

Approximation by some linear positive operators in polynomial weighted spaces

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Abstract. We consider certain linear positive operators L_n in polynomial weighted spaces of functions of one variable and study approximation properties of these operators, including theorems on the degree of approximation.

1. Introduction

1.1. M. BECKER in his paper [1] studied approximation problems for functions $f \in C_p$ and Szász–Mirakyan operators

$$S_n(f; x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad (1)$$

$$x \in R_0 := [0, +\infty), \quad n \in N := \{1, 2, \dots\},$$

where C_p with some fixed $p \in N_0 := \{0, 1, 2, \dots\}$ is a polynomial weighted space generated by the weight function

$$w_0(x) := 1, \quad w_p(x) := (1 + x^p)^{-1}, \quad \text{if } p \geq 1, \quad (2)$$

i.e., C_p is the set of all real-valued functions f , continuous on R_0 and such that $w_p f$ is uniformly continuous and bounded on R_0 . The norm in C_p is

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defined by the formula

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x \in R_0} w_p(x) \|f(x)\|. \quad (3)$$

In [1] there were proved theorems on the degree of approximation of $f \in C_p$ by operators S_n defined by (1). From these theorems it was deduced that

$$\lim_{n \rightarrow \infty} S_n(f; x) = f(x) \quad (4)$$

for every $f \in C_p$, $p \in N_0$ and $x \in R_0$. Moreover, the convergence (4) is uniform on every interval $[x_1, x_2]$, $x_2 > x_1 \geq 0$.

The Szasz–Mirakyan operators are important in approximation theory. They have been studied intensively, and their connections with different branches of analysis, such that as numerical analysis. Recently in many papers were introduced various modifications of operators S_n . Approximation properties of modified Szasz–Mirakyan operators

$$S_n^q(f; x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n+q}\right)$$

for $x \in R_0$, $n \in N$, $q > 0$, in exponential weighted spaces were examined in [3]. Their extensions can be found in, e.g., [4], [5].

The actual construction of the operator S_n and S_n^q requires estimation of infinite series which in a certain sense restricts their usefulness from the computational point of view. Thus the question arises, whether S_n , S_n^q and their generalizations cannot be replaced by a finite sum. In connection with this question we introduce the operators (8) Moreover, we shall prove that the order of approximation of $f \in C_p$ by L_n (defined by (8) is better than (25) and $(L_n(f))$ converges uniformly to f on R_0 . This together with the simple form of the operator makes the results, given in the present paper, more helpful e.g. in numerical methods.

1.2. In this paper we introduce certain linear positive operators and study their approximation properties. To this end, let C_p be the space given above and let $f \in C_p^1 := \{f \in C_p : f' \in C_p\}$, where f' is the first derivative of f . For $f \in C_p$ we define the modulus of continuity $\omega_1(f; \cdot)$ as usual ([2]) by the formula

$$\omega_1(f; C_p; t) := \sup_{0 \leq h \leq t} \|\Delta_h f(\cdot)\|_p, \quad t \in R_0, \quad (5)$$

where $\Delta_h f(x) := f(x+h) - f(x)$, for $x, h \in R_0$. From the above it follows that

$$\lim_{t \rightarrow 0+} \omega_1(f; C_p; t) = 0, \quad (6)$$

for every $f \in C_p$. Moreover, if $f \in C_p^1$, then there exists $M_1 = \text{const.} > 0$ such that

$$\omega_1(f; C_p; t) \leq M_1 \cdot t \quad \text{for } t \in R_0. \quad (7)$$

1.3. We introduce the following

Definition. Let $p \in N_0$ be a fixed number. For functions $f \in C_p$ we define the operators

$$\begin{aligned} L_n(f; x) &:= \frac{1}{(1 + (x + n^{-1})^2)^n} \\ &\times \sum_{k=0}^n \binom{n}{k} (x + n^{-1})^{2k} f\left(\frac{k}{n} \cdot \frac{1 + (x + n^{-1})^2}{x + n^{-1}}\right), \end{aligned} \quad (8)$$

$x \in R_0, n \in N$. □

Similarly as S_n , the operator L_n is linear and positive. In §2 we shall prove that L_n is an operator from the space C_p into C_p for every fixed $p \in N_0$.

For $t \in R_0$ and $n \in N$

$$(1 + t^2)^n = \sum_{k=0}^n \binom{n}{k} t^{2k}. \quad (9)$$

From (8) and (9) we derive the following formulas

$$L_n(1; x) = 1,$$

$$L_n(t; x) = x + n^{-1},$$

$$L_n(t^2; x) = (x + n^{-1})^2 \left[1 + \frac{n^{-1}}{(x + n^{-1})^2} \right],$$

$$L_n(t^3; x) = (x + n^{-1})^3 \left[1 + \frac{3n^{-1} - n^{-2}}{(x + n^{-1})^2} + \frac{n^{-2}}{(x + n^{-1})^4} \right],$$

$$L_n(t^4; x) = (x + n^{-1})^4 \left[1 + \frac{6n^{-1} - 4n^{-2} + n^{-3}}{(x + n^{-1})^2} + \frac{7n^{-2} - 4n^{-3}}{(x + n^{-1})^4} + \frac{n^{-3}}{(x + n^{-1})^6} \right], \quad (10)$$

for all $n \in N$ and $x \in R_0$.

2. Main results

2.1. From formulas (8), (9) and $L_n(t^k; x)$, $1 \leq k \leq 4$, given in the above we obtain

Lemma 1. *For all $x \in R_0$ and $n \in N$ we have*

$$\begin{aligned} L_n(t - x; x) &= n^{-1}, \\ L_n((t - x)^2; x) &= n^{-2} + n^{-1}, \\ L_n((t - x)^3; x) &= n^{-2} \left(3 - x + \frac{1}{x + n^{-1}} \right), \\ L_n((t - x)^4; x) &= n^{-2} \left(7 - \frac{4x}{x + n^{-1}} + \frac{n^{-1}}{(x + n^{-1})^2} \right. \\ &\quad \left. + n^{-1}(x^2 - 4x + 2) + n^{-2}(2x - 3) + n^{-3} \right). \quad \square \end{aligned}$$

Next we shall prove

Lemma 2. *Let $s \in N$ be a fixed number. Then there exist coefficients $\alpha_{s,j}(n)$, depending only on s, j, n and bounded with respect to n such that*

$$L_n(t^s; x) = (x + n^{-1})^s \sum_{j=1}^s \frac{\alpha_{s,j}(n)}{(x + n^{-1})^{2(j-1)}} \quad (11)$$

for all $n \in N$ and $x \in R_0$. Moreover, $\alpha_{s,1}(n) = 1$, $\alpha_{s,s}(n) = n^{1-s}$ and $\alpha_{s,j}(n) = O(1/n^{j-1})$ for $j = 1, 2, \dots, s$.

PROOF. We shall use mathematical induction for s . The formula (11) for $1 \leq s \leq 4$ is given in above. Let (11) hold for $f(x) = x^j$, $1 \leq j \leq s$, with fixed $s \in N$. We shall prove (11) for $f(x) = x^{s+1}$. From (8) and (9) it follows that

$$\begin{aligned}
L_n(t^{s+1}; x) &= \frac{x + n^{-1}}{n(1 + (x + n^{-1})^2)^{n-1}} \\
&\quad \times \sum_{j=0}^{n-1} \binom{n}{j} (n-j)(x + n^{-1})^{2j} \left(\frac{j+1}{n} \cdot \frac{1 + (x + n^{-1})^2}{x + n^{-1}} \right)^s \\
&= \frac{x + n^{-1}}{(1 + (x + n^{-1})^2)^{n-1}} \\
&\quad \times \sum_{j=0}^n \binom{n}{j} (x + n^{-1})^{2j} \left(\frac{j+1}{n} \cdot \frac{1 + (x + n^{-1})^2}{x + n^{-1}} \right)^s \\
&\quad - \frac{x + n^{-1}}{n(1 + (x + n^{-1})^2)^{n-1}} \\
&\quad \times \sum_{j=0}^n \binom{n}{j} (x + n^{-1})^{2j} j \left(\frac{j+1}{n} \cdot \frac{1 + (x + n^{-1})^2}{x + n^{-1}} \right)^s \\
&= \frac{x + n^{-1}}{(1 + (x + n^{-1})^2)^{n-1}} \\
&\quad \times \sum_{j=0}^n \binom{n}{j} (x + n^{-1})^{2j} \left(\frac{1 + (x + n^{-1})^2}{x + n^{-1}} \right)^s n^{-s} \sum_{\mu=0}^s \binom{s}{\mu} j^\mu \\
&\quad - \frac{x + n^{-1}}{n(1 + (x + n^{-1})^2)^{n-1}} \\
&\quad \times \sum_{j=0}^n \binom{n}{j} (x + n^{-1})^{2j} \left(\frac{1 + (x + n^{-1})^2}{x + n^{-1}} \right)^s n^{-s} \sum_{\mu=0}^s \binom{s}{\mu} j^{\mu+1} \\
&= (x + n^{-1})(1 + (x + n^{-1})^2) \\
&\quad \times \sum_{\mu=0}^s \binom{s}{\mu} \left(\frac{1 + (x + n^{-1})^2}{x + n^{-1}} \right)^{s-\mu} n^{\mu-s} L_n(t^\mu; x)
\end{aligned}$$

$$\begin{aligned}
& - (x + n^{-1})^2 L_n(t^{s+1}; x) - (x + n^{-1})^2 \\
& \times \sum_{\mu=0}^{s-1} \binom{s}{\mu} \left(\frac{1 + (x + n^{-1})^2}{x + n^{-1}} \right)^{s-\mu} n^{\mu-s} L_n(t^{\mu+1}; x).
\end{aligned}$$

Consequently

$$\begin{aligned}
L_n(t^{s+1}; x) &= (x + n^{-1}) \sum_{\mu=0}^s \binom{s}{\mu} \left(\frac{1 + (x + n^{-1})^2}{x + n^{-1}} \right)^{s-\mu} n^{\mu-s} L_n(t^{\mu}; x) \\
& - (x + n^{-1}) \sum_{\mu=0}^{s-1} \binom{s}{\mu} \left(\frac{1 + (x + n^{-1})^2}{x + n^{-1}} \right)^{s-\mu-1} n^{\mu-s} L_n(t^{\mu+1}; x).
\end{aligned}$$

From these we obtain

$$\begin{aligned}
L_n(t^{s+1}; x) &= (x + n^{-1}) \left(\frac{1 + (x + n^{-1})^2}{x + n^{-1}} \right)^s n^{-s} \\
& + (x + n^{-1}) \sum_{\mu=1}^s \binom{s}{\mu} \left(\frac{1 + (x + n^{-1})^2}{x + n^{-1}} \right)^{s-\mu} n^{\mu-s} L_n(t^{\mu}; x) \\
& - (x + n^{-1}) \sum_{\mu=1}^s \binom{s}{\mu-1} \left(\frac{1 + (x + n^{-1})^2}{x + n^{-1}} \right)^{s-\mu} n^{\mu-s-1} L_n(t^{\mu}; x).
\end{aligned}$$

By our assumption we get

$$\begin{aligned}
L_n(t^{s+1}; x) &= (x + n^{-1}) \left(\frac{1 + (x + n^{-1})^2}{x + n^{-1}} \right)^s n^{-s} \\
& + (x + n^{-1}) \sum_{\mu=1}^s \left\{ \binom{s}{\mu} n^{\mu-s} - \binom{s}{\mu-1} n^{\mu-s-1} \right\} \left(\frac{1 + (x + n^{-1})^2}{x + n^{-1}} \right)^{s-\mu} \\
& \times (x + n^{-1})^{\mu} \sum_{j=1}^{\mu} \frac{\alpha_{\mu,j}(n)}{(x + n^{-1})^{2(j-1)}} = (x + n^{-1})^{s+1} \left\{ \left(\frac{1 + (x + n^{-1})^2}{(x + n^{-1})^2} \right)^s n^{-s} \right. \\
& + \sum_{\mu=1}^s \sum_{j=1}^{\mu} \left\{ \binom{s}{\mu} n^{\mu-s} - \binom{s}{\mu-1} n^{\mu-s-1} \right\} \\
& \times \left. \left(\frac{1 + (x + n^{-1})^2}{(x + n^{-1})^2} \right)^{s-\mu} \frac{\alpha_{\mu,j}(n)}{(x + n^{-1})^{2(j-1)}} \right\}
\end{aligned}$$

$$\begin{aligned}
&= (x + n^{-1})^{s+1} \left\{ \left(\frac{1 + (x + n^{-1})^2}{(x + n^{-1})^2} \right)^s n^{-s} + \sum_{\mu=1}^s \sum_{j=1}^{\mu} \sum_{k=0}^{s-\mu} \binom{s-\mu}{k} \right. \\
&\quad \left. \times (x + n^{-1})^{2(\mu-s-j+k+1)} \left\{ \binom{s}{\mu} n^{\mu-s} - \binom{s}{\mu-1} n^{\mu-s-1} \right\} \alpha_{\mu,j}(n) \right\}.
\end{aligned}$$

Since $\alpha_{s,1}(n) = 1$, $\alpha_{s,s}(n) = n^{1-s}$ and $\alpha_{s,j}(n) = O(1/n^{j-1})$ for $j = 1, 2, \dots, s$, we have for $\mu = 1, 2, \dots, s$

$$\begin{aligned}
&\left\{ \binom{s}{\mu} n^{\mu-s} - \binom{s}{\mu-1} n^{\mu-s-1} \right\} \alpha_{\mu,j}(n) = O(1/n^{j+s-\mu-1}), \quad j=2, 3, \dots, \mu, \\
&\sum_{\mu=1}^s \left\{ \binom{s}{\mu} n^{\mu-s} - \binom{s}{\mu-1} n^{\mu-s-1} \right\} \alpha_{\mu,1}(n) = 1 - n^{-s}.
\end{aligned}$$

From the above and by elementary calculations we can write

$$\begin{aligned}
&\sum_{\mu=1}^s \sum_{j=1}^{\mu} \sum_{k=0}^{s-\mu} \binom{s-\mu}{k} (x + n^{-1})^{2(\mu-s-j+k+1)} \\
&\times \left\{ \binom{s}{\mu} n^{\mu-s} - \binom{s}{\mu-1} n^{\mu-s-1} \right\} \alpha_{\mu,j}(n) = 1 - n^{-s} + \sum_{\mu=2}^s \frac{\beta_{s,\mu}(n)}{(x + n^{-1})^{2(\mu-1)}},
\end{aligned}$$

where $\beta_{s,\mu}(n)$ are coefficients depending only on s, μ, n and bounded with respect to n and $\beta_{s,\mu}(n) = O(1/n^{\mu-1})$ for $\mu = 2, \dots, s$. Consequently we have

$$\begin{aligned}
L_n(t^{s+1}; x) &= (x + n^{-1})^{s+1} \left\{ \sum_{\mu=0}^s \frac{n^{-s}}{(x + n^{-1})^{2(s-\mu)}} \right. \\
&\quad \left. + 1 - n^{-s} + \sum_{\mu=2}^s \frac{\beta_{s,\mu}(n)}{(x + n^{-1})^{2(\mu-1)}} \right\} \\
&= (x + n^{-1})^{s+1} \left\{ 1 + \sum_{\mu=2}^s \frac{n^{-s}}{(x + n^{-1})^{2(s-\mu+1)}} \right. \\
&\quad \left. + \sum_{\mu=2}^s \frac{\beta_{s,\mu}(n)}{(x + n^{-1})^{2(\mu-1)}} + \frac{n^{-s}}{(x + n^{-1})^{2s}} \right\}
\end{aligned}$$

$$= (x + n^{-1})^{s+1} \sum_{\mu=1}^{s+1} \frac{\alpha_{s+1,\mu}(n)}{(x + n^{-1})^{2\mu-2}}$$

and $\alpha_{s+1,1}(n) = 1$, $\alpha_{s+1,s+1}(n) = n^{-s}$, $\alpha_{s+1,j}(n) = O(1/n^{j-1})$ for $j = 1, 2, \dots, s+1$, which proves (11) for $f(x) = x^{s+1}$. Hence the proof of (11) is completed. $\square$

Lemma 3. *Let $p \in N_0$ be a fixed number. Then there exists a positive constant $M_2 \equiv M_2(p)$, depending only on the parameter p such that*

$$\|L_n(1/w_p(t); \cdot)\|_p \leq M_2, \quad n \in N. \quad (12)$$

Moreover, for every $f \in C_p$ we have

$$\|L_n(f; \cdot)\|_p \leq M_2 \|f\|_p, \quad n \in N. \quad (13)$$

Formula (8) and inequality (13) show that L_n , $n \in N$, is a positive linear operator from the space C_p into C_p , for every $p \in N_0$.

PROOF. The inequality (12) is obvious for $p = 0$ by (2), (3) and (10). Let $p \in N$. By (2) and (8)–(11) we have

$$\begin{aligned} w_p(x) L_n(1/w_p(t); x) &= w_p(x) \{1 + L_n(t^p; x)\} \\ &= \frac{1}{1+x^p} + \frac{(x+n^{-1})^p}{1+x^p} \sum_{j=1}^p \frac{\alpha_{p,j}(n)}{(x+n^{-1})^{2j-2}}. \end{aligned}$$

For $x \in [1, +\infty)$, we get using Lemma 2

$$w_p(x) L_n(1/w_p(t); x) \leq 1 + \sum_{k=0}^p \binom{p}{k} \frac{x^{p-k}}{1+x^p} \sum_{j=1}^p \alpha_{p,j}(n) \leq M_2(p).$$

Let $x \in [0, 1)$ and

$$g(x) := (x + n^{-1})^{p+2-2j}. \quad (14)$$

We remark that g on $[0, 1)$ is an increasing function for $1 \leq j < (p+2)/2$ and a decreasing function for $(p+2)/2 < j \leq p$. From this we immediately obtain

$$\frac{\alpha_{p,j}(n)}{(x+n^{-1})^{2j-2-p}} \leq \frac{\alpha_{p,j}(n)}{(1+n^{-1})^{2j-2-p}} \leq \alpha_{p,j}(n), \quad 1 \leq j < (p+2)/2,$$

$$\frac{\alpha_{p,j}(n)}{(x+n^{-1})^{2j-2-p}} \leq \frac{\alpha_{p,j}(n)}{n^{-2j+2+p}} \leq \frac{n^{j-1}\alpha_{p,j}(n)}{n^{-j+1+p}}, \quad (p+2)/2 < j \leq p.$$

Applying Lemma 2, we get

$$w_p(x)L_n(1/w_p(t); x) \leq 1 + \sum_{j=1}^p \frac{\alpha_{p,j}(n)}{(x+n^{-1})^{2j-2-p}} \leq M_2(p)$$

for $x \in [0, 1]$, $n \in N$, where $M_2(p)$ is a positive constant depending only upon p . Therefore, the proof of inequality (12) is completed.

The formulas (8)–(9) and (2) imply

$$\|L_n(f(t); \cdot)\|_p \leq \|f\|_p \|L_n(1/w_p(t); \cdot)\|_p, \quad n \in N,$$

for every $f \in C_p$. Applying (12), we obtain (13). $\square$

Lemma 4. *Let $p \in N_0$ be a fixed number. Then there exists a positive constant $M_3 \equiv M_3(p)$ such that*

$$\left\| L_n \left(\frac{(t - \cdot)^2}{w_p(t)}; \cdot \right) \right\|_p \leq \frac{M_3}{n} \quad \text{for all } n \in N. \quad (15)$$

PROOF. The formulas given in Lemma 1 and (2), (3) imply (15) for $p = 0$.

By (2) and (10) we have

$$L_n((t-x)^2/w_p(t); x) = L_n((t-x)^2; x) + L_n(t^p(t-x)^2; x),$$

for $p, n \in N$. If $p = 1$, then by the equality we get

$$\begin{aligned} L_n((t-x)^2/w_1(t); x) &= L_n((t-x)^2; x) + L_n(t(t-x)^2; x) \\ &= L_n((t-x)^3; x) + (1+x)L_n((t-x)^2; x), \end{aligned}$$

which by (2), (3) and Lemma 1 yields (15) for $p = 1$.

Let $p \geq 2$. Applying Lemma 2, we get

$$w_p(x)L_n(t^p(t-x)^2; x) = w_p(x) \{ L_n(t^{p+2}; x) - 2xL_n(t^{p+1}; x) + x^2L_n(t^p; x) \}$$

$$\begin{aligned}
&= w_p(x) \left\{ (x + n^{-1})^{p+2} \sum_{j=1}^{p+2} \frac{\alpha_{p+2,j}(n)}{(x + n^{-1})^{2(j-1)}} - 2x(x + n^{-1})^{p+1} \right. \\
&\quad \times \sum_{j=1}^{p+1} \frac{\alpha_{p+1,j}(n)}{(x + n^{-1})^{2(j-1)}} + x^2(x + n^{-1})^p \sum_{j=1}^p \frac{\alpha_{p,j}(n)}{(x + n^{-1})^{2(j-1)}} \left. \right\} \\
&= w_p(x)(x + n^{-1})^p \left\{ n^{-2} + (x + n^{-1})^2 \sum_{j=2}^{p+2} \frac{\alpha_{p+2,j}(n)}{(x + n^{-1})^{2(j-1)}} \right. \\
&\quad \left. + 2x(x + n^{-1}) \sum_{j=2}^{p+1} \frac{\alpha_{p+1,j}(n)}{(x + n^{-1})^{2(j-1)}} + x^2 \sum_{j=2}^p \frac{\alpha_{p,j}(n)}{(x + n^{-1})^{2(j-1)}} \right\},
\end{aligned}$$

which by (2) and Lemma 2 implies for $x \in [1, +\infty)$

$$\begin{aligned}
w_p(x)L_n(t^p(t-x)^2; x) &\leq n^{-1} \frac{(1+x)^p}{1+x^p} \left\{ 1 + \sum_{j=2}^{p+2} \frac{n\alpha_{p+2,j}(n)}{(x + n^{-1})^{2(j-2)}} \right. \\
&\quad \left. + 2 \sum_{j=2}^{p+1} \frac{n\alpha_{p+1,j}(n)}{(x + n^{-1})^{2(j-2)}} + \sum_{j=2}^p \frac{n\alpha_{p,j}(n)}{(x + n^{-1})^{2(j-2)}} \right\} \leq \frac{M_4(p)}{n}, \quad n \in N.
\end{aligned}$$

Let $x \in [0, 1)$. Applying Lemma 2 and arguing as in the proof of Lemma 3, we easily obtain

$$w_p(x)L_n(t^p(t-x)^2; x) \leq \frac{M_4(p)}{n}, \quad n \in N.$$

Thus the proof is completed. $\square$

2.2. Now we shall give approximation theorems for L_n .

Theorem 1. *Let $p \in N_0$ be a fixed number. Then there exists a positive constant $M_5 \equiv M_5(p)$ such that for every $f \in C_p^1$ we have*

$$\|L_n(f; \cdot) - f(\cdot)\|_p \leq \frac{M_5}{\sqrt{n}} \|f'\|_p, \quad n \in N. \quad (16)$$

PROOF. Let $x \in R_0$ be a fixed point. Then for $f \in C_p^1$ we have

$$f(t) - f(x) = \int_x^t f'(u) du, \quad t \in R_0.$$

From this and by (8) and (10) we get

$$L_n(f(t); x) - f(x) = L_n\left(\int_x^t f'(u) du; x\right), \quad n \in N.$$

But by (2) and (3) we have

$$\left| \int_x^t f'(u) du \right| \leq \|f'\|_p \left(\frac{1}{w_p(t)} + \frac{1}{w_p(x)} \right) |t - x|, \quad t \in R_0,$$

which implies

$$\begin{aligned} & w_p(x) |L_n(f; x) - f(x)| \\ & \leq \|f'\|_p \left\{ L_n(|t - x|; x) + w_p(x) L_n\left(\frac{|t - x|}{w_p(t)}; x\right) \right\} \end{aligned} \quad (17)$$

for $n \in N$. By the Hölder inequality and by (10) and Lemmas 1, 3, 4 it follows that

$$\begin{aligned} & L_n(|t - x|; x) \leq \{L_n((t - x)^2; x) L_n(1; x)\}^{1/2} \leq \sqrt{\frac{2}{n}}, \\ & w_p(x) L_n\left(\frac{|t - x|}{w_p(t)}; x\right) \\ & \leq w_p(x) \left\{ L_n\left(\frac{(t - x)^2}{w_p(t)}; x\right) \right\}^{1/2} \left\{ L_n\left(\frac{1}{w_p(t)}; x\right) \right\}^{1/2} \leq \frac{M_7(p)}{\sqrt{n}} \end{aligned}$$

for $n \in N$. From this and by (17) we immediately obtain (16). $\square$

Theorem 2. Let $p \in N_0$ be a fixed number. Then there exists $M_8 \equiv M_8(p)$ such that for every $f \in C_p$ and $n \in N$ we have

$$\|L_n(f; \cdot) - f(\cdot)\|_p \leq M_8 \omega_1\left(f; C_p; \frac{1}{\sqrt{n}}\right). \quad (18)$$

PROOF. We use the Steklov function f_h of $f \in C_p$

$$f_h(x) := \frac{1}{h} \int_0^h f(x+t)dt, \quad x \in R_0, \quad h > 0. \quad (19)$$

From (19) we get

$$f_h(x) - f(x) = \frac{1}{h} \int_0^h \Delta_t f(x)dt,$$

$$f'_h(x) = \frac{1}{h} \Delta_h f(x), \quad x \in R_0, \quad h > 0,$$

which imply

$$\|f_h - f\|_p \leq \omega_1(f; C_p; h), \quad (20)$$

$$\|f'_h\|_p \leq h^{-1} \omega(f; C_p; h), \quad (21)$$

for $h > 0$. From this we deduce that $f_h \in C_p^1$ if $f \in C_p$ and $h > 0$. Hence we can write

$$\begin{aligned} w_p(x) |L_n(f; x) - f(x)| &\leq w_p(x) \{ |L_n(f - f_h; x)| + |L_n(f_h; x) - f_h(x)| \\ &\quad + |f_h(x) - f(x)| \} := A_1(x) + A_2(x) + A_3(x), \end{aligned}$$

for $n \in N$, $h > 0$ and $x \in R_0$. From (13) and (20) we get

$$\|A_1\|_p \leq M_2 \|f_h - f\|_p \leq M_2 \omega_1(f; C_p; h),$$

$$\|A_3\|_p \leq \omega_1(f; C_p; h).$$

By Theorem 1 and (21) it follows that

$$\|A_2\|_p \leq \frac{M_5}{\sqrt{n}} \|f'_h\|_p \leq \frac{M_5}{\sqrt{nh}} \omega_1(f; C_p; h).$$

Consequently

$$\|L_n(f; \cdot) - f(\cdot)\|_p \leq \left(1 + M_2 + \frac{M_5}{\sqrt{nh}}\right) \omega_1(f; C_p; h).$$

Now, for fixed $n \in N$, setting $h = \frac{1}{\sqrt{n}}$, we obtain

$$\|L_n(f; \cdot) - f(\cdot)\|_p \leq M_8(p) \omega_1\left(f; C_p; \frac{1}{\sqrt{n}}\right)$$

and we complete the proof. $\square$

From Theorem 1 and Theorem 2 we derive following two corollaries:

Corollary 1. For $f \in C_p$, $p \in N_0$, we have

$$\lim_{n \rightarrow \infty} \|L_n(f; \cdot) - f(\cdot)\|_p = 0. \quad \square$$

Corollary 2. If $f \in C_p^1$, $p \in N_0$, then

$$\|L_n(f; \cdot) - f(\cdot)\|_p = O(1/\sqrt{n}). \quad \square$$

2.3. Finally, we shall give the Voronovskaya type theorem for L_n .

Theorem 3. Let $f \in C_p^2 := \{f \in C_p : f', f'' \in C_p\}$. Then

$$\lim_{n \rightarrow \infty} n \{L_n(f; x) - f(x)\} = f'(x) + \frac{1}{2}f''(x) \quad (22)$$

for every $x > 0$.

PROOF. Let $x > 0$ be a fixed point. Then by the Taylor formula we have

$$f(t) = f(x) + f'(x)(t - x) + \frac{1}{2}f''(x)(t - x)^2 + \varepsilon(t; x)(t - x)^2$$

for $t \in R_0$, where $\varepsilon(t) \equiv \varepsilon(t; x)$ is a function belonging to C_p and $\varepsilon(x) = 0$. Hence by (8) and (10) we get

$$\begin{aligned} L_n(f; x) &= f(x) + f'(x)L_n(t - x; x) + \frac{1}{2}f''(x)L_n((t - x)^2; x) \\ &\quad + L_n(\varepsilon(t)(t - x)^2; x), \quad n \in N, \end{aligned} \quad (23)$$

which by Lemma 1 yields

$$\begin{aligned} &\lim_{n \rightarrow \infty} n \{L_n(f; x) - f(x)\} \\ &= f'(x) + \frac{1}{2}f''(x) + \lim_{n \rightarrow \infty} n L_n(\varepsilon(t)(t - x)^2; x). \end{aligned} \quad (24)$$

By the Hölder inequality we have

$$|L_n(\varepsilon(t)(t - x)^2; x)| \leq \{L_n(\varepsilon^2(t); x)\}^{1/2} \{L_n((t - x)^4; x)\}^{1/2}.$$

The properties of ε and Corollary 1 imply that

$$\lim_{n \rightarrow \infty} L_n(\varepsilon^2(t); x) = \varepsilon^2(x) = 0.$$

From this and by Lemma 1 we get

$$\lim_{n \rightarrow \infty} nL_n(\varepsilon(t)(t-x)^2; x) = 0$$

and (22) follows from (24). $\square$

Remark. In [1] it was proved that if $f \in C_p$, $p \in N_0$, then for the Szasz–Mirakyan operators S_n (defined by (1)) one has the following inequality

$$w_p(x)|S_n(f; x) - f(x)| \leq M_9 \omega_2 \left(f; C_p; \sqrt{\frac{x}{n}} \right), \quad x \in R_0, \quad n \in N_0,$$

where $M_9 = \text{const.} > 0$ and $\omega_2(f; \cdot)$ is the modulus of smoothness defined by the formula

$$\omega_2(f; C_p; t) := \sup_{0 \leq h \leq t} \|\Delta_h^2 f(\cdot)\|_p, \quad t \in R_0,$$

where $\Delta_h^2 f(x) := f(x) - 2f(x+h) + f(x+2h)$. In particular, if $f \in C_p^1$, $p \in N_0$, then

$$w_p(x)|S_n(f; x) - f(x)| \leq M_{10} \sqrt{\frac{x}{n}}, \quad (25)$$

for $x \in R_0$ and $n \in N$ ($M_{10} = \text{const.} > 0$).

Theorem 1, Theorem 2 and Corollary 2 in our paper show that the operators L_n , $n \in N$, give better degree of approximation of functions $f \in C_p$ and $f \in C_p^1$ than S_n .

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