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On some sufficient conditions of supersolvability of finite groups

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Abstract. A subgroup H is said to be c-supplemented in a finite group G if there exists a subgroup K of G such that $HK = G$ and $H \cap K$ is contained in $\mathrm{Core}_G(H)$. We determine the structure of a finite group G with the minimal subgroups of the generalized Fitting subgroup of some normal subgroups of G c-supplemented in G, generalizing some known results.

1. Introduction

All groups considered in this paper will be finite. We use $M < \cdot G$ to indicate that M is a maximal subgroup of G.

We say, following [5], a subgroup H of a group G is c-supplemented (in G) if there exists a subgroup K of G such that $HK = G$ and $H \cap K \leq H_G = \mathrm{Core}_G(H)$, and K is called a c-supplement of H in G.

Recall that a subgroup H of G is said to be c-normal (in G) if there exists a normal subgroup N of G such that $HN = G$ and $H \cap N \leq H_G$ ([4]). A subgroup H is said to be complemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. One can easily see that c-supplementation is a generalization of c-normality and complementation,

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that is, to remove the normal supplementation assumption in c-normality and to remove the trivial intersection assumption in complementation. There are examples to show that c-supplementation does not imply c-normality or complementation ([5]).

Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a formation provided that (i) if $G \in \mathcal{F}$ and $H \lhd G$, then $G/H \in \mathcal{F}$, and (ii) if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for normal subgroups M, N of G. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$ (see [1, Ch VI]). Throughout this paper $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation.

Let $\mathcal{P}$ be the set of prime numbers. A formation function is a function f defined on $\mathcal{P}$ such that $f(p)$ is a, possibly empty, formation. A chief factor H/K of a group G is f-central in G if $G/C_G(H/K) \in f(p)$ for all primes p dividing $|H/K|$. $\mathcal{F}$ is local if and only if there exists a formation function f such that $\mathcal{F}$ is the class of all groups with f-central factors. We write $\mathcal{F} = LF(f)$ and say that f is a local definition of $\mathcal{F}$. The Theorem of Gaschutz–Lubeseder–Schmid [1, IV 4.6] states that the non-empty saturated formations are the local ones. A chief factor H/K of a group G is said to be $\mathcal{F}$-central in G, $\mathcal{F}$ a saturated formation and f an integrated and local definition of $\mathcal{F}$, if H/K is f-central in G; H/K is $\mathcal{F}$-eccentric otherwise. A maximal subgroup M of G is called $\mathcal{F}$-normal in G if $G/M_G \in \mathcal{F}$ and $\mathcal{F}$-abnormal otherwise. M is said to be $\mathcal{F}$-critical in G if $Soc(G/M_G)$ is the unique minimal normal subgroup of G/M_G, M is $\mathcal{F}$-abnormal in G, and $G = MF'(G)$, where $F'(G)/\Phi(G) = \mathrm{Soc}(G/\Phi(G))$. By [13, Theorem 3.5], if G does not belong to $\mathcal{F}$, then G has an $\mathcal{F}$-critical maximal subgroup.

For a formation $\mathcal{F}$, each group G has a smallest normal subgroup N such that G/N is in $\mathcal{F}$. This uniquely determined normal subgroup of G is called the $\mathcal{F}$-residual subgroup of G and is denoted by $G^{\mathcal{F}}$.

Definition. Let p be a prime and G be a group. We define:

$$\mathcal{P}_p(G) = \{x \mid x \in G,\ |x| = p\},$$

$$\mathcal{P}(G) = \bigcup_{p \in \pi(G)} \mathcal{P}_p(G),$$

Let x be an element of G. We say that x is c-supplemented in G if $\langle x \rangle$ is c-supplemented in G.

2. Preliminary results

In this section, we give some results that are needed in this paper.

Lemma 2.1 (5, Lemma 2.1)**.** *Let G be a group. Then*

(1) *If H is c-supplemented in G, $H \leq M \leq G$, then H is c-supplemented in M;*

(2) *Let $N \triangleleft G$ and $N \leq H$. Then H is C-supplemented in G if and only if H/N is c-supplemented in G/N;*

(3) *Let π be a set of primes. Let N be a normal π'-subgroup and let H be a π-subgroup of G. If H is c-supplemented in G, then HN/N is c-supplemented in G/N.*

(4) *Let $H \leq G$ and $L \leq \Phi(H)$. If L is c-supplemented in G, then $L \triangleleft G$ and $L \leq \Phi(G)$.*

Let G be a group. The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. $F^*(G)$ is an important subgroup of G and it is a natural generalization of $F(G)$. The definition and important properties can be found in [6, X 13]. We would like to give the following basic facts which we will use in our proof.

Lemma 2.2. *Let G be a group and M a subgroup of G.*

(1) *If M is normal in G, then $F^*(M) \leq F^*(G)$;*

(2) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = \mathrm{soc}(F(G)C_G(F(G))/F(G))$;*

(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is soluble, then $F^*(G) = F(G)$.*

(4) *$C_G(F^*(G)) \leq F(G)$;*

(5) *Suppose K is a subgroup of G contained in $Z(G)$, then $F^*(G/K) = F^*(G)/K$.*

Proof. (1)–(4) can be found in [6, X 13].

(5) Denote $F^*(G/K) = L/K$. Consider a chief series of G of the form

$$G = G_0 > \cdots > G_{m-1} > G_m = K > G_{m+1} > \cdots > G_n = 1.$$

By the definition of the generalized Fitting subgroup, $\forall x \in L$, $\overline{x} = xK$ induces an inner automorphism on the chief factor $\overline{G_{i-1}}/\overline{G_i} = (G_{i-1}/K)/(G_i/K)$ of G/K, for $i = 1, 2, \ldots, m$, thus x induces an inner automorphism on the chief factor G_{i-1}/G_i of G, for $i = 1, 2, \ldots, m$. Since $K \leq Z(G)$, the automorphism induced by x on the chief factor G_{i-1}/G_i of G is identity, for $i = m+1, \ldots, n$, so x induces an inner automorphism on the chief factor G_{i-1}/G_i of G, for $i = 1, 2, \ldots, n$. Hence by [6, X, Lemma 13.1], x induces an inner automorphism on any chief factor of G. Thus $x \in F^*(G)$, i.e., $L \leq F^*(G)$. Obviously $F^*(G) \leq L$, hence $F^*(G/K) = F^*(G)/K$. □

Lemma 2.3. *Let G be a group. Assume that N is a normal subgroup of G ($N \neq 1$) and $N \cap \Phi(G) = 1$. Then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in $F(N)$. In particular, if $\Phi(G) = 1$, then $F(G)$ the direct product of minimal normal subgroups of G which are contained in $F(G)$.*

Proof. Refer to [1] Chapter 3. □

Next we give two properties of Fitting subgroup.

Lemma 2.4. *Suppose N, L are two normal subgroups of a group G.*

(1) *If L is nilpotent, then $F(NL) = F(N)L$.*

(2) *If $L \leq \Phi(G)$, then $F(NL/L) = F(N)L/L$.*

Proof. (1) By the hypothesis, it is easy to see that $F(N)L$ is a nilpotent normal subgroup of NL, so $F(N)L \leq F(NL)$. On the other hand, $F(NL) = F(NL) \cap NL = (F(NL) \cap N)L$, then $F(NL) \cap N$ is a nilpotent normal subgroup of N, thus is contained in $F(N)$. So $F(NL) \leq F(N)L$, the equality holds.

(2) Denote $F(NL/L){=}K/L$. Since $F(NL)/L$ is nilpotent, $F(NL)/L{\leq}$ K/L. On the other hand, K/L is nilpotent, $L \leq \Phi(G)$, [1, p. 220, Satz 3.5] implies that K is nilpotent, thus $K \leq F(NL)$, so $F(NL/L) = F(NL)/L = F(N)L/L$ by (1). □

Lemma 2.5. *Let G be a group with a normal subgroup N such that G/N is supersolvable. Suppose that $\Phi(G) = 1$ and $F^*(G) = F(G)$. If for any maximal subgroup M of G, either $F(N) \leq M$ or $F(N) \cap M <\cdot F(N)$, then G is supersolvable.*

Proof. If $N = 1$, obviously the theorem holds. So assume that $N \neq 1$, then Lemma 2.2 implies that $F^*(N) \leq F^*(G) = F(G)$, so $F^*(N) = F(N) \neq 1$. By Lemma 2.3 and the hypothesis, we have $F(N) = L_1 \times L_2 \times$

$\cdots \times L_s$, where L_i is a minimal normal subgroup of G contained in $F(N)$. Again by Lemma 2.3 we can write that $F(G) = F(N) \times H_1 \times H_2 \times \cdots \times H_r$, where H_i is a minimal normal subgroup of G contained in $F(G)$ but not contained in $F(N)$ and $F(G)$ is abelian. $\forall i$, $L_i \nleq \Phi(G)$ as $\Phi(G) = 1$, so there exists a maximal subgroup M_i of G such that $L_i \nleq M_i$. It follows that $G = L_i M_i$. Since L_i is abelian, $L_i \cap M_i \lhd L_i M_i = G$. The minimality of L_i implies that $L_i \cap M_i = 1$. Since $F(N) = F(N) \cap L_i M_i = L_i(F(N) \cap M_i)$, $F(N) \cap M_i < \cdot F(N)$ by hypotheses, then the nilpotence of $F(N)$ implies that $[F(N) : F(N) \cap M_i]$ is a prime.

Since $L_i \leq F(N), G = M_i F(N)$, thus $|L_i| = [G : M_i] = [F(N) : F(N) \cap M_i]$ is a prime. So $G/C_G(L_i)$ is abelian, then $G' \leq C_G(L_i)$, $\forall i$.

Since $H_j N/N$ is the minimal normal subgroup of the supersolvable group G/N, we have that $H_j \cong H_j N/N$ has also prime order. So $G/C_G(H_j)$ is abelian, then $G' \leq C_G(H_j)$. Therefore

$$G' \leq \left(\bigcap_{j=1}^{r} C_G(H_j) \right) \cap \left(\bigcap_{i=1}^{s} C_G(L_i) \right) = C_G(F(G)).$$

By Lemma 2.2(4), we have $C_G(F^*(G)) \leq F(G)$, thus $G' \leq F(G)$ by hypotheses.

For any maximal subgroup M of G, by hypotheses we have either $F(N) \leq M$ or $F(N) \cap M < \cdot F(N)$. If $F(N) \cap M < \cdot F(N)$, then $G = MF(N)$, thus $[G : M] = [F(N) : F(N) \cap M]$. The nilpotence of $F(N)$ implies $[F(N) : F(N) \cap M]$ is a prime, so $[G : M]$ is also a prime. If $F(N) \leq M$ but $F(G) \nleq M$, then there exists a minimal normal subgroup H_i such that $H_i \nleq M$, so $G = H_i M$. Thus $[G : M] = |H_i|$ is a prime. If $F(G) \leq M$, then $M \geq G'$, thus $M \lhd G$ and G/M has no trivial proper subgroup, so $|G : M|$ is also a prime. Therefore G is supersolvable by the well-known Huppert Theorem. $\square$

Lemma 2.6. *Let G be a finite group. Suppose $G = PM$, where P is a normal p-subgroup of G and M is a maximal subgroup of G. Then $P \cap M \lhd G$.*

Proof. First we have $N_G(P \cap M) \geq M$ by the normality of P. Obviously $P \nleq M$, so $P \cap M < P$. Since P is a p-group, P has a subgroup P_1 such that $P \cap M$ a proper normal subgroup of P_1, so $N_G(P \cap M) \geq$

$\langle M, P_1\rangle = G$ as M is a maximal subgroup of G and $P_1 \nleq M$. Thus $P \cap M \lhd G$. □

The following Theorem is a generalization of [14, Theorem 3.7].

Theorem 2.7. *Let G be a group with a normal subgroup N such that G/N is supersolvable. If every element of prime order of N is c-supplemented in G and N is quaternion-free, then G is supersolvable.*

Proof. Assume that the result is false and let G be a counterexample of minimal order.

(1) Every proper subgroups of G is supersolvable. Furthermore

 (a) There exists a normal Sylow p-subgroup of G such that $G = PR$ and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$;

 (b) If $p > 2$, then the exponent of P is p. When $p = 2$, the exponent of P is 2 or 4, especially in this case, every proper subgroup of G is nilpotent and $\Phi(P) \leq Z(G)$.

 Denote $K = G^{\mathcal{U}}$. Let M be a maximal subgroup of G. It is clear that $M/M \cap K$ is supersolvable and hence $M^{\mathcal{U}} \leq M \cap K$. By Lemma 2.1, every element of $\mathcal{P}(M^{\mathcal{U}})$ is c-supplemented in M, obviously, $M^{\mathcal{U}}$ is quaternion-free, so that M satisfies the hypotheses of G. The minimal choice of G yields that M is supersolvable. This holds for every maximal subgroup M of G. Hence we have that G is not supersolvable but every proper subgroup of G is supersolvable. [1, VI, §, Ex. 1] implies (1)(a) and (1)(b).

(2) $K = P$. Since G/P is supersolvable, we have that $K \leq P$. Then $K\Phi(P)/\Phi(P)$ is a normal subgroup of $G/\Phi(P)$ contained in $P/\Phi(P)$. Since $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, we have that either $K\Phi(P) = P$ or $K \leq \Phi(P)$. If $K < \Phi(P)$, then K is actually contained in $\Phi(G)$ and $G/\Phi(G)$ is supersolvable. Hence G is supersolvable, a contradiction, and so we have that $P = K$.

(3) $\Phi(P) \neq 1$. Otherwise P is elementary abelian and hence, by (1), every element of P lies in $\mathcal{P}(P)$. Our hypotheses claims that every element of P is c-supplemented in G. Let $1 \neq x \in P$. Then there exists $K \leq G$ such that $\langle x > K = G$ and $\langle x\rangle \cap K \leq \langle x\rangle_G$. Then $P = \langle x\rangle(P \cap K)$. Since P is abelian, we have that $P \cap K \lhd G$. By (1), P is a minimal

normal subgroup of G when $\Phi(P) = 1$. Therefore $P \cap K = 1$ or $P \leq M$. In both case, we have that $\langle x \rangle = P$ and therefore G is supersolvable, a contradiction.

(4) $p = 2$. Assume that $p > 2$. Then by (1)(b) every element of P is c-supplemented in G. Moreover, By Lemma 2.1(4), $\Phi(P)$ is contained in $\Phi(G)$ and $\Phi(P) \lhd G$. Next we see that the hypotheses of Theorem holds in $G/\Phi(P)$. Let $x \in P - \Phi(P)$. By hypotheses there exists a subgroup M of G such that $G = \langle x \rangle M$ and $\langle x \rangle \cap M \leq \langle x \rangle_G$. If $\langle x \rangle = \langle x \rangle_G$, then (1) implies that $P = \langle x \rangle \Phi(P) = \langle x \rangle$. Then G is supersolvable, a contradiction. And so we have that $\langle x \rangle \cap M = 1$. Hence M is a maximal subgroup of G because $o(x) = p$. This implies that $G/\Phi(P) = \langle x \rangle \Phi(P)/\Phi(P) \cdot M/\Phi(P)$ and $(\langle x \rangle \Phi(P)/\Phi(P)) \cap (M/\Phi(P)) = \overline{1}$ and $\langle x \rangle \Phi(P)/\Phi(P)$ is c-supplemented in $G/\Phi(P)$. Since $(G/\Phi(P))^{\mathcal{U}} \leq K\Phi(P)/\Phi(P)$, we have that $(G/\Phi(P))^{\mathcal{U}}$ is quaternion-free. The minimal choice of G (notice that $\Phi(P) \neq 1$ by (3)) implies that $G/\Phi(P)$ is supersolvable. Since $\Phi(P) \leq \Phi(G)$, we have that $G/\Phi(G)$ is supersolvable and so is G, a contradiction.

(5) Final contradiction.

If $\exp(P) = 2$, then P is elementary abelian, contrary to (3), so $\exp(P) = 4$. Thus the order of every element of $P - \Phi(P)$ is 4. Pick $a, b \in P - \Phi(P)$ such that the order of $c = [a, b]$ is 2. Denote $R = \langle a, b \rangle / \langle a^2 b^2, ca^2 \rangle$, then $\overline{a^2} = \overline{b^2} = [\overline{a}, \overline{b}] = \overline{c} \neq 1$, $o(\overline{a}) = o(\overline{b}) = 4$, $o(\overline{c}) = 2$. So R is quaternion and a section of P, it is contrary to the hypothesis that K is quaternion-free. These complete our proof. □

Lemma 2.8. *Suppose N is a normal subgroup of a group G. If every element of $\mathcal{P}(F(N))$ is c-supplemented in G and $F(N)$ is quaternion-free, then for any maximal subgroup M of G, there holds either $F(N) \leq M$ or $F(N) \cap M < \cdot F(N)$.*

Proof. For any maximal subgroup M of G, we first indicate that as in the proof of Lemma 2.3 when $F(N) \nleq M$, $F(N) \cap M < \cdot F(N)$ is equivalent to $[G : M]$ is a prime.

Suppose that $F(N) \nleq M$, then there exists a prime p such that $O_p(N) \nleq M$, therefore $G = O_p(N)M$. We go through our discussion with two cases.

(i) p is an odd prime.

Assume that there exists an element of order p of $O_p(N)$, x_1 say, such that x_1 is not normal in G. Since x_1 is c-supplemented in G by the hypotheses, there exists a subgroup of G, M_1 say, such that $G = \langle x_1\rangle M_1$ and $\langle x_1\rangle \cap M_1 = 1$. Then we have that $O_p(N) = \langle x_1\rangle(O_p(N) \cap M_1)$, where $O_p(N) \cap M_1$ is normal in G by Lemma 2.6, so we can write

$$G = \langle x_1\rangle[(O_p(N) \cap M_1)M],$$

where $[(O_p(N) \cap M_1)M$ is a subgroup of G containing the maximal subgroup M. If $(O_p(N) \cap M_1)M = M$, then $G = \langle x_1\rangle M$, so $[G : M]$ is a prime. Thus the lemma holds. So we assume $(O_p(N) \cap M_1)M = G$, denote $P_1 = O_p(N) \cap M_1$, thus $G = P_1M$, P_1 is normal in G by Lemma 2.6 and $x_1 \notin P_1$.

If there exists an element of order p of P_1, x_2 say, such that x_2 is not normal in G, then there exists a subgroup of G, M_2 say, such that $G = \langle x_2\rangle M_2$ and $\langle x_2\rangle \cap M_2 = 1$ as x_2 is c-supplemented in G by the hypotheses. Then $P_1 = \langle x_2\rangle(P_1 \cap M_2)$, $P_2 \cap M_2$ is normal in G by Lemma 2.6, so we can write $G = \langle x_2\rangle[(P_1 \cap M_2)M]$, where $[(P_1 \cap M_2)M$ is a subgroup of G containing the maximal subgroup M. If $(P_1 \cap M_2)M = M$, then $G = \langle x_2\rangle M$. It follows that $[G : M]$ is a prime. Thus the lemma holds. So assume $(P_1 \cap M_2)M = G$, denote $P_2 = P_1 \cap M_2$, thus $G = P_2M$, P_2 is a normal subgroup of G contained in $O_p(N)$ by Lemma 2.6 and $x_1, x_2 \notin P_2$.

Repeating this method, at end we have that either $[G : M]$ is a prime or there exists a normal p-subgroup of G contained in $O_p(N)$, P say, such that $G = PM$ and every element of order p of P is normal in G. In particular, P is a PN-group ([11]).

If P has an element x of order p such that $x \notin M$, then $G = \langle x\rangle M$ as M is a maximal subgroup of G and x is normal in G. Thus $[G : M]$ is a prime, the lemma holds. So we can assume that $\Omega_1(P) \leq M$. Consider the factor group $G/\Omega_1(P) = P/\Omega_1(P) \cdot M/\Omega_1(P)$. $\forall x\Omega_1(P) \in \Omega_1(P/\Omega_1(P)) = \Omega_2(P)/\Omega_1(P)$, then $x^p \in \Omega_1(P)$, thus $\langle x^p\rangle \lhd G$ by the properties of P. So $\forall g \in G$, there exists an integer i such that $(x^p)^g = (x^p)^i = (x^i)^p = (x^g)^p$. Since x^g, x^i lie in $\Omega_2(P)$ which is a PN-group with exponent of p^2, now [11, Th 1(iii)] implies that $(x^g x^{-i})^p = 1$, so $x^g x^{-i} \in \Omega_1(P)$, therefore $\langle x\Omega_1(P)\rangle \lhd G/\Omega_1(P)$. If $\langle x\Omega_1(P)\rangle \nleq M/\Omega_1(P)$, then $G/\Omega_1(P) = \langle x\Omega_1(P)\rangle M/\Omega_1(P)$, it follows that $[G : M] = [G/\Omega_1(P) : M/\Omega_1(P)]$ is a prime, the lemma holds. Thus we can assume that $\Omega_1(P/\Omega_1(P)) \leq M/\Omega_1(P)$. This leads that $\Omega_2(P) \leq M$.

Again consider the factor group $G/\Omega_2(P) = P/\Omega_2(P) \cdot M/\Omega_2(P)$. Using the same method as in above, we get that either $[G : M]$ is a prime or $\Omega_3(P) \leq M$, ..., repeating the method above, at end we have that $[G : M]$ is a prime as P is finite. So the lemma holds in this case.

(ii) $p = 2$.

Suppose that M_2 is a Sylow 2-subgroup of M, then $O_2(N)M_2$ is a Sylow 2-subgroup of G as $G = O_2(N)M$. Suppose that H is a maximal subgroup of $O_2(N)M_2$ containing M_2, then $H = H \cap (O_2(N)M_2) = (H \cap O_2(N))M_2$.

$\forall q \neq 2$, $\forall M_q \in S_q(M)$, since $O_2(N)M_q/O_2(N)$ is supersolvable, every element of $\mathcal{P}(O_2(N))$ is c-supplemented in G, thus is c-supplemented in $O_2(N)M_q$. Since $O_2(N) \subseteq F(N)$, $O_2(N)$ is quaternion-free by hypotheses, now Lemma 2.7 implies that $O_2(N)M_q$ is supersolvable, thus $O_2(N)M_q = O_2(N) \times M_q$. So $O^2(M) \leq C_G(O_2(N))$, therefore $(H \cap O_2(N))M_{2'}$ is a group, so is $(H \cap O_2(N))M$.

Since $M_2 \cap O_2(N) \leq M \cap (H \cap O_2(N)) \leq M \cap O_2(N) \leq O_2(N) \cap M_2$, we have that $M \cap O_2(N) = M_2 \cap O_2(N)$. Thus $(H \cap O_2(N)) \cap M = H \cap (O_2(N) \cap M) = H \cap (O_2(N) \cap M_2) = O_2(N) \cap M_2 = O_2(N) \cap M$. Therefore

$$|(H \cap O_2(N))M| = \frac{|H \cap O_2(G)| \cdot |M|}{|(H \cap O_2(N)) \cap M|} < \frac{|O_2(N)| \cdot |M|}{|M \cap O_2(N)|} = |G|,$$

so $(H \cap O_2(N))M < G$, this implies $(H \cap O_2(N))M = M$ as M is a maximal subgroup of G, so $H \cap O_2(N) \leq M$, $H \cap O_2(N) \leq M \cap O_2(N)$. This time,

$$[G : M] = \frac{|O_2(N)|}{|O_2(N) \cap M|} \leq \frac{|O_2(N)|}{|O_2(N) \cap H|} = |HO_2(N)/H| \leq 2,$$

as H is a maximal subgroup of Sylow 2-subgroup of G. Therefore $[G : M] = 2$ is a prime.

These complete the proof of the lemma. $\square$

3. Main results

Theorem 3.1. *Suppose G is a solvable group with a normal subgroup N such that G/N is supersolvable. If every element of $\mathcal{P}(F(N))$ is c-supplemented in G and $F(N)$ is quaternion-free, then G is supersolvable.*

Proof. Since G is solvable, $F^*(G)=F(G)$ by Lemma 2.2. If $\Phi(G)=1$, then By Lemma 2.5 and Lemma 2.8, G is supersolvable. Assume that $\Phi(G) \neq 1$. We consider the factor group $\overline{G} = G/\Phi(G)$. Since $\Phi(\overline{G}) = 1$, $F^*(\overline{G}) = F(\overline{G})$ as $\overline{G}$ is solvable. For any maximal subgroup $M/\Phi(G)$ of $\overline{G}$, obviously M is also a maximal subgroup of G, by Lemma 2.8 we have either $M \geq F(N)$ or $M \cap F(N)$ is a maximal subgroup of $F(N)$. If $F(N) \leq M$, then $F(\overline{N}) = F(N\Phi(G)/\Phi(G)) = F(N\Phi(G))/\Phi(G) = F(N)\Phi(G)/\Phi(G) \leq M/\Phi(G)$ by Lemma 2.4. If $F(N) \cap M < \cdot F(N)$, then $[F(N) : F(N) \cap M]$ is a prime. Since $[F(\overline{N}) : F(\overline{N}) \cap \overline{M}] = [F(N)\Phi(G) : F(N)\Phi(G) \cap M] = [F(N)\Phi(G) : (F(N) \cap M)\Phi(G)] = [F(N) : F(N) \cap M]$ is a prime, $F(\overline{N}) \cap \overline{M}$ is a maximal subgroup of $F(\overline{N})$. Therefore $\overline{G}$ satisfies the hypotheses of Lemma 2.5, thus $\overline{G}$ is supersolvable, so G is supersolvable. □

We want to delete the hypotheses of solvability of G in Theorem 3.1, but we should replace the Fitting subgroup $F(N)$ with the generalized Fitting subgroup $F^*(N)$.

Theorem 3.2. *Suppose G is a group with a normal subgroup N such that G/N is supersolvable. $F^*(N)$ is the generalized Fitting subgroup of N. If every element of prime order of $F^*(N)$ is c-supplemented in G and $F^*(N)$ is quaternion-free, then G is supersolvable.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. Then we have:

(1) Every proper normal subgroup of G is supersolvable.

If H is a proper normal subgroup of G, then we have that $H/H \cap N \cong HN/N$, thus $H/H \cap N$ is supersolvable. Since $F^*(H \cap N) \leq F^*(N)$ by Lemma 2.2, we have that every element of $\mathcal{P}^*(F(H \cap N))$ is c-supplemented in G, thus is c-supplemented in H by Lemma 2.1. So H satisfies the hypotheses of the theorem. The minimal choice of G implies that H is supersolvable.

Since $F(N) \leq F^*(N)$, Lemma 2.8 and Theorem 3.1 imply that:

(2) $\forall M < \cdot G$, there holds either $F(N) \leq M$ or $F(N) \cap M < \cdot F(N)$. Furthermore, G is not solvable.

(3) $G = G' = O^q(G) = N$, $\forall q \in \pi(G)$.

If $G' < G$, then G' is supersolvable by (1). Since G/G' is abelian, we have that G is solvable, contrary to (2).

If $O^q(G) < G$, then $O^q(G)$ is supersolvable by (1), thus G is solvable as $G/O^q(G)$ is a q-group, contrary to (2).

Again If $N < G$, then N is supersolvable by (1). Since G/N is supersolvable, we have that G is solvable, contrary to (2) too.

(4) $F^*(G) = F^*(N) = F(G) = F(N) < G$.

If $F^*(G) = G$, then every element of prime order or order 4 of G is c-supplemented in G. Lemma 2.7 implies that G is supersolvable, a contradiction. So $F^*(G) < G$, $F^*(G)$ is supersolvable by (1). In particular, $F^*(G) = F^*(F^*(G)) = F(F^*(G)) \leq F(G)$. So we have that $F^*(G) = F(G)$.

(5) $\Phi(G) = 1$.

Assume that $\Phi(G) \neq 1$. Suppose Q is any Sylow q-subgroup of $\Phi(G)$, for every element x of $\mathcal{P}^*(Q)$, x is c-supplemented in G by hypotheses, thus there exists a subgroup L of G such that $G = \langle x \rangle L$ and $\langle x \rangle \cap L \leq \langle x \rangle_G$. Since $x \in \Phi(G)$, we have that $G = L$ and $\langle x \rangle$ is normal in G. Thus $G/C_G(\langle x \rangle)$ is abelian, this implies that $G = G' \leq C_G(\langle x \rangle)$, i.e., G centralizes every element of $\mathcal{P}^*(Q)$. By HUPPERT's result ([1, Satz IV 5.12]), we get every q'-element of G centralizes Q. Thus $G/C_G(Q)$ is a q-group. Then $G = O^q(G) \leq C_G(Q)$, so $Q \leq Z(G)$, therefore $\Phi(G) \leq Z(G)$.

Consider the factor group $\overline{G} = G/\Phi(G)$. Since $\Phi(\overline{G}) = 1$, and by Lemma 2(5), $F^*(\overline{G}) = F^*(G)/\Phi(G) = F(G)/\Phi(G) = F(G/\Phi(G)) = F(\overline{G})$. For any maximal subgroup $M/\Phi(G)$ of $\overline{G}$, obviously M is also a maximal subgroup of G, so it holds that $F(N) \leq M$ or $F(N) \cap M < \cdot F(N)$ by (2). If $F(N) \leq M$, then $F(\overline{N}) = F(N\Phi(G)/\Phi(G)) = F(N\Phi(G))/\Phi(G) = F(N)\Phi(G)/\Phi(G) \leq M/\Phi(G)$. If $F(N) \cap M < \cdot F(N)$, then $[F(N) : F(N) \cap M]$ is a prime. Since $[F(\overline{N}) : F(\overline{N}) \cap \overline{M}] = [F(N)\Phi(G) : F(N)\Phi(G) \cap M] = [F(N)\Phi(G) : (F(N) \cap M)\Phi(G)] = [F(N) : F(N) \cap M]$ is a prime, $F(\overline{N}) \cap \overline{M}$ is a maximal subgroup of $F(\overline{N})$. We have either $M/\Phi(G) \geq F(G)/\Phi(G) = F(G/\Phi(G))$ or $M/\Phi(G) \cap F(G/\Phi(G)) = M \cap F(G)/\Phi(G)$ is a maximal subgroup of $F(G/\Phi(G))$. So $\overline{G}$ satisfies the hypotheses of Lemma 2.5, thus $\overline{G}$ is supersolvable, therefore G is supersolvable, a contradiction.

(6) The final contradiction.

By (1), (2), (4) and (5), G satisfies the hypotheses of Lemma 2.5, G is supersolvable. The final contradiction.

These complete the proof of our theorem. □

Corollary 3.3. *Let G be a group. If every element of prime order of $F^*(G)$ is c-supplemented in G and $F^*(G)$ is quaternion-free, then G is supersolvable.*

4. Generalization to formations

The following is a special case of [12, Theorem 2]:

Lemma 4.1. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Assume G is a group with a normal subgroup N of prime order such that $G/N \in \mathcal{F}$, then $G \in \mathcal{F}$.*

First we generalize Theorem 2.7 with formation, it is also a generalization of [14, Theorem 3.7] with c-supplementment.

Theorem 4.2. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Assume G is a group with a normal subgroup N such that $G/N \in \mathcal{F}$. If every element of $\mathcal{P}(N)$ is c-supplemented in G and N is quaternion-free, then G belongs to $\mathcal{F}$.*

Proof. Without losing generality we can assume $N = G^{\mathcal{F}}$.

Suppose that the result is false and let G be a counterexample of minimal order. Then $G \notin \mathcal{F}$ and $G^{\mathcal{F}} \neq 1$. Suppose that $G^{\mathcal{F}}$ is not 2-nilpotent. Then $G^{\mathcal{F}}$ has a subgroup K such that K is not 2-nilpotent but every proper subgroup of K is 2-nilpotent. Then by [1, IV satz 5.4], $K = K_2K_q$ where K_2 is a normal Sylow 2-subgroup and K_q is a non-normal cyclic Sylow q-subgroup for some odd prime q. Let $\mathcal{H}$ be the saturated formation of 2-nilpotent groups. Then $K^{\mathcal{H}}$ is contained in K_2 and every chief factor of K below $K^{\mathcal{H}}$ is $\mathcal{H}$-eccentric by [14, Lemma 3.6]. Let E be a minimal normal subgroup of K contained in $Z(K^{\mathcal{H}})$. If $EK_q < K$, then E is central in K and so $\mathcal{H}$-central, a contradiction. Hence $K^{\mathcal{H}} = K_2$ is a minimal normal subgroup of K. Pick an element x of order 2 in K_2, then x is c-supplemented in G by hypotheses, so there exists a subgroup H of G such that $G = \langle x\rangle H$ and $\langle x\rangle \cap H \leq \langle x\rangle_G$. If $\langle x\rangle_G = \langle x\rangle$, then obviously $K_2 = \langle x\rangle$. If $\langle x\rangle_G = 1$, then $[G : H] = 2$, so H is normal in G, so $K_2 \cap H$ is normal in K. Thus $K_2 \cap H = 1$ or K_2 by the minimality of K_2. If $K_2 \cap H = 1$, then $K_2 = K_2 \cap G = K_2 \cap (\langle x\rangle H) = \langle x\rangle(K_2 \cap H) = \langle x\rangle$. If $K_2 \cap H = K_2$, then $K_2 \leq H$, so $\langle x\rangle \cap H = \langle x\rangle = \langle x\rangle_G$, a contradiction. So we have that K_2 is a cyclic group of order 2. Then $[K : K_q] = 2$, so K_q is normal in K. Thus K is 2-nilpotent, a contradiction. Consequently $G^{\mathcal{F}}$ is 2-nilpotent. In particular, $G^{\mathcal{F}}$ is solvable. Let M be a maximal subgroup of G such that $G = MG^{\mathcal{F}}$ and $G/M_G \notin \mathcal{F}$. Then $G = MF(G)$. By [12, Proposition 1], $G^{\mathcal{F}}$ is a p-group for some prime p. By [15, Theorem 4.2], $p = 2$ and $(G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$. Moreover $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ by [12, Proposition 1]. If $\exp(G^{\mathcal{F}}) = 2$, then every element of $G^{\mathcal{F}}$ is c-supplemented in G by hypotheses, now [15, Theorem 4.2] implies that $G \in \mathcal{F}$, a contradiction. So $\exp(G^{\mathcal{F}}) = 4$. Then the order of every element of $G^{\mathcal{F}} - \Phi(G^{\mathcal{F}})$ is 4. Pick $a, b \in G^{\mathcal{F}} - \Phi(G^{\mathcal{F}})$ such that the order of $c = [a, b]$ is 2. Denote $R = \langle a, b\rangle/\langle a^2b^2, ca^2\rangle$, then $\overline{a^2} = \overline{b^2} = [\overline{a}, \overline{b}] = \overline{c} \neq 1, o(\overline{a}) = o(\overline{b}) = 4, o(\overline{c}) = 2$. So R is quaternion and a section of $G^{\mathcal{F}}$, it is contrary to the hypothesis that $G^{\mathcal{F}}$ is quaternion-free. These complete our proof. $\square$

For purpose of giving generalizations of Theorem 3.1 and Theorem 3.2 in word of formations, we give a lemma which is a generalization of Lemma 2.5.

Lemma 4.3. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Assume G is a group with a solvable normal subgroup such $G/N \in \mathcal{F}$. If for any maximal subgroup M of G, either $F(N) \leq M$ or $F(N) \cap M < \cdot F(N)$, then $G \in \mathcal{F}$.*

Proof. We prove the lemma by induction on $|G|$.

Case 1. $N \cap \Phi(G) \neq 1$.

Denote $N_1 = N \cap \Phi(G)$ and $\overline{G} = G/N_1$. Then $F(N/N_1) = F(N)/N_1$ by Lemma 2.4. Since $(G/N_1)/(N/N_1) \cong G/N$, we have that $(G/N_1)/(N/N_1)$ lies in $\mathcal{F}$ and N/N_1 is solvable. Suppose that M/N_1 is an arbitrary maximal subgroup of G/N_1, then $M < \cdot G$, so it holds either $F(N) \leq M$ or $F(N) \cap M < \cdot F(N)$ by hypotheses. If $F(N) \leq M$, then $F(\overline{N}) = F(N/N_1) =$

$F(N)/N_1 \leq M/N_1$. If $F(N) \cap M < \cdot F(N)$, then $[F(N) : F(N) \cap M]$ is a prime. Since $[F(\overline{N}) : F(\overline{N}) \cap \overline{M}] = [F(N) : F(N) \cap M]$ is a prime, we have that $F(\overline{N}) \cap \overline{M}$ is a maximal subgroup of $F(\overline{N})$. Therefore $\overline{G}$ satisfies the hypotheses of the Theorem, thus $\overline{G} \in \mathcal{F}$ by induction, so $G \in \mathcal{F}$ as $\mathcal{F}$ is saturated.

Case 2. $N \cap \Phi(G) = 1$.

By Lemma 2.3, we have we have $F(N) = L_1 \times L_2 \times \cdots \times L_s$, where L_i is a minimal normal subgroup of G contained in $F(N)$. $\forall i$, $L_i \nleq \Phi(G)$ as $N \cap \Phi(G) = 1$, so there exists a maximal subgroup M_i of G such that $L_i \nleq M_i$. It follows that $G = L_i M_i$. Since L_i is abelian, we have that $L_i \cap M_i \lhd L_i M_i = G$. The minimality of L_i implies that $L_i \cap M_i = 1$. Since $F(N) = F(N) \cap L_i M_i = L_i(F(N) \cap M_i)$, $F(N) \cap M_i < \cdot F(N)$ by hypotheses, the nilpotence of $F(N)$ implies that $[F(N) : F(N) \cap M_i]$ is a prime.

Since $L_i \leq F(N)$, we have that $G = M_i F(N)$ and thus $|L_i| = [G : M_i] = [F(N) : F(N) \cap M_i]$ is a prime. So $G/C_G(L_i)$ is abelian, so $G/C_G(L_i) \in \mathcal{F}$. Then $G/C_G(F(N)) = G/\bigcap_{i=1}^{s} C_G(L_i) \in \mathcal{F}$ as $\mathcal{F}$ is a formation. So $G/C_N(F(N)) = G/N \cap C_G(F(N)) \in \mathcal{F}$. Since N is solvable by hypotheses, we have that $C_N(F(N)) \leq F(N)$, so we get that $G/F(N) \in \mathcal{F}$.

Consider the factor group G/L_1. Since $(G/L_1)/(F(N)/L_1) \cong G/F(N)$, we have that $(G/L_1)/(F(N)/L_1)$ lies in and $F(N)/L_1$ is solvable. Suppose that M/L_1 is an arbitrary maximal subgroup of G/L_1, then $M < \cdot G$, so it holds either $F(N) \leq M$ or $F(N) \cap M < \cdot F(N)$ by hypotheses. If $F(N) \leq M$, then $F(F(N)/L_1)) = F(N)/L_1) \leq M/N_1$. If $F(N) \cap M < \cdot F(N)$, then $[F(N) : F(N) \cap M]$ is a prime. Since $[F(F(N)/L_1) : F(F(N)/L_1) \cap M/L_1] = [F(N)/L_1 : F(N)/L_1 \cap M/L_1] = [F(N) : F(N) \cap M]$ is a prime, we have that $F(F(N)/L_1) \cap M/L_1$ is a maximal subgroup of $F(F(N)/L_1)$. Therefore $G/L_1, F(N)/L_1$ satisfy the hypotheses of the Theorem, thus $G/L_1 \in \mathcal{F}$ by induction. If $F(N)$ contains another minimal normal subgroup of G, L_2 say, then $G/L_2 \in \mathcal{F}$, then $L_1 \cap L_2 = 1$, it follows that $G = G/(L_1 \cap L_2) \in \mathcal{F}$. The lemma holds. So we assume that $F(N) = L_1$, then $G/L_1 \in \mathcal{F}$, now applying Lemma 4.1, we get $G \in \mathcal{F}$.

These complete our proof. $\square$

Theorem 4.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Assume G is a group with a solvable normal subgroup such $G/N \in \mathcal{F}$. If every* *element of $\mathcal{P}(F(N))$ is c-supplemented in G and $F(N)$ is quaternion-free, then G belongs to $\mathcal{F}$.*

Proof. By Lemma 2.8 and hypotheses, we have that for any maximal subgroup M, there holds either $F(N) \leq M$ or $F(N) \cap M < \cdot F(N)$. Now applying Lemma 4.3, it follows that $G \in \mathcal{F}$. $\square$

Theorem 4.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Assume G is a group with a normal subgroup such $G/N \in \mathcal{F}$. If every element of $\mathcal{P}(F^*(N))$ is c-supplemented in G and $F^*(N)$ is quaternion-free, then G belongs to $\mathcal{F}$.*

Proof. By the hypotheses every element of $\mathcal{P}(F^*(N))$ is c-suplemented in G, thus is c-supplemented in N, and $F^*(N)$ is quaternion-free by hypotheses. Now Corollary 3.3 implies that N is supersolvable, so $F^*(N) = F(N)$. Then $G \in \mathcal{F}$ by Theorem 4.4. $\square$

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