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Some non-commutative products of distributions

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Abstract. Let g be a distribution and let $g_n = (g * \delta_n)(x)$, where $\delta_n(x)$ is a certain sequence converging to the Dirac delta-function. The product $f.g$ of two distributions f and g is defined to be the limit of the sequence $\{fg_n\}$, provided its limit h exists in the sense that

$$\lim_{n\to\infty} \langle f(x)g_n(x), \varphi(x)\rangle = \langle h(x), \varphi(x)\rangle$$

for all functions φ in $\mathcal{D}$. It is proved that

$$(\operatorname{sgn} x|x|^\lambda \ln^p |x|).(|x|^\mu \ln^q |x|) = \operatorname{sgn} x|x|^{\lambda+\mu} \ln^{p+q} |x|,$$

$$(|x|^\lambda \ln^p |x|).(\operatorname{sgn} x|x|^\mu \ln^q |x|) = \operatorname{sgn} x|x|^{\lambda+\mu} \ln^{p+q} |x|$$

for $-2 < \lambda + \mu \leq -1$ and $p, q = 0, 1, 2, \dots$

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Now let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

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Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

If now f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t)\rangle$$

for $n = 1, 2, \ldots$ It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [2].

Definition 1. Let f and g be distributions in $\mathcal{D}'$ for which on the interval (a, b), f is the k-th derivative of a locally summable function F in $L^p(a, b)$ and $g^{(k)}$ is a locally summable function in $L^q(a, b)$ with $1/p+ 1/q = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a, b) by

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [Fg^{(i)}]^{(k-i)}.$$

It follows easily that the products $(|x|^\lambda \ln^p |x|)(|x|^\mu \ln^q |x|)$ and exist by Definition 1 and

$$(|x|^\lambda \ln^p |x|)(|x|^\mu \ln^q |x|) = |x|^{\lambda+\mu} \ln^{p+q} |x|, \tag{1}$$

$$(\operatorname{sgn} x|x|^\lambda \ln^p |x|)(|x|^\mu \ln^q |x|) = \operatorname{sgn} x|x|^{\lambda+\mu} \ln^{p+q} |x|, \tag{2}$$

$$(|x|^\lambda \ln^p |x|)(\operatorname{sgn} x|x|^\mu \ln^q |x|) = \operatorname{sgn} x|x|^{\lambda+\mu} \ln^{p+q} |x|, \tag{3}$$

$$(\operatorname{sgn} x|x|^\lambda \ln^p |x|)(\operatorname{sgn} x|x|^\mu \ln^q |x|) = |x|^{\lambda+\mu} \ln^{p+q} |x| \tag{4}$$

for $\lambda + \mu > -1$ and $p, q = 0, 1, 2, \ldots$

The following definition for the non-commutative product of two distributions was given in [3] and generalizes Definition 1.

Definition 2. Let f and g be distributions in $\mathcal{D}'$ and let $g_n(x) = (g * \delta_n)(x)$. We say that the product $f.g$ of f and g exists and is equal to the distribution h on the interval (a, b) if

$$\lim_{n\to\infty} \langle f(x)g_n(x), \varphi(x)\rangle = \langle h(x), \varphi(x)\rangle$$

for all functions φ in $\mathcal{D}$ with support contained in the interval (a,b).

It was proved that if the product fg exists by Definition 1, then it exists by Definition 2 and $fg = f.g$.

The following theorem is easily proved.

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the products $f.g$ and $f.g'$ (or $f'.g$) exists. Then the product $f'.g$ (or $f.g'$) exists and*

$$(f.g)' = f'.g + f.g'. \tag{5}$$

The next theorem was proved in [4].

Theorem 2. *The product $(x^r \ln^p |x|).(x^{-r-1} \ln^q |x|)$ exists and*

$$(x^r \ln^p |x|).(x^{-r-1} \ln^q |x|) = x^{-1} \ln^{p+q} |x| \tag{6}$$

for $r = 0, \pm 1, \pm 2, \dots$ and $p, q = 0, 1, 2, \dots$

We now prove the following theorem which generalizes equations (2), (3) and (6).

Theorem 3. *The products $(\operatorname{sgn} x|x|^\lambda \ln^p |x|).(|x|^\mu \ln^q |x|)$ and $(|x|^\lambda \ln^p |x|).(\operatorname{sgn} x|x|^\mu \ln^q |x|)$ exist and*

$$(\operatorname{sgn} x|x|^\lambda \ln^p |x|).(|x|^\mu \ln^q |x|) = \operatorname{sgn} x|x|^{\lambda+\mu} \ln^{p+q} |x|, \tag{7}$$

$$(|x|^\lambda \ln^p |x|).(\operatorname{sgn} x|x|^\mu \ln^q |x|) = \operatorname{sgn} x|x|^{\lambda+\mu} \ln^{p+q} |x| \tag{8}$$

for $-2 < \lambda + \mu \le -1$ and $p, q = 0, 1, 2, \dots$

Proof. We first of all prove equation (7) when $\lambda > -1$. Putting

$$(|x|^\mu \ln^q |x|)_n = (|x|^\mu \ln^q |x|) * \delta_n(x),$$

we have

$$\int_{-a}^{a} (\operatorname{sgn} x|x|^\lambda \ln^p |x|)(|x|^\mu \ln^q |x|)_n \, dx = 0, \tag{9}$$

since the integrand is odd.

Further, if ψ is an arbitrary continuous function, we have

$$\int_{-a}^{a} (\operatorname{sgn} x|x|^\lambda \ln^p |x|)(|x|^\mu \ln^q |x|)_n x\psi(x) \, dx$$

$$= \int_{-a}^{a} |x|^{\lambda+1} \ln^p |x| (|x|^\mu \ln^q |x|)_n \psi(x)\, dx$$

and it follows that

$$\begin{aligned}
\lim_{n\to\infty} &\int_{-a}^{a} (\operatorname{sgn} x|x|^\lambda \ln^p |x|)(|x|^\mu \ln^q |x|)_n x\psi(x)\, dx \\
&= \int_{-a}^{a} |x|^{\lambda+\mu+1} \ln^{p+q} |x| \psi(x)\, dx \qquad (10) \\
&= \int_{-a}^{a} \operatorname{sgn} x|x|^{\lambda+\mu} \ln^{p+q} |x| x\psi(x)\, dx,
\end{aligned}$$

since on using equation (1), the sequence $\{|x|^{\lambda+1} \ln^p |x|(|x|^\mu \ln^q |x|)_n\}$ converges in the distributional sense to the locally summable function $|x|^{\lambda+\mu+1} \ln^{p+q} |x|$.

Now let φ be an arbitrary function in $\mathcal{D}$ and choose a so that $\operatorname{supp} \varphi \subset [-a, a]$. By the mean value theorem, we have

$$\varphi(x) = \varphi(0) + x\varphi'(\xi x),$$

where $0 < \xi < 1$. Then

$$\begin{aligned}
\langle (\operatorname{sgn} x|x|^\lambda \ln^p |x|)&(|x|^\mu \ln^q |x|)_n, \varphi(x)\rangle \\
&= \int_{-\infty}^{\infty} \operatorname{sgn} x|x|^\lambda \ln^p |x| (|x|^\mu \ln^q |x|)_n \varphi(x)\, dx \\
&= \varphi(0) \int_{-a}^{a} \operatorname{sgn} x|x|^\lambda \ln^p |x| (|x|^\mu \ln^q |x|)_n\, dx \\
&\quad + \int_{-a}^{a} \operatorname{sgn} x|x|^\lambda \ln^p |x| (|x|^\mu \ln^q |x|)_n x\varphi'(\xi x)\, dx.
\end{aligned}$$

Using equations (9) and (10), it follows that

$$\begin{aligned}
\lim_{n\to\infty} \langle (\operatorname{sgn} x|x|^\lambda \ln^p |x|)&(|x|^\mu \ln^q |x|)_n, \varphi(x)\rangle \\
&= \int_{-a}^{a} \operatorname{sgn} x|x|^{\lambda+\mu} \ln^{p+q} |x| x\varphi'(x)\, dx \\
&= \int_{-\infty}^{\infty} \operatorname{sgn} x|x|^{\lambda+\mu} \ln^{p+q} |x| [\varphi(x) - \varphi(0)]\, dx \\
&= \langle \operatorname{sgn} x|x|^{\lambda+\mu} \ln^{p+q} |x|, \varphi(x)\rangle
\end{aligned}$$

for arbitrary φ in $\mathcal{D}$, proving equation (7) for $\lambda > -1$, $-2 < \lambda + \mu \leq -1$ and $p, q = 0, 1, 2, \ldots$

Equation (8) follows similarly for $\lambda > -1$, $-2 < \lambda + \mu \leq -1$ and $p, q = 0, 1, 2, \ldots$

When $-2 < \lambda, \lambda + \mu \leq -1$, we have from equation (1)

$$|x|^{\lambda+1}(|x|^{\mu} \ln^q |x|) = |x|^{\lambda+\mu+1} \ln^q |x| \tag{11}$$

for $q = 0, 1, 2 \ldots$ Differentiating equation (11) we get

$$\begin{aligned}(\lambda + 1)&(\operatorname{sgn} x|x|^{\lambda}).(|x|^{\mu} \ln^q |x|)\\ &+ \mu|x|^{\lambda+1}.(\operatorname{sgn} x|x|^{\mu-1} \ln^q |x|) + q|x|^{\lambda+1}.(\operatorname{sgn} x|x|^{\mu-1} \ln^{q-1} |x|)\\ = (\lambda &+ \mu + 1) \operatorname{sgn} x|x|^{\lambda+\mu} \ln^q |x| + q \operatorname{sgn} x|x|^{\lambda+\mu} \ln^{q-1} |x|.\end{aligned} \tag{12}$$

Using Theorem 1 and equation (7), which has been proved for $\lambda > -1$, it follows that

$$(\operatorname{sgn} x|x|^{\lambda}).(|x|^{\mu} \ln^q |x|) = \operatorname{sgn} x|x|^{\lambda+\mu} \ln^q |x|. \tag{13}$$

Equation (7) therefore holds for $-2 < \lambda$, $\lambda + \mu \leq -1$, $p = 0$ and $q = 0, 1, 2, \ldots$

It follows similarly that

$$|x|^{\lambda}.(\operatorname{sgn} x|x|^{\mu} \ln^q |x|) = \operatorname{sgn} x|x|^{\lambda+\mu} \ln^q |x|. \tag{14}$$

Equation (8) therefore holds for $-2 < \lambda$, $\lambda + \mu \leq -1$, $p = 0$ and $q = 0, 1, 2, \ldots$

Now suppose that equations (7) and (8) hold for some k with $-k < \lambda$, $-2 < \lambda + \mu \leq -1$ and $p, q = 0, 1, 2, \ldots$ This is certainly true when $k = 1$. Also suppose that equations (7) and (8) hold for some p with $-k - 1 \leq \lambda$, $-2 < \lambda + \mu \leq -1$ and $q = 0, 1, 2, \ldots$ This is also true when $k = 1$ and $p = 0$. Then with $-k - 1 < \lambda$ and $-2 \leq \lambda + \mu < -1$, it follows from equation (1) that

$$(|x|^{\lambda+1} \ln^{p+1} |x|)(|x|^{\mu} \ln^q |x|) = |x|^{\lambda+\mu+1} \ln^{p+q+1} |x|. \tag{15}$$

Differentiating equation (15), we get

$$\begin{aligned}
&(\lambda+1)(\operatorname{sgn} x|x|^{\lambda}\ln^{p+1}|x|).(|x|^{\mu}\ln^{q}|x|)\\
&\qquad+(p+1)(\operatorname{sgn} x|x|^{\lambda}\ln^{p}|x|).(|x|^{\mu}\ln^{q}|x|)\\
&\qquad+\mu(|x|^{\lambda+1}\ln^{p+1}|x|).(\operatorname{sgn} x|x|^{\mu-1}\ln^{q}|x|)\\
&\qquad+q(|x|^{\lambda+1}\ln^{p+1}|x|).(\operatorname{sgn} x|x|^{\mu-1}\ln^{q-1}|x|)\\
&\quad=(\lambda+\mu+1)\operatorname{sgn} x|x|^{\lambda+\mu}\ln^{p+q+1}|x|+(p+q+1)\operatorname{sgn} x|x|^{\lambda+\mu}\ln^{p+q}|x|.
\end{aligned}$$

Using our assumptions and Theorem 1, it follows that

$$(\operatorname{sgn} x|x|^{\lambda}\ln^{p+1}|x|).(|x|^{\mu}\ln^{q}|x|)=\operatorname{sgn} x|x|^{\lambda+\mu}\ln^{p+q}|x|$$

giving equation (7) for $p+1$ and $-k-1<\lambda$.

Similarly, differentiation of the equation

$$(\operatorname{sgn} x|x|^{\lambda+1}\ln^{p+1}|x|)(\operatorname{sgn} x|x|^{\mu}\ln^{q}|x|)=|x|^{\lambda+\mu+1}\ln^{p+q+1}|x|,$$

using our assumptions and Theorem 1, it follows that

$$(|x|^{\lambda}\ln^{p+1}|x|).(\operatorname{sgn} x|x|^{\mu}\ln^{q}|x|)=\operatorname{sgn} x|x|^{\lambda+\mu}\ln^{p+q}|x|$$

giving equation (8) for $p+1$ and $-k-1<\lambda$.

Equations (7) and (8) now follow by induction for all λ, μ, with $-2<\lambda+\mu\leq 1$ and $p,q=0,1,2,\ldots$, completing the proof of the theorem. $\square$

We finally consider what happens if $\lambda+\mu\leq 2$. With $\lambda>-1$, the sequence $\{(|x|^{\mu}\ln^{q}|x|)_n\}$ will converge to the distribution $|x|^{\mu}\ln^{q}|x|$ and the integral

$$\int_{-n}^{n}\operatorname{sgn} x|x|^{\lambda+\mu}\ln|x|^{p+q}x\psi(x)\,dx$$

in equation (10) will in general be divergent. This means that

$$\lim_{n\to\infty}\langle(\operatorname{sgn} x|x|^{\lambda}\ln^{p}|x|)(|x|^{\mu}\ln^{q}|x|)_n,\varphi(x)\rangle$$

cannot exist for all functions φ. The product $(\operatorname{sgn} x|x|^{\lambda}\ln^{p}|x|).(|x|^{\mu}\ln^{q}|x|)$ will therefore not exist in this case.

Similarly, the product $(|x|^{\lambda}\ln^{p}|x|).(\operatorname{sgn} x|x|^{\mu}\ln^{q}|x|)$ will not exist if $\lambda+\mu<-2$.

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