{\mathbb{K}} + 1)^{r_{\mathbb{K}}+1} (\log(3d_{\mathbb{K}}))^{3r_{\mathbb{K}}+3} R_{\mathbb{K}}.$$

Modifying γ if necessary, we may assume that $\max_{1 \leq i \leq r_{\mathbb{K}}} |l_i| < w$.

If $|a_0x| \leq M(g) + 1$ then

$$2^m \leq |y|^m \leq (2M(g) + 1)^n,$$

and Theorem 1 is proved.

Otherwise, from $|a_0x| > M(g) + 1$ it follows that $|a_0x - \beta_i| > 1$ for $i = 1, \dots, d_{\mathbb{K}}$. Thus we have

$$\begin{aligned} |a_0^{n-1}by^m|^{h_{\mathbb{K}}} &\geq \max_{1 \leq i \leq d_{\mathbb{K}}} |a_0x - \beta_i|^{h_{\mathbb{K}}} \\ &\geq |a_0x - \beta_j|^{h_{\mathbb{K}}} \geq |\varepsilon'^{(j)}| \prod_{i=1}^{r_{\mathbb{K}}} |\varepsilon_i^{(j)}| |\alpha^{(j)}|^{-1} |\beta^{(j)}| |\gamma^{(j)}| \\ &\geq |\overline{\varepsilon'}|^{-d_{\mathbb{K}}+1} |\overline{\varepsilon_1}|^{-w} \dots |\overline{\varepsilon_{r_{\mathbb{K}}}}|^{-w} |\overline{\alpha}|^{-1} |\overline{\beta}|^{-d_{\mathbb{K}}+1} |\overline{\gamma}|^w. \end{aligned}$$

Here $|\overline{\nu}|$ denotes the house of the algebraic number ν , i.e. the maximum of the absolute values of its conjugates, and j is the appropriate index for which $|\gamma^{(j)}| = |\gamma|$. Supposing $m \geq n + 1$ (otherwise Theorem 1 follows), the last inequality yields

$$h(\gamma) \leq 2.182c_6 d_{\mathbb{K}}^2 \log_* |y|,$$

with the same c_6 as above. We may assume that $|a_0x| \geq \frac{1}{2}|y|^{\frac{m}{n}}$, or else we obtain

$$|a_0x| + M(g) \geq |y|^{\frac{m}{n}},$$

and Theorem 1 follows. Thus we get

$$|a_0x - \beta_i| \geq \frac{1}{4}|y|^{\frac{m}{n}} \quad (1 \leq i \leq d_{\mathbb{K}}). \quad (13)$$

We may suppose that

$$\frac{|\beta_i - \beta_j|}{|a_0x - \beta_i|} \geq \frac{|\beta_2 - \beta_1|}{|a_0x - \beta_2|}, \quad 1 \leq i, j \leq d_{\mathbb{K}}, \quad i \neq j.$$

Hence we have

$$\prod_{\substack{1 \leq i, j \leq d_{\mathbb{K}} \\ \beta_i \neq \beta_j}} \frac{|\beta_i - \beta_j|}{|a_0x - \beta_i|} \leq \frac{4^{d_{\mathbb{K}}(d_{\mathbb{K}}-1)} |\Delta(g)|}{|y|^{\frac{md_{\mathbb{K}}(d_{\mathbb{K}}-1)}{n}}}. \quad (14)$$

If $\left(\frac{a_0x - \beta_1}{a_0x - \beta_2}\right)^{h_{\mathbb{K}}} = 1$, then $\frac{\beta_1 - \beta_2}{a_0x - \beta_1}$ is an algebraic integer. Thus

$$\left| N_{\mathbb{L}/\mathbb{Q}} \left(\frac{\beta_1 - \beta_2}{a_0x - \beta_1} \right) \right| = \left| \frac{N_{\mathbb{L}/\mathbb{Q}}(\beta_1 - \beta_2)}{(N_{\mathbb{K}/\mathbb{Q}}(a_0x - \beta_1))^s} \right| \geq 1,$$

with $\mathbb{L} = \mathbb{Q}(\beta_1, \beta_2)$ and $[\mathbb{L} : \mathbb{K}] = s$. Combining this last inequality with (13), by $s \leq d_{\mathbb{K}}$ we obtain

$$\begin{aligned} |\Delta(g)|^{n^2} &\geq |N_{\mathbb{L}/\mathbb{Q}}(\beta_1 - \beta_2)| \geq |N_{\mathbb{K}/\mathbb{Q}}(a_0x - \beta_1)|^s \\ &\geq \left| \left(\frac{1}{4} |y|^{m/n} \right)^{d_{\mathbb{K}}} \right|^s \geq 2^{d_{\mathbb{K}}s(m/n-2)} \geq 2^{(2m/n)-2n^2}, \end{aligned}$$

which implies Theorem 1.

If $(\frac{a_0x-\beta_1}{a_0x-\beta_2})^{h_{\mathbb{K}}} \neq 1$, then we may assume that $|y|^{\frac{m}{2n}} \geq 2|\Delta(g)|h_{\mathbb{K}}$ (otherwise we would obtain a much better estimate for m). So by (14) we get

$$\begin{aligned} &\log \left| \left(\frac{a_0x - \beta_1}{a_0x - \beta_2} \right)^{h_{\mathbb{K}}} - 1 \right| \\ &\leq \log \left(h_{\mathbb{K}} \left| \frac{a_0x - \beta_1}{a_0x - \beta_2} - 1 \right| \right) \leq -\frac{m}{2n} \log_* |y|. \end{aligned} \tag{15}$$

In the case $\left| \left(\frac{a_0x - \beta_1}{a_0x - \beta_2} \right)^{h_{\mathbb{K}}} - 1 \right| > \frac{1}{3}$ one can obtain a very good bound for m by (15). Otherwise, using Lemma 2, (4) and (9), we get

$$\begin{aligned} &\left| \left(\frac{a_0x - \beta_1}{a_0x - \beta_2} \right)^{h_{\mathbb{K}}} - 1 \right| \\ &= \left| \left(\frac{\varepsilon_1}{\varepsilon_1^{(2)}} \right)^{l_1} \cdots \left(\frac{\varepsilon_{r_{\mathbb{K}}}}{\varepsilon_{r_{\mathbb{K}}}^{(2)}} \right)^{l_{r_{\mathbb{K}}}} \frac{\varepsilon' \beta / \alpha}{\varepsilon'^{(2)} \beta^{(2)} / \alpha^{(2)}} \left(\frac{\gamma}{\gamma^{(2)}} \right)^w - 1 \right| \\ &\geq \frac{1}{2} \left| b_0 \log(-1) + \sum_{i=1}^{r_{\mathbb{K}}} l_i \log \frac{\varepsilon_i}{\varepsilon_i^{(2)}} + \log \frac{\varepsilon' \beta / \alpha}{\varepsilon'^{(2)} \beta^{(2)} / \alpha^{(2)}} + w \log \frac{\gamma}{\gamma^{(2)}} \right| \\ &\geq \exp \left\{ -c_7(n) M(f)^{3n-3} (\log_* M(f))^{3n-1} \log_*^2 |b| \log_* |y| \log_* m \right\}, \end{aligned}$$

where b_0 is an integer with $|b_0| \leq w(r_{\mathbb{K}} + 1)$ and

$$c_7(n) = 2^{23.1n+48.418} n^{7n+14} \log n.$$

Here the superscript $^{(2)}$ denotes the image under the isomorphism $\mathbb{Q}(\beta_1) \rightarrow \mathbb{Q}(\beta_2)$. The comparison of this lower bound with (15) completes the proof in the first case.

In the second case f has only rational roots. Hence, all the zeros of g are integral. Let β_1 and β_2 be two distinct roots of g , of multiplicities k_1 and k_2 , respectively. Repeating the argument used in the first case one gets

$$u_i(a_0x - \beta_i) = v_i y_i^w, \quad i = 1, 2 \quad (16)$$

where $w = \frac{m}{(m, k_1 k_2)}$, $u_i, v_i, y_i \in \mathbb{Z}$, $|y_i| \geq 2$ and $|u_i| \leq |\Delta(g)^n|$, $|v_i| \leq |a_0^{n-1}b|$ ($i = 1, 2$). We may suppose that $|y_2| \geq |y_1|$. Set $\Lambda_1 = \log \frac{v_1 u_2}{v_2 u_1} + w \log \left(\frac{y_1}{y_2} \right)$. From (16) we deduce

$$\left| \frac{u_2(\beta_2 - \beta_1)}{v_2 y_2^w} \right| = \left| \frac{v_1 u_2}{v_2 u_1} \left(\frac{y_1}{y_2} \right)^w - 1 \right| \geq \frac{1}{2} |\Lambda_1|.$$

Using Lemma 2 again we have

$$\frac{m}{\log m} < 2^{41} n^5 \log_* M(f) \log_* |b|,$$

and Theorem 1 is proved. $\square$

PROOF of Theorem 2. Let $k = F(x) - by^m$. We apply Theorem 1 with $f(x) = F(x) - k$. Combining

$$M(f) \leq \sqrt{(n+1)} H(f)$$

with

$$H(f) \leq H(F) + |k| \leq 2H(F)|k|$$

and expressing $|k|$, we obtain the lower bound for $|F(x) - by^m|$ stated in the theorem. $\square$

PROOF of Theorem 3. Set $k = F(x) - by^m$ and let a_0 be the leading coefficient of F . By applying Lemma 3 to the equation

$$Q(a_0x) = a_0^{n-1}by^m$$

with $Q(x) = a_0^{n-1}(F(\frac{x}{a_0}) - k)$ and $a = a_0^{n-1}b$, and using the inequalities

$$|a_0| \leq H(F - k) \leq 2H(F)|k|,$$

we obtain a bound for $|x|$, hence for $|F(x)|$, in terms of $H(F)$, b , n , k and m . Namely, we get

$$\log_* \log_* |F(x)| \leq c_8 n^3 m \log_* m \log_* H(F) \log_* |b| \log_* |k|,$$

where c_8 is an effectively computable absolute constant. Further, from Theorem 2 we have

$$m^{\frac{1}{3n}} \leq 2^{8+\frac{56}{3n}} n^{\frac{23}{6}+\frac{17}{3n}} H(F) \log_*^{\frac{5}{6n}} |b| |k|.$$

Combining these estimates, Theorem 3 easily follows. $\square$

ACKNOWLEDGEMENTS. The author is grateful to Professors KÁLMÁN GYÖRY, ÁKOS PINTÉR and LAJOS HAJDU for their help and numerous valuable remarks.

References

- [1] A. BAKER and G. WÜSTHOLZ, Logarithmic forms and group varieties, *J. Reine Angew. Math.* **442** (1993), 19–62.
- [2] B. BRINDZA, Zeros of polynomials and exponential Diophantine equations, *Compositio Math.* **61** (1987), 135–157.
- [3] A. BÉRCZES, B. BRINDZA and L. HAJDU, On power values of polynomials, *Publ. Math. Debrecen* **53** (1998), 375–381.
- [4] B. BRINDZA, J.-H. EVERTSE and K. GYÖRY, Bounds for the solutions of some diophantine equations in terms of the discriminant, *J. Austral Math. Soc.* **51** (1991), 8–26.
- [5] Y. BUGEAUD, Sur la distance entre deux puissances pures, *C. R. Acad. Sci. Paris* **322** (1996), 1119–1121.
- [6] Y. BUGEAUD, Bornes effectives pour les solutions des equations en S -unités et des equations de Thue–Mahler, *J. Number Theory* **71** (1998), 227–244.
- [7] Y. BUGEAUD, On the greatest prime factor of $ax^m + by^n$, Number Theory: Diophantine, Computational and Algebraic Aspects, (K. Györy, A. Pethő and V. T. Sós, eds.), *Walter de Gruyter, Berlin – New York*, 1998, 115–122.
- [8] Y. BUGEAUD and L. HAJDU, Lower bounds for the difference $ax^n - by^m$, *Acta Math. Hungar.* **87** (2000), 279–286.
- [9] K. GYÖRY, Sur les polynômes à coefficients entiers et de discriminant donné II, *Publ. Math. Debrecen* **21** (1974), 125–144.
- [10] H. W. LENSTRA JR., Algorithms in algebraic number theory, *Bull. Amer. Math. Soc.* **26** (1992), 211–244.
- [11] K. MAHLER, An inequality for the discriminant of a polynomial, *Michigan Math. J.* **11** (1964), 257–262.
- [12] E. M. MATVEEV, An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers, II, *Izvestiya: Mathematics* **64** (2000), 1217–1269.

- [13] A. SCHINZEL and R. TIJDEMAN, On the equation $y^m = P(x)$, *Acta Arith.* **31** (1976), 199–204.
- [14] T. N. SHOREY, On the greatest prime factor of $ax^n + by^m$ jour *Acta Arith.*, Vol. 36, 1980, 21–25.
- [15] R. TIJDEMAN, Applications of the Gelfond–Baker method to rational number theory, *Colloq. Math. Soc. János Bolyai* 13, 1974, 399–416.
- [16] J. TURK, On the difference between perfect powers, *Acta Arith.* **45** (1986), 289–307.

ISTVÁN PINK
INSTITUTE OF MATHEMATICS AND INFORMATICS
UNIVERSITY OF DEBRECEN
4010 DEBRECEN P.O. BOX 12
HUNGARY

E-mail: pinki@math.klte.hu

(Received September 13, 2002; revised December 12, 2002)