

A consequence of the Proper Forcing Axiom in topology

By MILOŠ S. KURILIĆ (Novi Sad) and ALEKSANDAR PAVLOVIĆ (Novi Sad)

Abstract. If $\langle L, < \rangle$ is a dense linear order without end-points and if $A_1, A_2 \subset L$ are disjoint dense subsets of L , then $\mathcal{O}_{A_1 A_2}$ denotes the topology on L generated by the closed intervals $[a_1, a_2]$, where $a_1 \in A_1$ and $a_2 \in A_2$. It is proved that under the Proper Forcing Axiom each two spaces of the form $\langle \mathbb{R}, \mathcal{O}_{A_1 A_2} \rangle$, where A_1 and A_2 are $\aleph_1$ -dense subsets of reals, are homeomorphic.

0. Introduction

The standard topology on a linearly ordered set $\langle L, < \rangle$ (generated by the family of all open intervals) is extensively investigated. It is T_5 , collectionwise normal, connected iff the order is continuous, compact iff each subset of L has the supremum, there holds $\chi(L) \leq d(L) \leq w(L) \leq |L|$, etc. (see [4]).

If $\langle L, < \rangle$ is a dense linear order without end-points and if $A_1, A_2 \subset L$ are disjoint dense subsets of L , then $\mathcal{O}_{A_1 A_2}$ denotes the topology on L generated by the closed intervals $[a_1, a_2]$, where $a_1 \in A_1$, $a_2 \in A_2$ and $a_1 < a_2$.

Modifying the methods used in the theory of linearly ordered spaces it can be proved that in general, the spaces $\langle L, \mathcal{O}_{A_1 A_2} \rangle$ are zero-dimensional and collectionwise normal. Also, there holds $\chi(L, \mathcal{O}_{A_1 A_2}) \leq d(L, \mathcal{O}_{A_1 A_2}) \leq \min\{|A_1|, |A_2|\} \leq \max\{|A_1|, |A_2|\} = w(L, \mathcal{O}_{A_1 A_2}) \leq |L|$ and all the

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inequalities can be strict. If the sets A_1 and A_2 are countable, the space $\langle L, \mathcal{O}_{A_1 A_2} \rangle$ is metrizable.

Specially, if $\mathbb{R}$ is the real line, then the space $\langle \mathbb{R}, \mathcal{O}_{A_1 A_2} \rangle$ is a hereditarily separable, hereditarily Lindelöf, first countable T_6 -space.

The space $\langle \mathbb{R}, \mathcal{O}_{A_1 A_2} \rangle$ is second countable if and only if the sets A_1 and A_2 are countable. Such a space is a universal second countable zero-dimensional space. By the well known theorem of Cantor each two countable dense linear orders without end-points are isomorphic. If A_1, A_2 and B_1, B_2 are pairs of disjoint dense countable subsets of reals, then using Cantor's "back and forth" method it is easy to make an order-isomorphism from $A_1 \cup A_2$ onto $B_1 \cup B_2$ which maps A_1 onto B_1 (and A_2 onto B_2). Extending the isomorphism continuously we obtain an homeomorphism $F : \langle \mathbb{R}, \mathcal{O}_{A_1 A_2} \rangle \rightarrow \langle \mathbb{R}, \mathcal{O}_{B_1 B_2} \rangle$. So, each two spaces of the form $\langle \mathbb{R}, \mathcal{O}_{A_1 A_2} \rangle$, where $|A_1| = |A_2| = \aleph_0$, are homeomorphic.

Can this result be extended for uncountable cardinals? In this paper we will show that for $\aleph_1$ -dense sets A_1 and A_2 the answer in the affirmative is consistent with ZFC, namely we will prove

Theorem 1. *PFA $\Rightarrow$ All the spaces $\langle \mathbb{R}, \mathcal{O}_{A_1 A_2} \rangle$, where A_1 and A_2 are $\aleph_1$ -dense subsets of $\mathbb{R}$, are homeomorphic.*

In [2], using iterated forcing BAUMGARTNER proved the consistency of $MA +$ (today called) Baumgartner's Axiom, BA: "All $\aleph_1$ -dense sets of reals are order-isomorphic". A different proof was obtained by SHELAH (see [1]) and, amalgamating these two ideas, Baumgartner showed that the PFA implies BA (see [3]). In Corollary 8.3 of [7] TODORČEVIĆ presented an elegant proof of $PFA \Rightarrow BA$ using

Theorem 2 (Theorem 8.2 of [7]). (PFA) *If A and B are sets of reals of size $\aleph_1$, then there is an injection $f : A \rightarrow B$ which is the union of countably many increasing subfunctions.*

In fact, Todorčević deduced BA from $MA_{\aleph_1}$ for σ -centred posets + the consequence of the PFA given in the previous theorem.

For the proof of Theorem 1 we will modify the construction of Todorčević and use the following elementary fact.

Fact 1. If A and B are dense subsets of $\mathbb{R}$ and $f : A \rightarrow B$ is an order isomorphism, then the function $F : \mathbb{R} \rightarrow \mathbb{R}$ defined by $F(x) = \sup\{f(a) : a \in A \wedge a < x\}$ is the unique isomorphism which extends f .

The notation used in the paper is standard. The facts concerning forcing can be found in [5] or [6], while all the facts related to the PFA are contained in [3].

1. A combinatorial consequence of the PFA

A set $A \subset \mathbb{R}$ is called $\aleph_1$ -dense iff $|A \cap (x, y)| = \aleph_1$ for each $x, y \in \mathbb{R}$ satisfying $x < y$. In this section we extend Corollary 8.3 of [7].

Theorem 3. (PFA) *Let $\theta \geq 1$ be a countable ordinal, let $\{A_\alpha : \alpha < \theta\}$ and $\{B_\alpha : \alpha < \theta\}$ be two families of pairwise disjoint $\aleph_1$ -dense subsets of $\mathbb{R}$ and let $A = \bigcup_{\alpha < \theta} A_\alpha$ and $B = \bigcup_{\alpha < \theta} B_\alpha$. Then there exists an order isomorphism $f : A \rightarrow B$ such that $f[A_\alpha] = B_\alpha$, for each $\alpha < \theta$.*

PROOF. Let $\mathcal{B}$ denote the family of all open intervals with rational end-points. Using Theorem 2, for each $I, J \in \mathcal{B}$ and each $\alpha < \theta$ we choose injections $f_{I,J,\alpha} : A_\alpha \cap I \rightarrow B_\alpha \cap J$ and $g_{I,J,\alpha} : B_\alpha \cap J \rightarrow A_\alpha \cap I$ such that $f_{I,J,\alpha} = \bigcup_{k \in \omega} f_{I,J,\alpha,k}$ and $g_{I,J,\alpha} = \bigcup_{k \in \omega} g_{I,J,\alpha,k}$, where $f_{I,J,\alpha,k}$ and $g_{I,J,\alpha,k}$, $k \in \omega$, are increasing functions. Let

$$\mathbb{Q} = \left\{ p \subset \bigcup_{I,J \in \mathcal{B}} \bigcup_{\alpha < \theta} (f_{I,J,\alpha} \cup g_{I,J,\alpha}^{-1}) : p \text{ is a finite increasing function} \right\}.$$

Clearly the elements $p, q \in \mathbb{Q}$ are compatible if and only if $p \cup q$ is an increasing function. $\square$

Claim 1. The partial ordering $\langle \mathbb{Q}, \supset \rangle$ is σ -centred.

PROOF OF CLAIM 1. For the elements $I = (a, b)$ and $J = (c, d)$ of $\mathcal{B}$ we write $I \prec J$, if $b < c$. We will say that a finite sequence of rational open rectangles $\langle I_i \times J_i : i < r \rangle$ is increasing if $i < j < r$ implies $I_i \prec I_j$ and $J_i \prec J_j$. Clearly, the set $\mathcal{R}$ of all finite increasing sequences of rational open rectangles is countable. On the other hand, the set Φ of all

finite sequences of elements of the set $\bigcup_{I,J \in \mathcal{B}} \bigcup_{\alpha < \theta} \bigcup_{k \in \omega} \{f_{I,J,\alpha,k}, g_{I,J,\alpha,k}^{-1}\}$ is countable. Firstly we prove that $\mathbb{Q}$ is the union of the sets

$$\mathbb{Q}_{\langle I_i \times J_i : i < r \rangle}^{\langle \varphi_i : i < r \rangle} = \left\{ p \in \mathbb{Q} : p \subset \bigcup_{i < r} (I_i \times J_i) \cap \varphi_i \right\}, \quad (1)$$

where $r \in \omega$, $\langle I_i \times J_i : i < r \rangle \in \mathcal{R}$ and $\langle \varphi_i : i < r \rangle \in \Phi$. (Clearly, some of these sets are equal to $\{\emptyset\}$.) Let $p = \{\langle x_i, y_i \rangle : i < r\} \in \mathbb{Q}$, where $x_0 < x_1 < \dots < x_{r-1}$ (which implies $y_0 < y_1 < \dots < y_{r-1}$). Then it is easy to find an increasing sequence of rectangles $\langle I_i \times J_i : i < r \rangle \in \mathcal{R}$ such that $\langle x_i, y_i \rangle \in I_i \times J_i$, for $i < r$. According to the definition of $\mathbb{Q}$, for each $i < r$ there are $I'_i, J'_i \in \mathcal{B}$, $\alpha'_i < \theta$, $k'_i \in \omega$ and $\varphi_i \in \{f_{I'_i, J'_i, \alpha'_i, k'_i}, g_{I'_i, J'_i, \alpha'_i, k'_i}^{-1}\}$ such that $\langle x_i, y_i \rangle \in \varphi_i$. Now $p \subset \bigcup_{i < r} (I_i \times J_i) \cap \varphi_i$ and consequently $p \in \mathbb{Q}_{\langle I_i \times J_i : i < r \rangle}^{\langle \varphi_i : i < r \rangle}$.

Since there is countably many sets defined by (1) it remains to be proved they are centred. The functions $g_{I,J,\alpha,k}$ are increasing, hence the functions $g_{I,J,\alpha,k}^{-1}$ are increasing too, so $\bigcup_{i < r} (I_i \times J_i) \cap \varphi_i$ is an increasing function. Consequently, the union of a finite subset $\{p_k : k < n\}$ of $\mathbb{Q}_{\langle I_i \times J_i : i < r \rangle}^{\langle \varphi_i : i < r \rangle}$ is a finite increasing function, hence belongs to $\mathbb{Q}$ and clearly extends each p_k . Claim 1 is proved. $\square$

Claim 2. The sets $D_a = \{p \in \mathbb{Q} : a \in \text{dom}(p)\}$, $a \in A$, and the sets $D_b = \{p \in \mathbb{Q} : b \in \text{ran}(p)\}$, $b \in B$, are dense subsets of $\mathbb{Q}$.

PROOF OF CLAIM 2. Let $a \in A_\alpha$ for some $\alpha < \theta$ and let $q = \{\langle a_i, b_i \rangle : i < r\} \in \mathbb{Q} \setminus D_a$, that is $a \notin \text{dom}(q)$. W.l.o.g. we suppose $a_0 < a_1 < \dots < a_{r-1}$. If $a_i < a < a_{i+1}$ for some $i < r-1$, let $I, J \in \mathcal{B}$ where $a \in I \subset (a_i, a_{i+1})$ and $J \subset (b_i, b_{i+1})$. Then $a \in \text{dom}(f_{I,J,\alpha})$, so $\langle a, f_{I,J,\alpha}(a) \rangle \in I \times J$ and, since $p = q \cup \{\langle a, f_{I,J,\alpha}(a) \rangle\}$ is an increasing function, we have $p \in \mathbb{Q}$. Clearly $p \in D_a$ and $p \leq q$. If $a < a_0$ or $a > a_{r-1}$, we proceed similarly. For the sets D_b the proof is analogous. Claim 2 is proved.

Since the partial order $\mathbb{Q}$ is σ -centred, it is ccc and consequently proper, so, by the PFA there exists a filter $G \subset \mathbb{Q}$ intersecting all the sets mentioned in Claim 2. Thus, for $f = \bigcup G \subset A \times B$ we have $\text{dom}(f) = A$ and $\text{ran}(f) = B$. Clearly f is an increasing function thus it is an order isomorphism from A onto B . The fact that $f[A_\alpha] = B_\alpha$, for each $\alpha < \theta$ follows from the construction. $\square$

2. The proof of Theorem 1

Let the PFA hold and let A_1, A_2 and B_1, B_2 be pairs of disjoint $\aleph_1$ -dense subsets of $\mathbb{R}$. By Theorem 3, if $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, then there exists an order isomorphism $f : A \rightarrow B$ such that $f[A_i] = B_i$, for each $i \in \{1, 2\}$. Now, by Fact 1, there exists an order isomorphism $F : \mathbb{R} \rightarrow \mathbb{R}$ extending f . Clearly, if $a_1 \in A_1$, $a_2 \in A_2$, $b_1 \in B_1$ and $b_2 \in B_2$ where $a_1 < a_2$ and if $F(a_1) = b_1$ and $F(a_2) = b_2$, then

$$F[[a_1, a_2]] = [b_1, b_2] \quad \text{and} \quad F^{-1}[[b_1, b_2]] = [a_1, a_2].$$

By the first equality the mapping $F : \langle \mathbb{R}, \mathcal{O}_{A_1 A_2} \rangle \rightarrow \langle \mathbb{R}, \mathcal{O}_{B_1 B_2} \rangle$ is open, and by the second it is continuous. So, F is a homeomorphism.

Remark 1. Theorem 3 (and Theorem 1 as its consequence) can be proved without the PFA. Namely, in [3] a ccc partial ordering which adds an isomorphism between two $\aleph_1$ dense sets A and B generically is constructed under the CH. This construction can be modified for the case when $A = \bigcup_{n \in \omega} A_n$ and $B = \bigcup_{n \in \omega} B_n$, where A_n and B_n are $\aleph_1$ -dense subsets of $\mathbb{R}$. Then, as in [2], an iteration of length $\aleph_2$ gives a model in which the conclusion of Theorem 3 holds.

Also we note that Theorem 3 is proved under $\text{MA}_{\aleph_1}$ for σ -centred posets + the consequence of the PFA given in Theorem 2.

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MILOŠ S. KURILIĆ
INSTITUTE OF MATHEMATICS
UNIVERSITY OF NOVI SAD
TRG DOSITEJA OBRADOVIĆA 4, 21000 NOVI SAD
YUGOSLAVIA

E-mail: milos@im.ns.ac.yu

ALEKSANDAR PAVLOVIĆ
INSTITUTE OF MATHEMATICS
UNIVERSITY OF NOVI SAD
TRG DOSITEJA OBRADOVIĆA 4, 21000 NOVI SAD
YUGOSLAVIA

E-mail: apavlovic@im.ns.ac.yu

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