

On varieties defined by pseudocomplemented nondistributive lattices

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Abstract. Lattices with 1, where for each element $a \in L$ the interval $[a, 1]$ is pseudocomplemented, can be equipped with a binary operation “ $\circ$ ” similar to the operation of relative pseudocomplementation. These algebras $(L, \wedge, \vee, \circ, 1)$ form an arithmetical and 1-regular variety. We investigate the subvarieties and the congruence kernels in this variety. It is shown that all algebras $(L, \wedge, \vee, \circ, 1)$ where L is a finite sublattice of a free lattice can be characterized by a particular identity.

1. Introduction

A bounded lattice L is called *pseudocomplemented* if for any $x \in L$ there exists an element $x^* \in L$ with the property that

$$y \wedge x = 0 \quad \text{if and only if} \quad y \leq x^*.$$

In [4] were characterized lattices with greatest element 1 where for each element $a \in L$ the interval $[a, 1]$ is pseudocomplemented. It was shown that they can be equipped with a binary operation “ $\circ$ ” having similar properties as the operation of relative pseudocomplementation and that the class $\mathcal{P}$ of all these algebras $(L, \wedge, \vee, \circ, 1)$ is equational. Although lattices with relative pseudocomplementation are always distributive (see e.g. [2]), the

Mathematics Subject Classification: 06D15, 06B20.

Key words and phrases: relative pseudocomplement, $\wedge$ -semidistributive lattice, (L_n) -lattice, Brouwerian algebras, 1-regular variety.

above mentioned operation $\circ$ can be defined for nondistributive lattices, too. These preliminary results are discussed in Section 2.

An important subclass of the class of relatively pseudocomplemented lattices are *relative(L_n)-lattices* introduced in [11]. In Section 3, by the mean of the operation $\circ$, this notion is successfully extended to the nondistributive case. As a result we obtain new subvarieties of the variety $\mathcal{P}$. In Section 4 we apply our results to finite sublattices of free lattices. In Section 5 the congruence properties of $\mathcal{P}$ are investigated and we characterize the congruence kernels in the algebras of $\mathcal{P}$.

2. Preliminaries

Let L be a lattice with 1 and $x, y \in L$. The pseudocomplement of $x \vee y$ in the interval $[y, 1]$ (if it exists) is denoted by $x \circ y$ (see [4]), and it is called the *section pseudocomplement of x with respect to y* .

Lemma 2.1. *The following conditions are equivalent for a lattice L with 1.*

- (a) *For any $x, y \in L$ the section pseudocomplement $x \circ y$ exists in L .*
- (b) *Any principal filter $[a, 1]$ of L is a pseudocomplemented lattice.*
- (c) *For any $a \leq b$ the interval $[a, b]$ is a pseudocomplemented lattice.*

PROOF. The equivalence of (a) and (b) was proved in [4] and (c) $\implies$ (b) is clear. As any principal ideal of a pseudocomplemented lattice is also a pseudocomplemented lattice, (b) $\implies$ (c) is obvious. $\square$

A lattice L with 1 is called *sectionally pseudocomplemented*, if the section pseudocomplement $x \circ y$ exists for each $x, y \in L$.

Remark 2.2. We recall that $(L, \wedge, \vee, *, 1)$ is a *Brouwerian algebra* (or a *relatively pseudocomplemented lattice*) if $(L, \wedge, \vee, 1)$ is a lattice with 1 and having the property that for any $a, b, x \in L$,

$$a \wedge x \leq b \iff x \leq a * b.$$

The operation $\circ$ can be considered as an extension of $*$, since for $x \in [y, 1]$ we have $x * y = x \circ y$, whenever $x * y$ exists (see [4]). If L is a distributive lattice, then the operations $\circ$ and $*$ coincide (see [1]).

Example 2.3. The lattice N_5 (see Figure 1) is sectionally pseudocomplemented but not relatively pseudocomplemented (see [4]).

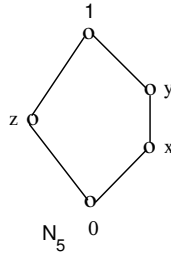


Figure 1

E.g. the relative pseudocomplement $y * x$ does not exist, however the section pseudocomplement $y \circ x$ exists and equals to x .

A lattice L is $\wedge$ -semidistributive if $a \wedge b_1 = a \wedge b_2$ implies $a \wedge b_1 = a \wedge (b_1 \vee b_2)$ for any $a, b_1, b_2 \in L$. A complete lattice L is called *completely $\wedge$ -semidistributive* if for any $b_i \in L, i \in I$ and $a \in L$ the relations $a \wedge b_i = y, i \in I$ imply $a \wedge (\bigvee \{b_i \mid i \in I\}) = y$. In view of [4], for any complete lattice L the conditions of the Lemma 2.1 are equivalent to the condition

(d) L is completely $\wedge$ -semidistributive.

A lattice L with 0 is called *0-distributive* if for any elements $a, b, c \in L$ $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$. Clearly, any principal filter of a $\wedge$ -semidistributive lattice is 0-distributive. According to [14; Theorem 1], an algebraic lattice is pseudocomplemented if and only if it is 0-distributive. These results leads us to the following

Proposition 2.4. *If L is an algebraic lattice, then the following conditions are equivalent:*

- (i) L is sectionally pseudocomplemented.
- (ii) L is completely $\wedge$ -semidistributive.
- (iii) L is $\wedge$ -semidistributive.
- (iv) Any principal filter $[a, 1]$ of L is a 0-distributive lattice.

PROOF. The implications (ii) $\implies$ (iii) $\implies$ (iv) are obvious and the equivalence (i) $\iff$ (ii) was established in [4]. As any principal filter

$[a, 1]$ of an algebraic lattice is also an algebraic lattice, the above mentioned result of [14] gives (iv) $\implies$ (b), whence using Lemma 2.1 we get (iv) $\implies$ (i). $\square$

We note that in the rest of the paper we deal with arbitrary sectionally pseudocomplemented lattices and in general we do not assume that they are algebraic or complete.

Let $\mathcal{P}$ denote the class of all algebras $(L, \wedge, \vee, \circ, 1)$, defined on sectionally pseudocomplemented lattices $(L, \wedge, \vee, 1)$. In [4] it was shown that the class $\mathcal{P}$ is determined by identities in signature $\{\wedge, \vee, \circ, 1\}$, namely by the lattice axioms and by the identities

- (1) $x \circ x = 1, 1 \circ x = x$
- (2) $((x \circ y) \circ y) \wedge (x \vee y) = x \vee y$
- (3) $(x \vee y) \circ y = x \circ y, y \vee (x \circ y) = x \circ y$
- (4) $([(x \vee z) \wedge (y \vee z)] \circ z) \wedge [(x \vee z) \wedge (y \circ z)] \circ z = x \circ z$

Thus $\mathcal{P}$ is a variety and, according to Remark 2.3, $\mathcal{P}$ contains as a subvariety the variety $\mathcal{B}$ of all Brouwerian algebras.

3. Hereditary weakly L_n -lattices

Definition 3.1. (i) Let L be a pseudocomplemented lattice and $n \geq 1$. We say that L is a *weakly L_n -lattice*, if it satisfies the equation:

$$(x_1 \wedge \dots \wedge x_n)^* \vee (x_1^* \wedge \dots \wedge x_n)^* \vee \dots \vee (x_1 \wedge \dots \wedge x_n^*)^* = 1. \quad (L_n)$$

If in addition L is distributive, then it is called an *(L_n) -lattice* [11].

(ii) L is called a *hereditary weakly (L_n) -lattice* if any principal filter $[a, 1]$ of it is a weakly (L_n) -lattice.

Notice, that for $n = 1$ Definition 3.1(i) gives $x^* \vee x^{**} = 1$ and we say that the lattice L is *weakly Stonean*. If L is a distributive lattice, then Definition 3.1(ii) implies that any interval $[a, b] \subseteq L$ is also an (L_n) -lattice. Lattices with this property were called in [11] *relative (L_n) -lattices*. (Relative (L_1) -lattices are known also under the name *relative Stone lattices*, see e.g. [10].)

$$(x_1 \wedge \dots \wedge x_n) * y \vee (x_1 * y \wedge \dots \wedge x_n * y) \vee \dots \vee (x_1 \wedge \dots \wedge x_n * y) * y = 1. \quad (L_n)'$$

Lemma 3.2. *If L is a sectionally pseudocomplemented lattice, then*

$$x \leq z \implies x \circ y \geq z \circ y.$$

- (i) L is a hereditary weakly (L_n) -lattice.
- (ii) $x \circ y$ is defined for all $x, y \in L$ and the algebra $(L, \wedge, \vee, \circ, 1)$ satisfies the equation:

$$(x_1 \wedge \dots \wedge x_n) \circ y \vee (x_1 \circ y \wedge \dots \wedge x_n \circ y) \vee \dots \vee (x_1 \wedge \dots \wedge x_n \circ y) \circ y = 1. \quad (P_n)$$

On the other hand, using the identity $(x \vee y) \circ y = x \circ y$ we get

$$\begin{aligned} x_1 \circ y \wedge \dots \wedge x_n &\leqslant (x_1 \vee y) \circ y \wedge \dots \wedge (x_n \vee y), \\ \vdots \\ x_1 \wedge \dots \wedge x_n \circ y &\leqslant (x_1 \vee y) \wedge \dots \wedge (x_n \vee y) \circ y. \end{aligned}$$

$$\begin{aligned} & (x_1 \wedge \dots \wedge x_n) \circ y \vee (x_1 \circ y \wedge \dots \wedge x_n) \circ y \vee \dots \vee (x_1 \wedge \dots \wedge x_n \circ y) \circ y \\ & \geq [(x_1 \vee y) \wedge \dots \wedge (x_n \vee y)] \circ y \vee [(x_1 \vee y) \circ y \wedge \dots \wedge (x_n \vee y)] \circ y \vee \dots \\ & \vee [(x_1 \vee y) \wedge \dots \wedge (x_n \vee y) \circ y] \circ y. \end{aligned}$$

Since $x_1 \vee y \geq y, \dots, x_n \vee y \geq y$, and $(x_1 \vee y) \circ y \geq y, \dots, (x_n \vee y) \circ y \geq y$, all these elements belong to the pseudocomplemented lattice $[y, 1]$. Let u^y

$$\begin{aligned} & [(x_1 \vee y) \wedge \dots \wedge (x_n \vee y)] \circ y = [(x_1 \vee y) \wedge \dots \wedge (x_n \vee y)]^y, \\ & [(x_1 \vee y) \circ y \wedge \dots \wedge (x_n \vee y)] \circ y = [(x_1 \vee y)^y \wedge \dots \wedge (x_n \vee y)^y]^y, \\ & \\ & [(x_1 \vee y) \wedge \dots \wedge (x_n \vee y) \circ y] \circ y = [(x_1 \vee y) \wedge \dots \wedge (x_n \vee y)^y]^y. \end{aligned}$$
$$\begin{aligned} & (x_1 \wedge \dots \wedge x_n) \circ y \vee (x_1 \circ y \wedge \dots \wedge x_n) \circ y \vee \dots \vee (x_1 \wedge \dots \wedge x_n \circ y) \circ y \\ & \quad \geq [(x_1 \vee y) \wedge \dots \wedge (x_n \vee y)]^y \vee [(x_1 \vee y)^y \wedge \dots \wedge (x_n \vee y)^y]^y \vee \dots \\ & \quad \vee [(x_1 \vee y) \wedge \dots \wedge (x_n \vee y)^y]^y = 1. \end{aligned}$$

(ii) $\implies$ (i): Assume that (P_n) holds in the algebra $(L, \wedge, \vee, \circ, 1)$ and take an $a \in L$. As $x \circ y$ is defined for all $x, y \in L$, in view of Lemma 2.1 the interval $[a, 1]$ is a pseudocomplemented lattice. Let $x_1, \dots, x_n \geq a$. Since for any $x \in [a, 1]$ we have $x \circ a = x^a$, we get

This equation shows that for any $a \in L$, the principal filter $[a]$ is an (L_n) -lattice. Thus L is a hereditary weakly (L_n) -lattice. $\square$

Indeed, it is not hard to see that any principal filter $[a]$ of N_5 satisfies the equality $x^a \vee (x^a)^a = 1$, therefore N_5 is a hereditary weakly (L_1) -lattice. In view of Theorem 3.3, $(N_5, \wedge, \vee, \circ, 1)$ satisfies (P_1) , too.

Let $\mathcal{P}_n$ denote the class of all algebras $(L, \wedge, \vee, \circ, 1)$ corresponding to hereditary weakly (L_n) -lattices. Because any $\mathcal{P}_n$ is a subclass of $\mathcal{P}$ determined by the identity (P_n) , any $\mathcal{P}_n$ is a subvariety of $\mathcal{P}$. Since in

the variety $\mathcal{B}$ of Brouwerian algebras the identity (P_n) is the same as $(L_n)'$, the subvarieties $\mathcal{B}_n = \mathcal{B} \cap \mathcal{P}_n$ of $\mathcal{B}$ consist of algebras $(L, \wedge, \vee, *, 1)$ corresponding to relative (L_n) -lattices. As in view of [12] relative (L_n) -lattices form a proper subclass of the class of relative (L_{n+1}) -lattices, we have $\mathcal{B}_n \subset \mathcal{B}_{n+1}$. Let $\mathcal{B}_{-1}$ and $\mathcal{B}_0$ be the class of all trivial Brouwerian algebras and the subclass of $\mathcal{B}$ determined by the identity $x \vee y \vee (x * y) = 1$, respectively. Clearly, $\mathbb{L} = (L, \wedge, \vee, *, 1) \in \mathcal{B}_0 \Leftrightarrow$ any $[y, 1]$ is complemented $\Leftrightarrow$ every interval of L is a Boolean lattice. Hence $\mathbb{L} \in \mathcal{B}_0$ if and only if the dual of L is a *generalized Boolean lattice* (see e.g. [9] or [1]).

Proposition 3.5. *The variety $\mathcal{P}$ contains an infinite chain of proper subvarieties $\mathcal{B}_{-1} \subset \mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{P}_1 \subset \dots \subset \mathcal{P}_n \subset \dots$ and $\mathcal{B}_0$ is the single minimal subvariety of $\mathcal{P}$.*

PROOF. The inclusions $\mathcal{B}_{-1} \subset \mathcal{B}_0 \subset \mathcal{B}_1, \mathcal{B}_1 \subseteq \mathcal{P}_1, \mathcal{P}_n \subseteq \mathcal{P}$ are obvious. As $(N_5, \wedge, \vee, \circ, 1)$ is in $\mathcal{P}_1 \setminus \mathcal{B}$ (see Example 3.4) $\mathcal{B}_1 \subset \mathcal{P}_1$ is also clear. Since for any $x_1, \dots, x_n, x_{n+1}, y \in L$ by Lemma 3.2 we get

$$\begin{aligned} & (x_1 \wedge \dots \wedge x_n \wedge x_{n+1}) \circ y \vee (x_1 \circ y \wedge \dots \wedge x_n \wedge x_{n+1}) \circ y \vee \dots \\ & \vee (x_1 \wedge \dots \wedge x_n \circ y \wedge x_{n+1}) \circ y \vee (x_1 \wedge \dots \wedge x_n \wedge x_{n+1} \circ y) \circ y \\ & \geq (x_1 \wedge \dots \wedge x_n) \circ y \vee (x_1 \circ y \wedge \dots \wedge x_n) \circ y \vee \dots \\ & \vee (x_1 \wedge \dots \wedge x_n \circ y) \circ y, \end{aligned}$$

the identity (P_n) implies (P_{n+1}) and this proves $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}$. Now $\mathcal{B} \cap \mathcal{P}_n \subset \mathcal{B} \cap \mathcal{P}_{n+1}$ implies $\mathcal{P}_n \neq \mathcal{P}_{n+1}$ and $\mathcal{P}_n \neq \mathcal{P}$.

It is known that $\mathcal{B}_0$ is a minimal variety generated by $\mathbb{B}_2$, the Brouwerian algebra defined on the chain with two elements (see e.g. [2]). Let $\mathcal{M}$ be a minimal subvariety of $\mathcal{P}$ and $\mathbb{A} = (A, \wedge, \vee, \circ, 1)$ a nontrivial algebra in $\mathcal{M}$. Then there exists an $a \in A \setminus \{1\}$. As $(\{a, 1\}, \wedge, \vee, \circ, 1)$ is subalgebra of $\mathbb{A}$ isomorphic to $\mathbb{B}_2$, we get $\mathcal{B}_0 = \mathcal{M}$. $\square$

4. Application to finite sublattices of a free lattice

In this section we show that any finite sublattice of a free lattice satisfies the equations (L_4) and (P_4) .

Since any finite sublattice L of a free lattice is an algebraic $\wedge$ -semi-distributive lattice (see e.g. Theorem 2.4 in [6]), according to Proposition 2.2 any principal filter of such a lattice L is a pseudocomplemented lattice. Moreover, we prove:

Theorem 4.1. *Any finite sublattice of a free lattice is a hereditary weakly (L_4) -lattice.*

PROOF. As any principal filter $[a]$ of a finite sublattice of a free lattice F is also a finite sublattice of F , it is enough to prove that each finite sublattice L of F satisfies the identity (L_4) .

In view of [13, Corollary 3.9], a lattice L satisfying the descending chain condition is a weakly (L_n) -lattice whenever under each join-irreducible element of L are at most n atoms. In [7] is proved that any finite sublattice L of a free lattice has a *breadth* at most 4 (see also [6, Corollary 5.5]), i.e. for any $n \geq 4$ and any finite set $\{a_1, a_2, \dots, a_n\}$ there exist $a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4} \in \{a_1, a_2, \dots, a_n\}$ such that $\bigvee_{i=1}^n a_i = a_{i_1} \vee a_{i_2} \vee a_{i_3} \vee a_{i_4}$.

Now, let L be a finite sublattice of a free lattice and p a join-irreducible element of L . We show that under p are at most 4 atoms.

Indeed, let us denote by $a_1, a_2, \dots, a_n$ the atoms of L which are under p . Then there are at most four atoms $a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4} \in [0, p]$ such that $\bigvee_{i=1}^n a_i = a_{i_1} \vee a_{i_2} \vee a_{i_3} \vee a_{i_4}$. If $n > 4$, then there exist an atom a_{i_0} , $i_0 \in \{1, \dots, n\}$ such that $a_{i_0} \notin \{a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}\}$. As L is $\wedge$ -semidistributive, the relations $a_{i_1} \wedge a_{i_0} = 0$, $a_{i_2} \wedge a_{i_0} = 0$, $a_{i_3} \wedge a_{i_0} = 0$ and $a_{i_4} \wedge a_{i_0} = 0$ imply $a_{i_0} = (a_{i_1} \vee a_{i_2} \vee a_{i_3} \vee a_{i_4}) \wedge a_{i_0} = 0$, – a contradiction.

Hence L satisfies the identity (L_4) . □

Using the above result and applying Theorem 3.3 we obtain:

Corollary 4.2. *If L is a finite sublattice of a free lattice, then L is sectionally pseudocomplemented and the algebra $(L, \wedge, \vee, \circ, 1)$ satisfies the equation (P_4) .*

5. On the congruence properties of the variety $\mathcal{P}$

In this section we shall study the congruence properties of algebras $\mathbb{L} = (L, \wedge, \vee, \circ, 1)$ from the variety $\mathcal{P}$.

Of course, the variety $\mathcal{P}$ is congruence distributive because its reduct to the signature $\{\wedge, \vee\}$ is a class of lattices. Moreover, $\mathcal{P}$ is also congruence permutable. Indeed, one can deduce that a Mal'cev term of $\mathcal{P}$ can be e.g.

$$p(x, y, z) = (y \circ x) \wedge (x \vee z) \wedge (y \circ z).$$

We recall that a variety is *arithmetical* if it is congruence distributive and congruence permutable at the same time. Thus we have:

Theorem 5.1. *The variety $\mathcal{P}$ is arithmetical.*

Let $\Theta \in \text{Con } \mathbb{L}$. The set $[1]_\Theta = \{x \in L \mid (1, x) \in \Theta\}$ is called the *kernel of Θ* . We say that the set $K \subseteq L$ is a *congruence kernel* if $K = [1]_\Theta$ for some $\Theta \in \text{Con } \mathbb{L}$.

Recall that $\mathbb{L}$ is *1-regular* if $[1]_\Theta = [1]_\Phi$ implies $\Theta = \Phi$ for each $\Theta, \Phi \in \text{Con } \mathbb{L}$. The following result was given by B. CSÁKÁNY in [5]:

Proposition 5.2. *A variety $\mathcal{V}$ with the constant 1 is 1-regular if and only if there exist $n \in \mathbb{N}$ and binary terms $b_1(x, y), \dots, b_n(x, y)$ such that $\mathcal{V}$ satisfies the equivalence*

$$b_1(x, y) = \dots = b_n(x, y) = 1 \iff x = y.$$

By using this proposition we can prove:

Theorem 5.3. *The variety $\mathcal{P}$ is 1-regular.*

PROOF. Take $n = 2$ and $b_1(x, y) = x \circ y$, $b_2(x, y) = y \circ x$. Of course, $b_1(x, x) = b_2(x, x) = 1$. Conversely, suppose $b_1(x, y) = b_2(x, y) = 1$.

Then $x \circ y = 1$ implies $(x \vee y)^y = 1$, i.e. $x \vee y = y$ and $y \circ x = 1$ implies $(x \vee y)^x = 1$, i.e. $x \vee y = x$, thus we get $x = y$. $\square$

Remark 5.4. (i) We can get also the Pixley term for arithmeciety of $\mathcal{P}$, which is $t(x, y, z) = [(x \circ y) \circ z] \wedge [(z \circ y) \circ x] \wedge (x \vee z)$.

(ii) Let us note, as shown in [3], that a variety $\mathcal{V}$ is 1-regular and permutable if and only if there exist $n \in \mathbb{N}$, binary terms $s_1(x, y), \dots, s_n(x, y)$ and a $(2 + n)$ -ary term q such that $\mathcal{V}$ satisfies the identities

$$s_i(x, x) = 1, \quad \text{for } i = 1, \dots, n$$

$$x = q(x, y, s_1(x, y), \dots, s_n(x, y)),$$

$$y = q(x, y, 1, \dots, 1).$$

To verify this Mal'cev condition, one can take $n = 2$, $s_1(x, y) = x \circ y$, $s_2(x, y) = y \circ x$ and

$$q(x, y, z, v) = (z \circ y) \wedge [(v \circ (y \circ x)) \circ x] \wedge (x \vee y).$$

This term q will be also important in the proof of the Theorem 5.7.

Since the variety $\mathcal{P}$ is 1-regular, every congruence $\Theta \in \text{Con } \mathbb{L}$ for $\mathbb{L} \in \mathcal{P}$ is determined by its kernel $[1]_\Theta$. Hence, our task is to determine the congruence kernels and get an explicit description of a congruence determined by a given kernel.

Let $K \subseteq L$ and let $t(x_1, \dots, x_n, y_1, \dots, y_k)$ be a term of $\mathbb{L} = (L, \wedge, \vee, \circ, 1)$ in two sorts of variables. We say that K is *y-closed with respect to t* if $t(a_1, \dots, a_n, b_1, \dots, b_k) \in K$ whenever $b_1, \dots, b_k \in K$ and for each $a_1, \dots, a_n \in L$.

For the sake of brevity, we introduce the notations:

$$Q_1 = q(x_1, x_2, y_1, y_2) = (y_1 \circ x_2) \wedge [(y_2 \circ (x_2 \circ x_1)) \circ x_1] \wedge (x_1 \vee x_2),$$

$$Q_2 = q(x_3, x_4, y_3, y_4) = (y_3 \circ x_4) \wedge [(y_4 \circ (x_4 \circ x_3)) \circ x_3] \wedge (x_3 \vee x_4).$$

Further, define the following terms in two sorts of variables:

$$t_1(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (Q_1 \circ Q_2) \circ (x_2 \circ x_4),$$

$$t_2(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (x_2 \circ x_4) \circ (Q_1 \circ Q_2),$$

$$t_3(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (Q_1 \wedge Q_2) \circ (x_2 \wedge x_4),$$

$$t_4(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (x_2 \wedge x_4) \circ (Q_1 \wedge Q_2),$$

$$t_5(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (Q_1 \vee Q_2) \circ (x_2 \vee x_4),$$

$$t_6(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (x_2 \vee x_4) \circ (Q_1 \vee Q_2).$$

Lemma 5.5. (i) Let $K = [1]_\Theta$ for some $\Theta \in \text{Con } \mathbb{L}$ and $t(x_1, \dots, x_n, y_1, \dots, y_k)$ be a term of $\mathbb{L}$ such that $t(x_1, \dots, x_n, 1, \dots, 1) = 1$. If $y_1, \dots, y_k \in K$ then $t(x_1, \dots, x_n, y_1, \dots, y_k) \in K$.

(ii) For the terms $t_1, \dots, t_6$ defined above, we have

$$t_i(x_1, x_2, x_3, x_4, 1, 1, 1, 1) = 1.$$

PROOF. The proof of (i) is elementary and hence omitted. (ii) is an easy consequence of $q(x_1, x_2, 1, 1) = x_2$ and $q(x_3, x_4, 1, 1) = x_4$. $\square$

Lemma 5.6. *Let K be a subset of L with $1 \in K$. Define the relation Φ_K on L by*

$$(x, y) \in \Phi_K \iff x \circ y \in K \quad \text{and} \quad y \circ x \in K. \quad (*)$$

Then $K = [1]_{\Phi_K}$.

PROOF. If $a \in K$, then $1 \circ a = a \in K$ and $a \circ 1 = 1 \in K$, thus by (*) $(1, a) \in \Phi_K$ and so $a \in [1]_{\Phi_K}$. Conversely, if $a \in [1]_{\Phi_K}$, then (*) gives $a = 1 \circ a \in K$. Thus $K = [1]_{\Phi_K}$. $\square$

A filter F of a lattice L is called *standard* if it is a standard element in the lattice $\mathcal{F}(L)$ of all filters of L , i.e. if the equality $[a] \wedge (F \vee [b]) = ([a] \wedge F) \wedge [a \vee b]$ holds in $\mathcal{F}(L)$.

Theorem 5.7. *Let $\mathbb{L} = (L, \wedge, \vee, \circ, 1) \in \mathcal{P}$ and $K \subseteq L$ with $1 \in K$. Then the following assertions are equivalent:*

- (i) K is a congruence kernel.
- (ii) K is y -closed with respect to the terms $t_1, \dots, t_6$.
- (iii) The relation Φ_K defined by (*) is a congruence of $\mathbb{L}$.
- (iv) K is a standard filter of L .

PROOF. (i) $\implies$ (ii): Assume that $K = [1]_{\Theta}$ for some $\Theta \in \text{Con } \mathbb{L}$. Then $1 \in K$ and, by Lemma 5.5, K is y -closed with respect to $t_1, \dots, t_6$.

(ii) $\implies$ (iii): Obviously, Φ_K is reflexive. Suppose $(a, b) \in \Phi_K$ and $(c, d) \in \Phi_K$ for some $a, b, c, d \in L$. Then $a \circ b, b \circ a, c \circ d, d \circ c \in K$. Since in view of Remark 5.4(ii) we have

$$a = q(a, b, s_1(a, b), s_2(a, b)) = q(a, b, a \circ b, b \circ a)$$

and

$$c = q(c, d, s_1(c, d), s_2(c, d)) = q(c, d, c \circ d, d \circ c),$$

and since K is y -closed with respect to t_1 , applying the term t_1 , we get $(a \circ c) \circ (b \circ d) = (q(a, b, a \circ b, b \circ a) \circ q(c, d, c \circ d, d \circ c)) \circ (b \circ d) = t_1(a, b, c, d, a \circ b, b \circ a, c \circ d, d \circ c) \in K$.

Analogously we can show $(b \circ d) \circ (a \circ c) \in K$ applying t_2 instead of t_1 . Hence, by $(*)$ we have also $(a \circ c, b \circ d) \in \Phi_K$.

Substituting t_3 and t_4 (instead of t_1 and t_2) in the above argument, we get $(a \wedge c, b \wedge d) \in \Phi_K$ and for t_5, t_6 we get $(a \vee c, b \vee d) \in \Phi_K$.

Together, Φ_K is a reflexive relation on L having the substitution property with respect to all operations of $\mathbb{L}$. As $\mathbb{L}$ belongs to a Mal'cev variety, by the theorem of WERNER [15] we obtain $\Phi_K \in \text{Con } \mathbb{L}$.

(iii) $\implies$ (i): Since by Lemma 5.6 we have $K = [1]_{\Phi_K}$ and since by assumption $\Phi_K \in \text{Con } \mathbb{L}$, K is a congruence kernel.

(i) $\implies$ (iv): Assume that Θ is a congruence of $\mathbb{L}$ such that $K = [1]_{\Theta}$. Since Θ is also a congruence of the lattice $(L, \wedge, \vee, 1)$, it is clear that $[1]_{\Theta}$ is a lattice filter. Suppose that $(x, y) \in \Theta$. Then $(x \vee y, x \wedge y) \in \Theta$, as Θ is a lattice congruence. Hence $(x \vee y) \circ (x \wedge y) \in K$, because Θ is a congruence on $\mathbb{L}$. Since

$$x \wedge y = (x \vee y) \wedge [(x \vee y) \circ (x \wedge y)],$$

K is a standard filter by [8, Theorem III.5].

(iv) $\implies$ (i): Assume that K is a standard filter of L and take $\Theta = \Theta[K]$ the smallest congruence on L generated by K . This exists by [8, Theorem III.5] and it is easy to check that $K = [1]_{\Theta}$. We have only to show that Θ is compatible with the binary operation $\circ$. It is enough to show that the factor-lattice L/Θ is sectionally pseudocomplemented. More precisely, we claim that $[a]_{\Theta} \circ [b]_{\Theta} = [a \circ b]_{\Theta}$ for any $a, b \in L$. Really,

$$([a]_{\Theta} \vee [b]_{\Theta}) \wedge [a \circ b]_{\Theta} = [a \vee b]_{\Theta} \wedge [a \circ b]_{\Theta} = [(a \vee b) \wedge (a \circ b)]_{\Theta} = [b]_{\Theta}$$

as Θ is a lattice congruence.

Now, take $[x]_{\Theta} \geq [b]_{\Theta}$ in L/Θ and suppose that

$$([a]_{\Theta} \vee [b]_{\Theta}) \wedge [x]_{\Theta} = [(a \vee b) \wedge x]_{\Theta} = [b]_{\Theta}.$$

Without loss of generality we can assume $x \geq b$ in L . Then $(a \vee b) \wedge x \geq b$ and in view of [8, Theorem III.5] there exists a $g \in K$ such that $(a \vee b) \wedge x \wedge g = b$ holds in L . It follows that $x \wedge g \leq a \circ b$, as L is sectionally pseudocomplemented. As $g \in K$, we have $[x \wedge g]_{\Theta} = [x]_{\Theta}$.

Finally, we obtain $[x]_{\Theta} \leq [a \circ b]_{\Theta}$ and hence $[a \circ b]_{\Theta} = [a]_{\Theta} \circ [b]_{\Theta}$, as claimed. $\square$

Corollary 5.8. *For any $\Theta \in \text{Con } \mathbb{L}$ we have $\Theta = \Phi_{[1]_{\Theta}}$.*

PROOF. Let $\Theta \in \text{Con } \mathbb{L}$ and take $K = [1]_{\Theta}$. Then by Theorem 5.7 we have $\Phi_K \in \text{Con } \mathbb{L}$ and Lemma 5.6 gives $[1]_{\Theta} = [1]_{\Phi_K}$. As $\mathbb{L}$ is an algebra of a congruence 1-regular variety, we get $\Theta = \Phi_K$, i.e. $\Theta = \Phi_{[1]_{\Theta}}$. $\square$

Problems

- 1) Characterize the subdirectly irreducible algebras in the varieties $\mathcal{P}_n$, $n \in \mathbb{N}$.
- 2) Characterize the lattice of subvarieties of $\mathcal{P}$.

ACKNOWLEDGMENTS. The authors thank to the referee for his/her valuable remarks which made the paper more readable and for completing Theorem 5.7. The first author gratefully acknowledges the support by the Council of Czech Government MSM 153100011. The second author gratefully thanks the support by Szécsenyi István Grant and by the Hungarian National Foundation for Scientific Research (Grant No. T029525 and T034137).

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(Received November 21, 2002; revised April 4, 2003)