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On the equation $1^k + 2^k + \ldots + x^k = y^n$

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To Professor L. Tamássy on his 80th birthday

Abstract. In our paper a survey is given on the title equation. General finiteness theorems, bounds on n and the number of solutions, complete solution for small values of k, and some generalizations and analogues are presented. The basic ideas of the most important proofs are also outlined. We note that in the proofs of the recent results virtually every technique of modern Diophantine analysis has been employed.

1. Introduction

A classical problem of LUCAS [24], from 1875, was whether the diophantine equation

$$1^2 + 2^2 + \cdots + x^2 = y^2 \tag{1}$$

has solutions in positive integers other than $(x, y) = (1, 1)$ and $(24, 70)$. Putative solutions by MORET–BLANC [26] and LUCAS [25] contained some errors and it was not until 1918 that WATSON [36] was able to completely

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solve equation (1), and to correctly answer the question, in the negative. For more recent and elementary proofs we refer to [23] and [1], [15], respectively. For a survey on this equation, see [4].

In 1956, SCHÄFFER [29] considered the more general equation

$$S_k(x) = y^n, \tag{2}$$

where, here and subsequently, we write

$$S_k(x) = 1^k + 2^k + \cdots + x^k \quad \text{for positive integer } k.$$

As is known, $S_k(x)$ can be expressed as a polynomial of degree $k+1$ with rational coefficients. Hence, for $n \geq 2$, (2) is in fact a special superelliptic equation; for general superelliptic equations, see [31] and the survey paper [12].

For every k and n, $(x, y) = (1, 1)$ is a solution of (2). SCHÄFFER [29] proved that if $k \geq 1$ and $n \geq 2$ are fixed, then apart from certain exceptions (k, n) (see (3) below), (2) has only finitely many solutions in integers $x, y \geq 1$. His proof was ineffective. In 1980, GYŐRY, TIJDEMAN and VOORHOEVE [19] established, in an effective form, a more general finiteness theorem in which the exponent n is also unknown. Later, various generalizations and analogues have been obtained by Győry, Tijdeman, Voorhoeve, Brindza, Dilcher, Urbanowicz and others. These results will be presented in Section 2.

In Section 3, some results of PINTÉR [28] as well as BRINDZA and PINTÉR [11] are formulated, which furnish explicit upper bounds, in terms of k, for n and for the number of solutions of (2) in integers $x, y \geq 1$, $n \geq 2$, respectively.

Finally, Section 4 is concerned with a conjecture of SCHÄFFER [29] on the solvability of (2). First some partial results of SCHÄFFER [29] are presented. Then two recent theorems of JACOBSON, PINTÉR and WALSH [20] and BENNETT, GYŐRY and PINTÉR [5] are stated which provide the complete solution of (2) for $n = 2$ and even $k \leq 58$, and for $k \leq 11$ and arbitrary $n \geq 2$, respectively.

2. Finiteness results and various generalizations

SCHÄFFER [29] proved the following.

Theorem A (SCHÄFFER [29]). *For fixed $k \geq 1$ and $n \geq 2$, (2) has at most finitely many solutions in positive integers x and y, unless*

$$(k, n) \in \{(1, 2), (3, 2), (3, 4), (5, 2)\}, \tag{3}$$

where, in each case, there are infinitely many such solutions.

SCHÄFFER [29] gave, for each (k, n) contained in the set (3), all the solutions. Further, he showed that in all other cases the number of solutions is bounded by a constant depending only on k. In Schäffer's proof the main line of attack is as follows. As is known,

$$S_k(x) = \frac{1}{k+1}(B_{k+1}(x+1) - B_{k+1}(0)),$$

where $B_{k+1}(x)$ denotes the $(k+1)$-th Bernoulli polynomial. This implies that if $k \geq 2$ is even, then $0, -1$ and $-1/2$ are simple zeros of $S_k(x)$, while for odd $k \geq 3$, 0 and -1 are double zeros of $S_k(x)$. Hence it follows that

$$S_k(x) = \begin{cases} x^2(x+1)^2 T_k(x)/C_k & \text{if } k > 1 \text{ is odd,} \\ x(x+1)(2x+1)T_k(x)/C_k & \text{if } k \geq 2 \text{ is even.} \end{cases}$$

Here C_k is a positive integer and $T_k(x)$ is a polynomial with integer coefficients. Schäffer proved that for odd $n \geq 3$ and for every solution x, y of (2), the numbers $x, x+1$ and, if k is even, $2x+1$ are all perfect nth powers multiplied by bounded factors, the bound depending only on k. Thus (2) leads to finitely many equations of the type

$$AX^n - BY^n = 1 \text{ in non-zero } X, Y \in \mathbb{Z}, \tag{4}$$

with fixed non-zero integers A, B, and the number of the resulting equations (4) is bounded above by a constant depending only on k. However, by theorems of NAGELL ([27], case $n = 3$) and DOMAR ([17], case $n \geq 5$) each of the equations (4) has at most two solutions, whence, for odd $n \geq 3$, Theorem A follows. Since the results utilized from [27] and [17] are ineffective, Schäffer's proof is also ineffective.

On applying Baker's method, Győry, Tijdeman and Voorhoeve [19] proved a more general and effective result in which the exponent n is also unknown.

Theorem B (Győry, Tijdeman and Voorhoeve [19]). *Let $k \geq 2$ and r be fixed integers with $k \notin \{3, 5\}$ if $r = 0$, and let s be a square-free odd integer. Then the equation*

$$sS_k(x) + r = y^n \tag{5}$$

in positive integers $x, y \geq 2$, $n \geq 2$ has only finitely many solutions, and all these can be effectively determined.

Of particular importance is the special case when $s = 1$ and $r = 0$.

Corollary (Győry, Tijdeman and Voorhoeve [19]). *For given $k \geq 2$ with $k \notin \{3, 5\}$, equation (2) has only finitely many solutions in integers $x, y \geq 1$, $n \geq 2$, and all these can be effectively determined.*

The following striking result is due to Voorhoeve, Győry and Tijdeman [35].

Theorem C (Voorhoeve, Győry and Tijdeman [35]). *Let $R(x)$ be a fixed polynomial with integer coefficients, and let $k \geq 2$ be a fixed integer such that $k \notin \{3, 5\}$. Then the equation*

$$S_k(x) + R(x) = y^n \tag{6}$$

in integers $x, y \geq 2$, $n \geq 2$ has only finitely many solutions, and an effective upper bound can be given for n.

In their proofs, the authors in [19] and [35] showed that the polynomials $sS_k(x) + r$ and $S_k(x) + R(x)$ has at least two distinct zeros. Then they applied a well-known theorem of Schinzel and Tijdeman [35] on superelliptic equations to derive an effective upper bound for n. Further, in [19], they proved that $sS_k(x) + r$ has at least three simple zeros, and then an effective result of Baker [2] concerning superelliptic equations completed the proof of Theorem B. In [35], a general statement was obtained on the multiplicities of zeros of $S_k(x) + R(x)$. This enabled the authors to apply an ineffective finiteness criterion of LeVeque [22] on the numbers of solutions of superelliptic equations to prove Theorem C.

Later, various generalizations and analogues of Theorems B and C have been established in an effective form. Some of them will now be presented.

Brindza [7] gave a common effective generalization of Theorems B and C. Set $A = \mathbb{Z}[X]$, $\kappa = (k+1)\prod_{p-1|(k+1)!} p$ (p prime), and

$$F(y) = Q_m y^m + \cdots + Q_1 y + Q_0 \in A[y].$$

Consider the equation

$$F(S_k(x)) = y^n \tag{7}$$

in integers $x, y \geq 2$, $n \geq 2$.

Theorem D (Brindza [7]). *If $Q_i(x) \equiv 0 \pmod{\kappa^i}$ for $i = 2, \ldots, m$; $Q_1(x) \equiv \pm 1 \pmod 4$, and $k \notin \{1, 2, 3, 5\}$ then all solutions of (7) satisfy $\max(x, y, n) < c_1$, where c_1 is an effectively computable constant depending only on F and k.*

The effective character of Theorem D is due to an effective version of LeVeque's theorem, established by Brindza [8] in 1984. In particular, if $k \notin \{1, 2, 3, 5\}$ and $Q_2(x) = \cdots = Q_n(x) = 0$ and $Q_1(x) = s$, where s is an odd integer, then, by Theorem D, all solutions of the equation

$$sS_k(x) + Q_0(x) = y^n$$

in integers $x, y, n \geq 2$ satisfy $\max(x, y, n) < c_2$, where c_2 is an effectively computable constant depending only on s and $Q_0(x)$. In case of $k > 5$, this generalizes Theorem B and provides, in a more general form, an effective version of Theorem C.

For $k \not\equiv 1 \pmod 4$, the assumption concerning s in Theorem B was weakened by Kano [21]. For further results on (5) with $s = 8$ and $n = 2$, we refer to [10] and [13].

Dilcher [16] proved character analogues of the results concerning equation (2). Let χ be a primitive quadratic residue class character with conductor $f = f_\chi$.

Theorem E (Dilcher [16]). *Let χ be a primitive quadratic character, and k a fixed positive integer. If k is sufficiently large, then the equation*

$$\chi(1)1^k + \chi(2)2^k + \ldots + \chi(xf)(xf)^k = y^n \tag{8}$$

has only finitely many solutions in integers $x, y \geq 1$ and $n \geq 2$, with effective upper bounds for x, y, n.

As an interesting special case ($f = 4$) we mention that for any integer $k \geq 3$ with $k \notin \{4, 5\}$, Theorem E gives effectively computable upper bounds for the solutions of the equation

$$1^k - 3^k + 5^k - \cdots + (4x-3)^k - (4x-1)^k = \pm y^n$$

in integers $x, y \geq 2$ and $n \geq 2$. For further generalizations of the character case, we refer to [32], [33] and [34].

In [32], Urbanowicz dealt with the general equation

$$f(1)1^k + f(2)2^k + \cdots + f(x)x^k + R(x) = y^n, \tag{9}$$

where $f : \mathbb{N}_0 \to \mathbb{Z}$ is a periodic function. For $f = 1$, this is just equation (6), while for f a quadratic character, (9) reduces to (8). In [32], the author gave some natural subclasses of all periodic functions f such that if $k \geq 4$ with $k \neq 5$, then for f from this subclass and for any $R(x) \in \mathbb{Z}[x]$, (9) possesses only finitely many solutions in integers $x \geq 1$, $y, n \geq 2$ for f from this subclass and for any $R(x) \in \mathbb{Z}[x]$. For example, all periodic functions $f : \mathbb{N}_0 \to \{\pm 1\}$ with period not divisible by 4 belong to the above mentioned subclass.

The proofs of the results of Brindza [7], Kano [21], Dilcher [16] and Urbanowicz [32], [33] and [34] are in fact based upon the classical approach of Schäffer, certain properties of generalized Bernoulli polynomials, some arguments of [19] and [35], and results of Schinzel and Tijdeman [30] and Brindza [8] on superelliptic equations.

3. Explicit upper bounds for n and for the number of solutions

In Theorems B to E, the upper bounds on n are not given explicitly. Using Baker's method, Pintér [28] obtained a rather sharp explicit upper bound for the exponent n.

Theorem F (PINTÉR [28]). *All the solutions x, y, n to equation (2) with $x > 10^3(k/2)^{k+5/2} = c(k)$, $y > 1$ and $n \geq 2$ satisfy*

$$n < c_3 k \log 2k, \tag{10}$$

where c_3 is an effectively computable absolute constant.

In the case $x \leq c(k)$ we have $2^n \leq y^n \leq c(k)^{k+1}$ and so, if $k > 2$, we get

$$n < 6(k+1)^2 \log \frac{k}{2}. \tag{11}$$

In [29], SCHÄFFER's bound on the number of solutions of (2) was not given explicitly. Applying a general result of EVERTSE and SILVERMAN [18] concerning superelliptic equations, one may derive the explicit bound $17^k n^{2k}$ for the number of solutions to (2), provided that this number is finite. BRINDZA [9] derived a better estimate, by showing that for any given $n \geq 3$ with $n \neq 4$, equation (2) has at most e^{7k} solutions. Further, for $k \leq 60$, he proved that (2) has at most e^{33} solutions.

Recently, BRINDZA and PINTÉR [11] proved that for even k and n=2, (2) has at most $\max\{c_4, 9^k\}$ solutions, where c_4 denotes an effectively computable absolute constant. Moreover, in the case when in (2) the exponent n is also unknown, they established the following

Theorem G (BRINDZA and PINTÉR [11]). *Apart from the case $(k, n) = (3, 4)$, equation (2) possesses at most $\max\{c_5, e^{3k}\}$ solutions in positive integers $x, y > 1$, and $n > 2$, where c_5 is an effectively computable absolute constant.*

To prove Theorem G, the authors use among other things (10), (11) and a refined version of SCHÄFFER's approach [29], and reduce (2) to a "small" number of equations of the form (4). Then they apply a recent estimate of BENNETT [3] on the number of solutions of (4).

4. On the resolution of equation (2)

The solution $(x, y) = (1, 1)$ of (2) is called *trivial*. As was seen above, for $(k, n) = (2, 2)$ the only non-trivial solution of (2) is $(x, y) = (24, 70)$. In 1956, SCHÄFFER [29] was able to prove that equation (2) has only the trivial solution in each of the following cases: $k \in \{1, 5\}$ and $n = 4$, $k = 3$ and $n = 8$, $k \in \{4, 6, 8, 9, 10\}$ and $n = 2$, $k \leq 11$ and $n \in \{3, 5\}$, $k \leq 11$ with $k \neq 10$ and $n \in \{29, 41, 53, 113, 173, 281, 509, 641\}$. Further, he formulated the following

Conjecture (SCHÄFFER [29]). *For $k \geq 1$ and $n \geq 2$ with (k, n) not in the set* (3)*, equation* (2) *has only one non-trivial solution, namely $(k, n, x, y) = (2, 2, 24, 70)$.*

Theorem B makes it possible, at least in principle, to determine all solutions of (2). However, the bounds provided by Theorem B are not given explicitly. On the other hand, even explicit values of the bounds which could be derived by Baker's method, would be too large for practical use. In general, for any fixed k, it seems to be hopeless to find all solutions of (2) by the present methods.

Recently, a considerable progress has been made concerning Schäffer's conjecture. The next theorem, achieved by JACOBSON, PINTÉR and WALSH [20], confirms the conjecture for $n = 2$ and for even k with $k \leq 58$.

Theorem H (JACOBSON, PINTÉR and WALSH [20]). *For $n = 2$ and even values of k with $k \leq 58$, equation* (2) *has only the trivial solution except in the case $k = 2$, when there is the anomalous solution $(x, y) = (24, 70)$.*

In [20], the authors used a computational approach for finding all integral solutions of (2) for $n = 2$ and for even values of k. After reducing this problem to that of finding integral solutions of a certain class of quartic equations of the form

$$b^2 X^4 - dY^2 = 1, \tag{12}$$

they combined some recent results of COHN [14] and BENNETT and WALSH [6] on equations (12) with the powerful computational machinery related to quadratic number fields. Using their approach, they found all integral

solutions for $k \leq 70$ assuming the Extended Riemann Hypothesis, and for $k \leq 58$, unconditionally.

It is an even more difficult problem to find all solutions of (2) when the exponent n is not fixed. By means of deep tools BENNETT, GYŐRY and PINTÉR [5] proved completely Schäffer's conjecture for $k \leq 11$ (which includes all the values considered by Schäffer) and, most importantly, for arbitrary n.

Theorem I (BENNETT, GYŐRY and PINTÉR [5]). *For $1 \leq k \leq 11$ and (k, n) not in the set (3), equation (2) has only the trivial solution, unless $k = 2$, in which case there is the additional solution $(n, x, y) = (2, 24, 70)$.*

Our method of proof, which may, with a modicum of effort, be extended to higher values of k, combines a wide variety of techniques, classical and modern, in Diophantine analysis.

To illustrate the basic idea of our proof, assume that (2) has a non-trivial solution with $k \leq 11$ and (k, n) not in (3). For n a power of 2, it suffices to prove Theorem I for $k \in \{7, 11\}$ in view of the previous results. In this case, equation (2) was reduced to elliptic curves, whose integral points were provided by the computational package MAGMA.

For odd prime n, equation (2) was reduced to equations of the form (4), where A, B are relatively prime positive integers with $B > A + 1$ and

$$AB \in \Big\{11, 14, 22, 30, 42, 66, 2 \cdot 5^{n-1}, 6 \cdot 5^{n-1}, 22 \cdot 5^{n-1}, \\ 66 \cdot 5^{n-1}, 2 \cdot 5^{(n\pm 1)/2}, 2 \cdot 3^{(n\mp 1)/2} \cdot 5^{(n\pm 1)/2}\Big\}.$$

Using a sharp lower bound for linear forms in logarithms of two algebraic numbers, we proved that $n < 4000$ in each of the equations (4) involved. It remained to treat the equations (4) considered above with $n < 4000$ prime.

For small values of n, this is readily accomplished via known computational techniques. For $n \leq 19$, we resolved the corresponding equations via MAGMA. For values of n greater than 100 or so, this is well out of range of current techniques, based on Baker's method and lattice basis reduction. For $A > 1$, we used a local method for solving the corresponding equations (4). Finally, for $A = 1$, a new technique was applied for handling

such equations, based upon classical work on Fermat-type equations and the theory of Frey curves, Galois representations and modular forms.

Remark. We note that it is a rare situation where one can explicitly solve superelliptic equations of as high degree as we encountered in [5]. Furthermore, this was accomplished by solving certain high degree Thue equations, itself being a notoriously difficult problem.

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