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On affine translation hypersurfaces of constant mean curvature

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Abstract. The purpose of this paper is to classify the affine translation hypersurfaces with nonzero constant mean curvature.

1. Introduction

An n-dimensional hypersurface in E^{n+1} is called a translation hypersurface if it is obtained as the graph of the function $F(x_1,\dots,x_n) = f_1(x_1)+\dots+f_n(x_n)$, where $f_1(x_1),\dots,f_n(x_n)$ are differentiable functions. A hypersurface is said to be minimal if its mean curvature is zero identically. As well known, a minimal translation surface in the 3-dimensional Euclidean space E^3 must be a plane or a surface which is the graph of the function $F(x_1,x_2) = \frac{1}{a}(\ln\cos(ax_1) - \ln\cos(ax_2))$, where a is a nonzero constant. For a translation hypersurface M with constant mean curvature in E^{n+1}, we have obtained [2]

(1) when M is minimal, then $F(x_1,\dots,x_n)$ is a linear function or

$$F(x_1,\dots,x_n) = \frac{1}{a}\ln\frac{\cos(ax_1)}{\cos(ax_2)\dots\cos(ax_k)} + c_{k+1}x_{k+1} + \dots + c_n x_n,$$

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(2) when M is not minimal, then

$$F(x_1, \dots, x_n) = -\frac{\sqrt{1+a^2}}{2H}\sqrt{1-4H^2x_1^2} + ax_2 + a_3x_3 + \cdots + a_nx_n,$$

where a, $c_{k+1}, \dots, c_n$, $a_3, \dots, a_n$ are constant. Naturally, one can consider the similar problem: to classify affine translation hypersufaces in the affine space R^{n+1}. Of course, the case of affine hypersurfaces in R^{n+1} is more complicated than that of hypersurfaces in E^{n+1}. In [7], F. Manhart studied nondegenerate affine minimal translation surfaces in an affine space R^3 and gave a complete classification.

As a generalization, W. Yang and D. Qiu [11] classified affine minimal translation hypersurfaces in R^{n+1}.

In [10] and [9], the first author of this paper and H. Pabel classified the affine translation surfaces with nonzero constant mean curvature in R^3, respectively.

The purpose of the present paper is to classify affine translation hypersurfaces in R^{n+1} with nonzero constant mean curvature. Our main result is:

Theorem. *Let M be an n-dimensional nondegenerate affine translation hypersurface with nonzero constant mean curvature in R^{n+1}. Then up to affine transformations, M is the graph of the following function:*

$$x_{n+1} = \alpha \int_{x_0}^{x_1} \left\{ \int_{t_0}^{t} \left(Hs^2 + a_1\right)^{-\frac{n+2}{n+1}} ds \right\} dt + a_2x_2^2 + \cdots + a_nx_n^2,$$

where $\alpha, a_1, \dots, a_n$ are constant and $H(\neq 0)$ is the affine mean curvature of M in R^{n+1}.

2. Preliminaries

Let $f : M \to R^{n+1}$ be an immersion of a connected differentiable n-manifold M into the affine space R^{n+1} equipped with usual flat connection D and a parallel volume element ω, and let ξ be an arbitrary local field of transversal vector to $f(M)$. For any vector fields X, Y on M, we write

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi, \tag{2.1}$$

$$D_X\xi = -f_*(SX) + \tau(X)\xi, \tag{2.2}$$

Thus we have an affine connection ∇, a symmetric tensor h of type $(0,2)$, a tensor S of type (1.1) and 1-form τ on M. We call h, S and τ the affine fundamental form, the affine shape operator and the transversal connection form, respectively. We define by $H = \frac{1}{n}\operatorname{trace} S$ the affine mean curvature of M. We call M affine minimal if H is zero identically. We define a volume element θ on M by

$$\begin{aligned}\theta(X_1,\dots,X_n) &= \omega(f_*(X_1),\dots,f_*(X_n),\xi)\\ &= \det(f_*(X_1),\dots,f_*(X_n),\xi),\end{aligned} \tag{2.3}$$

for any tangent vectors $X_1,\dots,X_n$ to M.

We say that f is nondegenerate if h is nondegenerate. This nondegeneracy does not depend on the choice of ξ. If f is nondegenerate, it is known that there is a unique ξ up to sign such that the corresponding induced connection ∇, the affine fundamental form h,and the induced volume element θ satisfy

(i) $\nabla\theta = 0$, thus (∇, θ) is an equiaffine structure on M.

(ii) $\theta = \omega_h$, $\omega_h(X_1,\dots,X_n) = |\det(h(X_i,X_j))|^{\frac{1}{2}}$ (volume element given by h).

We call such ξ the affine normal of f. Condition (i) implies that $\tau = 0$ so that $D_X\xi = -f_*(SX)$.

Let $x_{n+1} = F(x_1,\dots,x_n)$ be a differentiable function on a domain $D \subset R^n$. We shall determine the affine normal of an immersion

$$f : D \ni (x_1,\dots,x_n) \mapsto (x_1,\dots,x_n,F(x_1,\dots,x_n)) \in R^{n+1}.$$

We start with a tentative choice of transversal field $\xi = (0,\dots,0,1)$. Since $D_{\partial_i}\xi = 0$, we have $\tau = 0$. Denote by ∂_j the coordinate vector field $\partial/\partial x_j$. Then we have

$$f_*(\partial_1) = (1,0,\dots,0,F_1),\ \dots,\ f_*(\partial_n) = (0,\dots,0,1,F_n),$$

where $F_j = \partial F/\partial x_j$. Thus we get

$$D_{\partial_i} f_*(\partial_j) = (0,\dots,0,F_{ij}) = F_{ij}\xi, \quad F_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j},$$

and

$$\nabla_{\partial_i}(\partial_j) = 0, \quad h(\partial_i, \partial_j) = F_{ij}.$$

Thus the immersion is nondegenerate if and only if $\det(F_{ij}) \neq 0$. We find that

$$\theta(\partial_1, \ldots, \partial_n) = \det(f_*(\partial_1), \ldots, f_*(\partial_n), \xi) = 1.$$

Hence taking $\phi = |\det(F_{ij})|^{\frac{1}{n+2}}$, we can find Z such that $\phi\xi + Z = \bar{\xi}$ is the affine normal field and $\bar{\xi}$ is given by

$$\bar{\xi} = -\sum_{j,k} \left(F^{kj} \phi_j \right) f_*(\partial_k) + \phi\xi,$$

where $\phi_j = \partial\phi/\partial x_j$, (F^{ij}) is the inverse of the matrix (F^{ij}). From which we have

$$D_{\partial_i}\bar{\xi} = -\sum_{j,k} \partial_i \left(F^{kj} \phi_j \right) f_*(\partial_k)$$

and

$$S(\partial_i) = \sum_{j,k} \partial_i \left(F^{kj} \phi_j \right) \partial_k.$$

Hence we see that the affine mean curvature of M satisfies

$$H = \frac{1}{n} \sum_{i,j} \partial_i \left(F^{ij} \phi_j \right). \tag{2.4}$$

3. Proof of the main theorem

Throughout this section, we assume that M is a translation hypersurface, i.e., it is obtained as the graph of function $F(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_n(x_n)$, where $f_1, \ldots, f_n$ are differentiable functions. Thus we have

$$(F_{ij}) = \begin{pmatrix} f_1''(x_1) & & 0 \\ & \ddots & \\ 0 & & f_n''(x_n) \end{pmatrix},$$

$$(F^{ij}) = (F_{ij})^{-1} = \begin{pmatrix} (f_1''(x_1))^{-1} & & 0 \\ & \ddots & \\ 0 & & (f_n''(x_n))^{-1} \end{pmatrix},$$

and from the nondegeneracy assumption, we have

$$\phi = |\det(F_{ij})|^{\frac{1}{n+2}} = \left|f_1''(x_1)\dots f_n''(x_n)\right|^{\frac{1}{n+2}}$$

never vanishes, so the f_i'''s never vanish either. Let $G_i = \varepsilon_i f_i$ satisfy $G_i'' = \varepsilon_i f_i'' = |f_i''|$, where $\varepsilon_i = 1$ when $f_i'' > 0$ and $\varepsilon_i = -1$ when $f_i'' < 0$. Thus we get

$$\phi_i = \frac{1}{n+2} G_i''^{-1} G_i'''(G_1''\dots G_n'')^{\frac{1}{n+2}}$$

and

$$\phi_{ii} = \frac{1}{n+2}(G_1''\dots G_n'')^{\frac{1}{n+2}}\left(-\frac{n+1}{n+2}G_i''^{-2}G_i'''^{2} + G_i''^{-1}G_i^{(4)}\right).$$

Therefore, by a direct calculation we have

$$\begin{aligned}
nH &= \sum_i \partial_i\left(F^{ii}\phi_i\right) = \sum_i \left((\partial_i F^{ii})\phi_i + F^{ii}\phi_{ii}\right)\\
&= \frac{1}{n+2}(G_1''\dots G_n'')^{\frac{1}{n+2}}\\
&\quad \cdot \sum_i\left(-\frac{f_i'''}{f_i''^{2}}G_i'''G_i''^{-1} - \frac{n+1}{n+2}f_i''^{-1}G_i''^{-2}G_i'''^{2} + f_i''^{-1}G_i''^{-1}G_i^{(4)}\right)\\
&= \frac{1}{n+2}\left(G_1''\dots G_n''\right)^{\frac{1}{n+2}}\sum_i\left(-\frac{\varepsilon_i G_i'''^{2}}{G_i''^{3}} - \frac{n+1}{n+2}\frac{\varepsilon_i G_i'''^{2}}{G_i''^{3}} + \frac{\varepsilon_i G_i^{(4)}}{G_i''^{2}}\right)\\
&= \frac{1}{n+2}\left(G_1''\dots G_n''\right)^{\frac{1}{n+2}}\sum_i\left(-\frac{2n+3}{n+2}\frac{\varepsilon_i G_i'''^{2}}{G_i''^{3}} + \frac{\varepsilon_i G_i^{(4)}}{G_i''^{2}}\right)\\
&= \frac{1}{n+2}(G_1''\dots G_n'')^{\frac{1}{n+2}}\sum_i\frac{\varepsilon_i}{G_i''^{3}}\left(-\frac{2n+3}{n+2}G_i'''^{2} + G_i''G_i^{(4)}\right)\\
&= (G_2''\dots G_n'')^{\frac{1}{n+2}}\varepsilon_1 G_1''^{-\frac{3n+5}{n+2}}\left(-\frac{2n+3}{(n+2)^2}G_1'''^{2} + \frac{1}{n+2}G_1''G_1^{(4)}\right)\\
&\quad + (G_1''G_3''\dots G_n'')^{\frac{1}{n+2}}\varepsilon_2 G_2''^{-\frac{3n+5}{n+2}}\left(-\frac{2n+3}{(n+2)^2}G_2'''^{2} + \frac{1}{n+2}G_2''G_2^{(4)}\right)\\
&\quad + \dots\\
&\quad + (G_1''\dots G_{n-1}'')^{\frac{1}{n+2}}\varepsilon_n G_n''^{-\frac{3n+5}{n+2}}\left(-\frac{2n+3}{(n+2)^2}G_n'''^{2} + \frac{1}{n+2}G_n''G_n^{(4)}\right)
\end{aligned}$$

so we have

$$nH = (G_2'' \dots G_n'')^{\frac{1}{n+2}} Q_1(x_1) + \dots + (G_1'' \dots G_{n-1}'')^{\frac{1}{n+2}} Q_n(x_n), \tag{3.1}$$

where

$$Q_i(x_i) = \varepsilon_i G_i^{''-\frac{3n+5}{n+2}} \left(-\frac{2n+3}{(n+2)^2} G_i^{'''2} + \frac{1}{n+2} G_i'' G_i^{(4)} \right). \tag{3.2}$$

In order to prove our Theorem, we need the following Lemma.

Lemma. *If H is a non-zero constant, then there exists i $(i = 1, \dots, n)$ such that $G_i''' \neq 0$ and $G_j''' = 0$ for $j \neq i$.*

Proof. We first prove that if all $G_i''' \neq 0$, then $H = 0$. In fact, differentiating (3.1) with respect to x_i we get

$$\begin{aligned} 0 &= (G_1'' \dots G_{i-1}'' G_{i+1}'' \dots G_n'')^{\frac{1}{n+2}} Q_i'(x_i) \\ &+ \frac{1}{n+2} G_i^{''\frac{1}{n+2}-1} G_i''' \Big[(G_2'' \dots G_{i-1}'' G_{i+1}'' \dots G_n'')^{\frac{1}{n+2}} Q_1(x_1) \\ &+ \dots + (G_1'' \dots G_{i-1}'' G_{i+1}'' \dots G_{n-1}'')^{\frac{1}{n+2}} Q_n(x_n) \Big] \end{aligned}$$

from which we get

$$\begin{aligned} -\frac{(n+2)Q_i'(x_i)}{G_i^{''-\frac{n+1}{n+2}} G_i'''} &= \frac{Q_1(x_1)}{G_1^{''\frac{1}{n+2}}} + \dots + \frac{Q_{i-1}(x_{i-1})}{G_{i-1}^{''\frac{1}{n+2}}} \\ &+ \frac{Q_{i+1}(x_{i+1})}{G_{i+1}^{''\frac{1}{n+2}}} + \dots + \frac{Q_n(x_n)}{G_n^{''\frac{1}{n+2}}}. \end{aligned} \tag{3.3}$$

Differentiating (3.3) with respect to x_j, we can get easily that

$$\left(\frac{Q_j(x_j)}{G_j^{''\frac{1}{n+2}}} \right)'_{x_j} = 0, \qquad j \neq i.$$

In the other hand, changing i to k $(k \neq i)$ in (3.3) and differentiating with respect to x_j, we get

$$\left(\frac{Q_j(x_j)}{G_j^{''\frac{1}{n+2}}} \right)'_{x_j} = 0, \qquad j \neq k.$$

From the above two formulas, we can get

$$\left(\frac{Q_j(x_j)}{G_j''^{\frac{1}{n+2}}}\right)' = 0 \qquad j = 1, \dots, n. \tag{3.4}$$

And so

$$Q_j(x_j) = c_j G_j''^{\frac{1}{n+2}}, \qquad (j = 1, \dots, n), \tag{3.5}$$

where c_j 's are constant. From (3.5) we get

$$Q_j'(x_j) = \frac{1}{n+2} c_j G_j''^{-\frac{n+1}{n+2}} G_j'''. \tag{3.6}$$

Therefore, combining (3.3), (3.5) with (3.6) we can get

$$c_1 + \cdots + c_n = 0. \tag{3.7}$$

Combining (3.1), (3.5) with (3.7) we get

$$nH = (G_1'' \dots G_n'')^{\frac{1}{n+2}} (c_1 + \cdots + c_n) = 0,$$

and so $H = 0$.

Then we assume that $G_n'''(x_n) = 0$. In this case, $G_n'' = d_n = \text{constant}$ and $Q_n(x_n) = 0$. From (3.1) we get

$$\begin{aligned} nH d_n^{-\frac{1}{n+2}} &= (G_2'' \dots G_{n-1}'')^{\frac{1}{n+2}} Q_1(x_1) \\ &\quad + \cdots + \left(G_1'' \dots G_{n-2}''\right)^{\frac{1}{n+2}} Q_{n-1}(x_{n-1}). \end{aligned} \tag{3.8}$$

If $G_i'''(x_i) \neq 0$ $(i = 1, \dots, n-1)$, using the same method as above we can get $nH d_n^{-\frac{1}{n+2}} = 0$ and so $H = 0$. Thus continuing such process we can get

$$G_2'' = d_2, \dots, \; G_n'' = d_n$$

and

$$nH = (d_2 \dots d_n)^{\frac{1}{n+2}} Q_1(x_1)$$

i.e.

$$nH = (d_2 \dots d_n)^{\frac{1}{n+2}} \varepsilon_1 G_1''^{-\frac{3n+5}{n+2}} \left(-\frac{2n+3}{(n+2)^2} G_1'''^2 + \frac{1}{n+2} G_1'' G_1^{(4)}\right), \tag{3.9}$$

where $d_2, \ldots, d_n$ are constant and $G_1''' \neq 0$ so that $H \neq 0$. This completes the proof of Lemma.

Now we begin the proof of Theorem. Affine minimal translation hypersurfaces, that is the case $H = 0$ are classified in [7], [11]. Then by Lemma, we only need to treat with the case that $G_2''' = \cdots = G_n''' = 0$ and $G_1''' \neq 0$. Hence from (3.9) we get

$$HCG_1''^{3-\frac{1}{n+2}} = -\frac{2n+3}{n+2}G_1'''^2 + G_1''G_1^{(4)}, \tag{3.10}$$

where

$$C = n(n+2)\varepsilon_1(d_2 \ldots d_n)^{-\frac{1}{n+2}}.$$

Let $g(x_1) = G_1''(x_1)$ and $s = g'$. Then from (3.10) we have

$$g'' - \frac{2n+3}{n+2}\frac{1}{g}g'^2 = CHg^{\frac{2n+3}{n+2}}$$

and so

$$\frac{ds^2}{dg} - \frac{2(2n+3)}{n+2}\frac{1}{g}s^2 = 2CHg^{\frac{2n+3}{n+2}}. \tag{3.11}$$

Thus we get

$$s^2 = g^{\frac{2(2n+3)}{n+2}}\left(-\frac{2(n+2)}{n+1}CHg^{-\frac{n+1}{n+2}} + d\right),$$

where d is a constant. Then we have

$$g' = s = \pm g^{\frac{2n+3}{n+2}}\left(-\frac{2(n+2)}{n+1}CHg^{-\frac{n+1}{n+2}} + d\right)^{\frac{1}{2}}.$$

Let $g^{-\frac{n+1}{n+2}} = m$. Then

$$\pm\frac{n+1}{n+2}(am+d)^{\frac{1}{2}}dx_1 = dm, \tag{3.12}$$

where $a = -\frac{2(n+2)}{n+1}CH$.

From (3.12) we get

$$\frac{2}{a}(am+d)^{\frac{1}{2}} = \pm\frac{n+1}{n+2}x_1 + e$$

and

$$g = m^{-\frac{n+2}{n+1}} = \left[\frac{a}{4}\left(\frac{n+1}{n+2}x_1 \pm e\right)^2 - \frac{d}{a}\right]^{-\frac{n+2}{n+1}},$$

i.e.

$$g = \left[-\frac{n+2}{2(n+1)}CH\left(\frac{n+1}{n+2}x_1 \pm e\right)^2 + \frac{(n+1)d}{2(n+2)CH}\right]^{-\frac{n+2}{n+1}}, \tag{3.13}$$

where e is a constant. So we get

$$\begin{aligned} G_1(x_1) &= \int_{x_0}^{x_1}\left\{\int_{t_0}^{t}\left[AH\left(\frac{n+1}{n+2}s \pm B\right)^2 + a_1\right]^{-\frac{n+2}{n+1}} ds\right\} dt \\ &= \left[A\left(\frac{n+1}{n+2}\right)^2\right]^{-\frac{n+2}{n+1}} \int_{x_0}^{x_1}\left\{\int_{t_0}^{t}\left(Hs^2 + a_1\right)^{-\frac{n+2}{n+1}} ds\right\} dt, \end{aligned}$$

where

$$A = -\frac{(n+2)C}{2(n+1)}, \quad B = e, \quad a_1 = \frac{(n+1)d}{2(n+2)CH}.$$

Therefore, under equiaffine translation we get

$$x_{n+1} = \alpha\int_{x_0}^{x_1}\left\{\int_{t_0}^{t}\left(Hs^2 + a_1\right)^{-\frac{n+2}{n+1}} ds\right\} dt + a_2x_2^2 + \cdots + a_nx_n^2, \tag{3.14}$$

where $\alpha, a_2, \ldots, a_n$ are constant.

This completes the proof of Theorem. □

In particular, taking $a_1 = 0$ from (3.14) we get

$$x_{n+1} = A_1(x_1 + B_1)^{-\frac{2}{n+1}} + A_2x_1 + B_2 + a_2x_2^2 + \cdots + a_nx_n^2, \tag{3.15}$$

where A_i, B_i, a_i are constant.

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References

[1] W. Blaschke, Vorlesungen Uber Differentialgeometrie II, *Berlin*, 1923.

[2] C. Chen and H. Sun, On translation hypersurfaces with constant mean curvature in $(n+1)$-dimentional spaces, *J. Beijing Institute of Technology* **12**, no. 3 (2003) (*to appear*).

[3] F. Dillen, A. Martinez, F. Milan, F. G. Santos and L. Vrancken, On the pick invariant, the affine mean curvature and the Gauss curvature of affine surfaces, *Results in Mathematics* **20** (1991), 622–642.

[4] A. M. Li, U. Simon and G. Zhao, Global Affine Differential Geometry of Hypersurfaces, *W. De Gruyer*, *Berlin and New York*, 1993.

[5] H. Liu, Translation surfaces with constant mean curvature in 3-dimensional space, *J. Geom.* **64** (1999), 141–149.

[6] M. A. Magid, Timelike Thomsen surfaces, *Results in Mathematics* **20** (1991), 691–697.

[7] F. Manhart, Die affineminimalruckungfachen, *Arch. Math.* **44** (1985), 547–556.

[8] K. Nomizu and T. Sasaki, Affine Differential Geometry, *Cambridge University Press*, *Cambridge, New York*, 1994.

[9] H. Pabel, Translationsflachen in der Aquiaffinen Differentialgeometrie, *J. of Geom.* **40** (1991), 148–164.

[10] H. Sun, On affine translation surfaces of constant mean curvature, *Kumamoto J. Math.* **13** (2000), 49–57.

[11] W. Yang and D. Qiu, On affine minimal translation hypersurfaces in A^{n+1}, *J. Math. (PRC)* **12** (1992), 27–33.

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