

On collapsing iteration semigroups of set-valued functions

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Abstract. We introduce the notion of collapsing iteration semigroup of set-valued functions and study conditions under which a certain family of set-valued functions, naturally occurring in iteration theory, is such a semigroup.

Introduction

Given a set X a function $F : (0, \infty) \times X \rightarrow 2^X$ is said to be *set-valued iteration semigroup* if

$$F(s + t, x) = F(t, F(s, x)) \quad \text{for } x \in X \quad \text{and } s, t \in (0, \infty)$$

(here and in the sequel we write $F(t, A)$ for the image $F(\{t\} \times A)$ of $\{t\} \times A$; see also Section 2). This notion was introduced and studied under various assumptions by A. SMAJDOR in [2]. In the present paper we propose a more general notion of *collapsing iteration semigroup* postulating that $F : (0, \infty) \times X \rightarrow 2^X$ satisfies the condition

$$F(s + t, x) \subset F(t, F(s, x)) \quad \text{for } x \in X \quad \text{and } s, t \in (0, \infty).$$

Both above definitions take pattern by the classical notion of iteration semigroup intensively studied, among others, by M. C. ZDUN (see, for

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instance, [3]). It is well-known and easy to check that if α is a bijection mapping X onto $\mathbb{R}$ then the function $f : (0, \infty) \times X \rightarrow X$, given by

$$f(t, x) = \alpha^{-1}(\alpha(x) + t),$$

is an iteration semigroup:

$$f(s + t, x) = f(t, f(s, x)) \quad \text{for } x \in X \quad \text{and } s, t \in (0, \infty).$$

More generally we have the following observation.

Remark 1. Let α be a bijection mapping a set X onto an interval $I \subset \mathbb{R}$ with the right endpoint $q \in (-\infty, +\infty]$. Assume that $q \in I$ whenever q is finite. Then the function $f : (0, \infty) \times X \rightarrow X$, given by

$$f(t, x) = \alpha^{-1}(\min\{\alpha(x) + t, q\}), \quad (1)$$

is an iteration semigroup.

PROOF. Fix $s \in (0, \infty)$ and $x \in X$. If $\alpha(x) + s < q$ then

$$\min\{\min\{\alpha(x) + s, q\} + t, q\} = \min\{\alpha(x) + s + t, q\}$$

and if $\alpha(x) + s \geq q$ then

$$\begin{aligned} \min\{\min\{\alpha(x) + s, q\} + t, q\} &= \min\{q + t, q\} = q \\ &= \min\{\alpha(x) + s + t, q\} \end{aligned}$$

for every $t \in (0, \infty)$. Then

$$\begin{aligned} f(s + t, x) &= \alpha^{-1}(\min\{\alpha(x) + s + t, q\}) \\ &= \alpha^{-1}(\min\{\min\{\alpha(x) + s, q\} + t, q\}) \\ &= \alpha^{-1}(\min\{\alpha(\alpha^{-1}(\min\{\alpha(x) + s, q\})) + t, q\}) \\ &= \alpha^{-1}(\min\{\alpha(f(s, x)) + t, q\}) = f(t, f(s, x)) \end{aligned}$$

for every $t \in (0, \infty)$ which gives the desired equality. $\square$

It seems that for the first time this observation was made by M. C. ZDUN (cf. [3, Theorems 5.1–8.1]). As he proved there, (1) with homeomorphic α is a general form of the so called continuous iteration semigroups on an interval (cf. also [1, Theorem 1]).

The aim of the present paper is to introduce a set-valued counterpart of (1) and to find conditions under which such set-valued functions are collapsing iteration semigroups.

1. Preliminaries

In what follows, given sets X, Y and a set-valued function $F : X \rightarrow 2^Y$, we define images and preimages by F putting

$$F(U) := \bigcup_{x \in U} F(x)$$

for every $U \subset X$ and

$$F^{-1}(V) := \{x \in X : F(x) \cap V \neq \emptyset\}$$

for every $V \subset Y$.

Fix a set X and a set-valued function $A : X \rightarrow 2^{\mathbb{R}}$ with non-empty values. Put

$$S := A(X) \quad \text{and} \quad q := \sup S.$$

Given $x, y \in X$ and $t \in (0, \infty)$ we say that $A(y)$ is *t-attainable from* $A(x)$ if

$$[A(x) + t] \cap A(y) \neq \emptyset.$$

If $x \in X$, $t \in (0, \infty)$ and

$$\inf A(x) + t \geq q$$

$A(x)$ is called *t-coming out*.

Throughout this paper we will always assume that

(H) for every $s, t \in (0, \infty)$ and $x, z \in X$ such that $A(z)$ is $(s + t)$ -attainable from $A(x)$ there exists a $y \in X$ such that $A(y)$ is s -attainable from $A(x)$ and $A(z)$ is t -attainable from $A(y)$.

Proposition 1. (i) If S is an interval then (H) holds.

(ii) Assume that all values of A are open sets. If $(\inf S, \sup S) \subset \text{cl } S$ then (H) holds.

(iii) Assume that all values of A are intervals. If (H) holds then $(\inf S, \sup S) \subset \text{cl } S$.

PROOF. (i) Assume that S is an interval. Fix $s, t \in (0, \infty)$ and $x, z \in X$ such that $A(z)$ is $(s + t)$ -attainable from $A(x)$, that is

$$[A(x) + s] \cap [A(z) - t] \neq \emptyset.$$

Therefore there exists a real number u such that

$$u \in A(x) + s \quad \text{and} \quad u \in A(z) - t, \quad (2)$$

whence

$$u - s \in A(x) \subset S \quad \text{and} \quad u + t \in A(z) \subset S.$$

Since S is an interval and $u - s < u < u + t$ we have $u \in S$. Then there exists a $y \in X$ such that $u \in A(y)$. Hence and by (2) we get

$$A(y) \cap [A(x) + s] \neq \emptyset$$

and

$$[A(y) + t] \cap A(z) \neq \emptyset.$$

(ii) Now assume that $(\inf S, \sup S) \subset \text{cl } S$. Fix $s, t \in (0, \infty)$ and $x, z \in X$ such that

$$[A(x) + s + t] \cap A(z) \neq \emptyset.$$

Then $[A(x) + s] \cap [A(z) - t]$ is a non-empty open subset of $(\inf S, \sup S)$. Since the latter is contained in $\text{cl } S$ this means that

$$[A(x) + s] \cap [A(z) - t] \cap S \neq \emptyset$$

which completes the proof of (ii).

(iii) Assume that (H) holds. Suppose that $(\inf S, \sup S) \not\subset \text{cl } S$. Then there exists $v \in (\inf S, \sup S)$ and an open interval G such that $v \in G$ and

$$G \cap S = \emptyset. \quad (3)$$

Since $\inf S < v < \sup S$ and all values of A are intervals there exist $x, z \in X$ such that

$$u < v \quad \text{for } u \in A(x) \quad \text{and} \quad v < w \quad \text{for } w \in A(z).$$

We can find $s, t \in (0, \infty)$ such that

$$\emptyset \neq [A(x) + s] \cap [A(z) - t] \subset G. \quad (4)$$

Obviously

$$[A(x) + s + t] \cap A(z) \neq \emptyset.$$

Then, by (H), there is a $y \in X$ with

$$[A(x) + s] \cap A(y) \neq \emptyset \quad \text{and} \quad [A(z) - t] \cap A(y) \neq \emptyset. \quad (5)$$

Since $A(x) + s$, $A(z) - t$ and $A(y)$ are intervals, by (4) and (5), we have

$$[[A(x) + s] \cap [A(z) - t]] \cap A(y) \neq \emptyset.$$

Then, by (4), $G \cap S \neq \emptyset$ which contradicts (3). $\square$

The examples below show that none of the implications in Proposition 1 can be converted; also the assumptions made in (ii) and (iii) turn out to be essential.

Example 1. Let X be an arbitrary set and $A : X \rightarrow 2^{\mathbb{R}}$ be defined by

$$A(x) = (0, 2) \cup (3, 5).$$

Obviously $S = (0, 2) \cup (3, 5)$ and

$$(\inf S, \sup S) = (0, 5) \not\subset \text{cl } S.$$

We will show that (H) holds. Notice that

$$\{t \in (0, \infty) : [A(x_1) + t] \cap A(x_2) \neq \emptyset\} = (0, 5)$$

for every $x_1, x_2 \in X$. Fix $x, z \in X$ and let $s, t \in (0, \infty)$ be such that

$$[A(x) + s + t] \cap A(z) \neq \emptyset.$$

Then $s + t \in (0, 5)$ whence $s, t \in (0, 5)$. Therefore, taking any $y \in X$, we have $[A(x) + s] \cap A(y) \neq \emptyset$ and $A(y) \cap [A(z) - t] \neq \emptyset$.

Example 2. Let $X = \{1, 2, 3\}$ and $A : X \rightarrow 2^{\mathbb{R}}$ be defined by

$$A(x) = \begin{cases} (0, 2) & \text{for } x = 1, \\ (2, 3) & \text{for } x = 2, \\ \{3\} & \text{for } x = 3. \end{cases}$$

Obviously $S = (0, 2) \cup (2, 3]$ and $(\inf S, \sup S) = (0, 3) \subset \text{cl } S$. Notice that $[A(1) + 2] \cap A(3) \neq \emptyset$ but $A(3) - 1 = \{2\}$ and $2 \notin S$. Then $[A(3) - 1] \cap S = \emptyset$ and, consequently, (H) does not hold.

Corollary 1. (i) Assume that A is single-valued. Then (H) holds if and only if S is an interval.

(ii) Assume that all values of A are open intervals. Then (H) holds if and only if $(\inf S, \sup S) \subset \text{cl } S$.

PROOF. (i) Assume (H) and take $u, w \in S$ such that $u < w$. Therefore there exist points $x, z \in X$ such that $A(x) = \{u\}$, $A(z) = \{w\}$. Fix a $v \in (u, w)$ and put

$$s := v - u, \quad t := w - v.$$

Obviously $A(x) + s + t = A(z)$, i.e. $A(z)$ is $(s + t)$ -attainable from $A(x)$. Then, by (H), there exists a $y \in X$ such that $[A(x) + s] \cap A(y) \neq \emptyset$, i.e. $A(x) + s = A(y)$. On the other hand, $A(x) + s = \{v\}$. Thus $A(y) = \{v\}$ and, consequently, $v \in S$. This means that S is an interval and by Proposition 2(i) completes the proof of (i).

The second assertion follows immediately from Proposition 2(ii) and (iii). $\square$

For every $x \in X$ define

$$\tau(x) := \sup \{t \in [0, \infty) : [A(x) + t] \cap S \neq \emptyset\}.$$

Theorem 1. Let $x \in X$. If $t < \tau(x)$ then $[A(x) + t] \cap S \neq \emptyset$ and if $t > \tau(x)$ then $[A(x) + t] \cap S = \emptyset$ for every $t \in (0, \infty)$.

PROOF. To prove the first claim it suffices to show that if for some $t \in (0, \tau(x))$ condition $[A(x) + t] \cap S \neq \emptyset$ holds, then $[A(x) + s] \cap S \neq \emptyset$ for every $s \in (0, t)$. To this aim let $t \in (0, \tau(x))$ satisfy $[A(x) + t] \cap S \neq \emptyset$ and fix an $s \in (0, t)$. Then there exists a $z \in X$ such that

$$[A(x) + t] \cap A(z) \neq \emptyset.$$

Let $u := t - s$. Then $u \in (0, \infty)$ and

$$[A(x) + s + u] \cap A(z) \neq \emptyset.$$

By (H) there exists a $y \in X$ such that

$$[A(x) + s] \cap A(y) \neq \emptyset.$$

Thus we have shown that

$$[A(x) + s] \cap S \neq \emptyset,$$

which completes the proof in this case.

The second assertion follows directly from the definition of $\tau(x)$. $\square$

Lemma 1. *For every $x \in X$*

$$\tau(x) = q - \inf A(x).$$

PROOF. Fix an $x \in X$ and suppose that $\tau(x) < q - \inf A(x)$. Then $\tau(x) \neq \infty$ and $\tau(x) + \inf A(x) < q$ whence there exist s, t such that $s \in A(x) + \tau(x)$, $t \in S$ and $s < t$. Thus

$$t = s + (t - s) \in A(x) + (\tau(x) + (t - s)),$$

contrary to Theorem 1.

Now suppose that $\tau(x) > q - \inf A(x)$ for an $x \in X$. Therefore $\inf A(x) \neq -\infty$ and there exists a u such that $q < u < \inf A(x) + \tau(x)$. We can find an $s \in (0, \tau(x))$ such that $u = \inf A(x) + s$. Obviously $u \notin S$. Then for every $v \in A(x) + s$ we have $v \geq u$ and, consequently, $v \notin S$ which contradicts Theorem 1. $\square$

Corollary 2. *For every $x \in X$*

$$[A(x) + \tau(x)] \cap S \subset \{q\}.$$

PROOF. By Lemma 1 and definition of the number q we have

$$[A(x) + \tau(x)] \cap S = [A(x) + q - \inf A(x)] \cap S \subset [q, +\infty] \cap (-\infty, q] = \{q\}$$

for every $x \in X$. $\square$

Let $e : (0, \infty) \times X \rightarrow [0, \infty)$ be a function defined by

$$e(t, x) := \sup\{s \in [0, t] : [A(x) + s] \cap S \neq \emptyset\}.$$

Lemma 2. *For every $t \in (0, \infty)$ and $x \in X$*

$$e(t, x) = \min\{t, \tau(x)\}.$$

PROOF. If $t \leq \tau(x)$ then, by Theorem 1,

$$[A(x) + s] \cap S \neq \emptyset$$

for every $s \in [0, t)$ whence

$$e(t, x) = \sup\{s \in [0, t] : [A(x) + s] \cap S \neq \emptyset\} = t = \min\{t, \tau(x)\}.$$

If $t > \tau(x)$ then, again by Theorem 1, we have

$$\begin{aligned} e(t, x) &= \sup\{s \in [0, t] : [A(x) + s] \cap S \neq \emptyset\} \\ &= \sup\{s \in [0, \tau(x)] : [A(x) + s] \cap S \neq \emptyset\} = \tau(x) = \min\{t, \tau(x)\}. \end{aligned}$$

□

2. Collapsing iteration semigroups

In what follows, given a set-valued function $F : (0, \infty) \times X \rightarrow 2^X$, we put

$$F(t, U) := F(\{t\} \times U) = \bigcup_{x \in U} F(t, x)$$

whenever $U \subset X$; moreover, we will write $F^t(x)$ and $F^t(U)$ instead of $F(t, x)$ and $F(t, U)$, respectively.

Remark 2. If $F : (0, \infty) \times X \rightarrow 2^X$ is a single-valued collapsing iteration semigroup then

$$F(s + t, x) = F(t, F(s, x))$$

for every $s, t \in (0, \infty)$ and $x \in X$, that is F generates a classical iteration semigroup.

Remark 3. If S is an interval, $q \in S$ whenever q is finite and A is a single-valued bijection mapping X onto S then

$$A^{-1}(A(x) + e(t, x)) = A^{-1}(\min\{A(x) + t, q\})$$

for every $t \in (0, \infty)$ and $x \in X$.

PROOF. Assume that S is a interval and $A : X \rightarrow S$ is a single-valued bijection. Fix $t \in (0, \infty)$ and $x \in X$. Then, by Lemmas 2 and 1 and the equality $\inf\{A(x)\} = A(x)$, we have

$$\begin{aligned} A(x) + e(t, x) &= A(x) + \min\{t, \tau(x)\} \\ &= \min\{A(x) + t, A(x) + \tau(x)\} \\ &= \min\{A(x) + t, A(x) + q - \inf\{A(x)\}\} \\ &= \min\{A(x) + t, q\} \end{aligned}$$

which gives the desired property. $\square$

Now we pass to the problem of finding conditions under which a set-valued function of the form

$$F(t, x) := A^{-1}(A(x) + e(t, x)) \quad (6)$$

is a collapsing iteration semigroup.

Lemma 3. *Let $F : (0, \infty) \times X \rightarrow 2^X$ be given by (6) and let $t \in (0, \infty)$ and $x \in X$. If $t < \tau(x)$ then*

$$F(t, x) = A^{-1}(A(x) + t) \neq \emptyset$$

and if $t \geq \tau(x)$ then

$$F(t, x) = \begin{cases} A^{-1}(\{q\}), & \text{if } q \in S \text{ and } \inf A(x) \in A(x); \\ \emptyset & \text{otherwise.} \end{cases}$$

PROOF. The first assertion follows directly from Lemma 2 and Theorem 1. So assume that $t \geq \tau(x)$. Again by Lemma 2 we have

$$F^t(x) = A^{-1}(A(x) + \tau(x)) = A^{-1}((A(x) + \tau(x)) \cap S).$$

If $q \in S$ and $\inf A(x) \in A(x)$ then, by Lemma 1 and Corollary 2,

$$F^t(x) = A^{-1}(\{q\}).$$

If either $q \notin S$, or $\inf A(x) \notin A(x)$ then, again by Lemma 1 and Corollary 2, we have

$$[A(x) + \tau(x)] \cap S = \emptyset$$

whence $F^t(x) = \emptyset$. $\square$

Proposition 2. *Let $F : (0, \infty) \times X \rightarrow 2^X$ be given by (6), $x \in X$, and $s, t \in (0, \infty)$. If $s + t \leq \tau(x)$ then*

$$F^{s+t}(x) \subset F^t(F^s(x)). \quad (7)$$

PROOF. Assume that $s + t \leq \tau(x)$. To show the inclusion (7) fix a $z \in F^{s+t}(x)$. By (6) and Lemma 2 we have

$$z \in F^{s+t}(x) = A^{-1}(A(x) + (s + t))$$

that is

$$A(z) \cap [A(x) + (s + t)] \neq \emptyset.$$

Thus, by virtue of (H), there exists a $y \in X$ such that

$$A(z) \cap [A(y) + t] \neq \emptyset \quad \text{and} \quad A(y) \cap [A(x) + s] \neq \emptyset.$$

According to Theorem 1 we have $t \leq \tau(y)$, whence

$$z \in A^{-1}(A(y) + t) = F^t(y) \quad \text{and} \quad y \in A^{-1}(A(x) + s) = F^s(x)$$

that is $z \in F^t(F^s(x))$. □

Now we are in position to formulate the main result of the paper.

Theorem 2. *Let $F : (0, \infty) \times X \rightarrow 2^X$ be given by (6). Then F is a collapsing iteration semigroup if and only if either $q \notin S$, or, for every $x \in X$ and $s, t \in (0, \infty)$ such that $s + t > \tau(x)$ and $A(x)$ has the smallest element, at least one of the following conditions holds:*

- (i) *there exists a $y \in F(s, x)$ such that $A(y)$ is t -coming out and has the smallest element,*
- (ii) *for every $z \in A^{-1}(\{q\})$ there exists a $y \in F(s, x)$ such that $A(z)$ is t -attainable from $A(y)$.*

Before proving Theorem 2 we will derive the following consequence of it.

Theorem 3. *Let $F : (0, \infty) \times X \rightarrow 2^X$ be given by (6). Every of the following conditions is sufficient for F to be a collapsing iteration semigroup:*

- (i) $q \notin S$; (ii) $q = \infty$;

- (iii) *no value of A has the smallest element;*
- (iv) *A is single-valued.*

PROOF. It follows immediately from Theorem 2 that each of the conditions (i)–(iii) is sufficient for F to be a collapsing iteration semigroup.

Assume that A is single-valued and $q \in S$. Fix $s, t \in (0, \infty)$ and $x \in X$ such that $s + t > \tau(x)$. Then, by Lemma 1, we have

$$s + t > q - \inf A(x).$$

Of course $\inf A(x) \neq -\infty$ and

$$\inf A(x) + s + t > q. \quad (8)$$

First assume that $s < \tau(x)$. Thus, on account of Lemma 3,

$$F^s(x) = A^{-1}(A(x) + s) \neq \emptyset.$$

Fix a $y \in F^s(x)$. Then $A(y) \cap [A(x) + s] \neq \emptyset$, that is $A(x) + s = A(y)$ whence, by (8), $\inf A(y) + t > q$ which means that $A(y)$ is t -coming out.

Now consider the case $s \geq \tau(x)$. Then, by Lemma 3 and the assumptions,

$$F^s(x) = A^{-1}(\{q\}) \neq \emptyset.$$

Taking any $y \in F^s(x)$ we have $A(y) = \{q\}$, whence

$$\inf A(y) + t = q + t > q.$$

To complete the proof it is enough to use Theorem 2. □

PROOF OF THEOREM 2. Assume that F is a collapsing iteration semigroup and $q \in S$. Fix $x \in X$ and $s, t \in (0, \infty)$ such that

$$s + t > \tau(x) \quad \text{and} \quad \inf A(x) \in A(x).$$

Obviously $A^{-1}(\{q\}) \neq \emptyset$, so we can fix a $z \in A^{-1}(\{q\})$. By Lemma 3

$$F^{s+t}(x) = A^{-1}(\{q\}).$$

Thus $z \in F^{s+t}(x)$ and, consequently, there exists a $y \in F^s(x)$ such that

$$z \in F^t(y). \quad (9)$$

At first assume that $t \geq \tau(y)$. Then, by Lemma 1,

$$\inf A(y) + t \geq q.$$

Since $F^t(y) \neq \emptyset$ it follows from Lemma 3 that

$$F^t(y) = A^{-1}(\{q\}) \quad \text{and} \quad \inf A(y) \in A(y).$$

In such a way we have come to condition (i).

If $t < \tau(y)$ then, by Lemma 3,

$$F^t(y) = A^{-1}(A(y) + t)$$

whence, by (9),

$$A(z) \cap [A(y) + t] \neq \emptyset$$

and, consequently, (ii) holds true.

To prove the converse fix $x \in X$ and $s, t \in (0, \infty)$. If $s + t \leq \tau(x)$ the assertion follows from Proposition 2 so assume that $s + t > \tau(x)$. If $q \notin S$ then, by Lemma 3, $F^{s+t}(x) = \emptyset$ and (7) holds true. Thus we can assume that $q \in S$. To prove (7) fix a $z \in F^{s+t}(x)$. Since $F^{s+t}(x) \neq \emptyset$ we have, by Lemma 3,

$$F^{s+t}(x) = A^{-1}(\{q\}) \quad \text{and} \quad \inf A(x) \in A(x).$$

In particular,

$$z \in A^{-1}(\{q\}). \quad (10)$$

Assume (i). Then there exists a $y \in F^s(x)$ such that

$$\inf A(y) + t \geq q \quad \text{and} \quad \inf A(y) \in A(y).$$

Thus, according to Lemma 1, $\tau(y) = q - \inf A(y) \leq t$. Consequently, it follows from Lemma 3 that $F^t(y) = A^{-1}(\{q\})$ whence, by (10), we have $z \in F^t(y) \subset F^t(F^s(x))$.

Finally assume (ii). Then $[A(y) + t] \cap A(z) \neq \emptyset$ for a $y \in F^s(x)$, i.e.

$$z \in A^{-1}(A(y) + t). \quad (11)$$

In particular, the set $A^{-1}(A(y) + t)$ is non-empty, whence, by Theorem 1, $t \leq \tau(y)$ and, on account of (6) and Lemma 2,

$$A^{-1}(A(y) + t) = F^t(y).$$

Thus, by (11),

$$z \in F^t(y) \subset F^t(F^s(x))$$

which completes the proof. $\square$

Example 3. Let $X = [0, 1]$ and $A : X \rightarrow 2^{\mathbb{R}}$ be defined by

$$A(x) := [x - 1, x + 1].$$

Obviously $S = [-1, 2]$ and $q = 2$. Then, by Proposition 1(i), A satisfies condition (H). According to Lemma 1

$$\tau(x) = 3 - x \tag{12}$$

for every $x \in X$. Let $F : (0, \infty) \times X \rightarrow 2^X$ be given by (6). We will find the explicit formula for F .

Fix $t \in (0, \infty)$ and $x \in X$. If $t < \tau(x)$ then, by Lemma 3,

$$\begin{aligned} F^t(x) &= A^{-1}(A(x) + t) = A^{-1}([x - 1, x + 1] + t) \\ &= A^{-1}([x + t - 1, x + t + 1]) \\ &= \{y \in [0, 1] : [y - 1, y + 1] \cap [x + t - 1, x + t + 1] \neq \emptyset\} \\ &= \{y \in [0, 1] : x + t - 1 \leq y + 1 \text{ and } y - 1 \leq x + t + 1\} \\ &= [0, 1] \cap [x + t - 2, x + t + 2] \\ &= [\max\{x + t - 2, 0\}, 1]. \end{aligned}$$

If $t \geq \tau(x)$ then, again by Lemma 3,

$$F^t(x) = A^{-1}(\{2\}) = \{1\}.$$

Consequently, we have

$$F^t(x) = \begin{cases} [\max\{x + t - 2, 0\}, 1], & \text{if } x + t < 3, \\ \{1\}, & \text{if } x + t \geq 3, \end{cases} \tag{13}$$

for every $t \in (0, \infty)$ and $x \in X$. Observe that

$$1 \in F^t(x) \tag{14}$$

for every $t \in (0, \infty)$ and $x \in X$.

Now fix $s, t \in (0, \infty)$ and $x \in X$. If $s + t > \tau(x)$ then, by (13) and (14), we have $F^{s+t}(x) = \{1\} \subset F^t(F^s(x))$. Thus Proposition 2 implies that F is a collapsing iteration semigroup. Observe that F does not satisfy any of the assumptions (i)–(iv) of Theorem 3. Consequently, none of those conditions, which are sufficient for F to be a collapsing iteration semigroup, is necessary.

Observe also that $\tau(1) = 2$ and, by (13), $F^2(1) = \{1\}$ and $F^1(y) = [0, 1]$ for $y \in [0, 1]$. On the other hand, again by (13),

$$F^1(F^1(1)) = \bigcup_{y \in [0, 1]} F^1(y) = \bigcup_{y \in [0, 1]} [0, 1] = [0, 1].$$

Consequently, $F^1(F^1(1)) \not\subset F^2(1)$. This shows that F does not satisfy the condition

$$F^t(F^s(x)) \subset F^{s+t}(x) \quad \text{for } s, t \in (0, \infty) \quad \text{and } x \in X,$$

which could serve as a definition of expanding iteration semigroup.

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