

The property of smallness up to a complemented Banach subspace

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Abstract. This article investigates locally convex spaces which satisfy the property of smallness up to a complemented Banach subspace, the *SCBS* property, which was introduced by Djakov, Terzioğlu, Yurdakul and Zahariuta. It is proved that a bounded perturbation of an automorphism on a complete barrelled locally convex space with the *SCBS* is stable up to a Banach subspace. New examples are given, and the relation of the *SCBS* with the quasinormability is analyzed. It is proved that the Fréchet space l_{p+} does not satisfy the *SCBS*, therefore this property is not inherited by subspaces or separated quotients.

1. Introduction

Our terminology for locally convex spaces is standard and we refer the reader to [11] or [8]. For a locally convex space X , $U(X)$ denotes a basis of absolutely convex neighborhoods of the origin in X and for $U \in U(X)$, p_U is the gauge of U . A linear operator T from a locally convex space X into another Y is bounded if $T(U)$ is a bounded subset of Y for some $U \in U(X)$.

Mathematics Subject Classification: 46A03, 46A45.

Key words and phrases: The *SCBS* property, the conditions (QN) , (AN) , l -Köthe spaces, the space l_{p+} , bounded factorization property, Douady's lemma, complemented Banach subspaces.

The first author thanks TŰBÍTAK for the financial support and the second thanks V. Zahariuta, P. Djakov for the useful discussions about the condition (QN) . We are grateful to our referee for his (her) very useful comments and suggestions.

We say that a pair (X, Y) has *the bounded factorization property* and write $(X, Y) \in \mathcal{BF}$ if each continuous linear operator from X into X that factors over Y is bounded.

By GROTHENDIECK ([7], p. 107), a locally convex space X is said to be *quasi-normable* if for each $U \in U(X)$ there is $V \in U(X)$ so that for every $\epsilon > 0$ there exists a bounded subset A of X such that $V \subset A + \epsilon U$.

Following [5], we say that a locally convex space X satisfies *the property of smallness up to a complemented Banach subspace*, abbreviated from now on as the *SCBS* property, if for each bounded subset A of X , for each $U \in U(X)$, and for every $\epsilon > 0$, there are complementary subspaces B and E of X such that B is a Banach space and $A \subset B + \epsilon U \cap E$.

In [5], it was proved that all Banach-valued l_p -Köthe spaces have the *SCBS* property and the bounded perturbation of an automorphism on an l_p -Köthe space is stable up to some Banach basic subspace. This was essential there to get a modification of the generalized Douady lemma [16].

We still have no complete characterization of Fréchet or *DF*-spaces with the *SCBS* property. In this work we analyze the *SCBS* property, and show that more general form of Köthe spaces, say, l -Köthe spaces, some quasi-normable Fréchet spaces and the strong duals of some asymptotically normable Fréchet spaces have this property. We also get a characterization of quasi-normable l -Köthe spaces in terms of their basic Banach subspaces (see [2], [4]). Modifying Theorem 1 in [5] we obtain, as an extension of Proposition 3 there, that any infinite dimensional complemented Banach subspace of a c_0 -Köthe space is basic. This result may be considered as a partial answer to the well-known Pelczynski problem: Does a complemented subspace of a space with basis have a basis? Moreover, in this case we confirm the conjecture of BESSAGA [1] that each complemented subspace of a Köthe space is isomorphic to a basic subspace.

2. Results

Note that a locally convex direct sum of Banach spaces has the *SCBS* property trivially, since every bounded subset of it is contained in a finite sum of its components.

We follow DRAGILEV [6] to define l -Köthe spaces. First, we need:

Definition. Let $(X, \|\cdot\|)$ be a Banach space with a basis (x_n) and let (x'_n) denote the sequence of coefficient functionals. The norm $\|\cdot\|$ of X is called *monotonous* if the following implication holds: for any $x, y \in X$, $|x'_n(x)| \leq |x'_n(y)|$ ($\forall n \in \mathbb{N}$) implies $\|x\| \leq \|y\|$ (see also [9]).

It is known that every Banach space with an unconditional basis has a monotonous norm which is equivalent to its original norm. Indeed, it is enough to put

$$\|x\| = \sup_{|\alpha_k| \leq 1} \left| \sum_k x'_k(x) \alpha_k x_k \right|$$

where $|\cdot|$ denotes the original norm.

Throughout this work we denote by l a Banach sequence space in which the canonical system (e_n) is an unconditional basis, with a monotonous norm $\|\cdot\|$ satisfying $\|e_n\| = 1$ for each $n \in \mathbb{N}$. Let Λ be the class of all such spaces; in particular, l_p and c_0 are in this class.

Definition. Let $l \in \Lambda$ and $\|\cdot\|$ be a norm (monotonous in this work) in l . If $(a_{n,p})$ is a Köthe matrix, then the l -Köthe space $\lambda^l(a_{n,p})$ is the locally convex space of all sequences (t_n) , such that $(t_n a_{n,p}) \in l$ for any $p \in \mathbb{N}$, with the topology generated by the seminorms

$$|(t_n)|_p = \|(t_n a_{n,p})\|, \quad p \in \mathbb{N}.$$

Note that $|e_n|_p = a_{n,p} \|e_n\| = a_{n,p}$.

Proposition 1. *Every l -Köthe space has the SCBS property. In particular, every c_0 -Köthe space has the SCBS property.*

PROOF. Let $X = \lambda^l(a_{n,k})$ be a l -Köthe space. Let A be a bounded subset of X . Without loss of generality we may assume that

$$A = \left\{ x \in X : |x|_k = \|(x_i a_{i,k})\| = \sup_{|\alpha_i| \leq 1} \left| \sum_i \alpha_i x_i a_{i,k} e_i \right| \leq c_k \forall k \right\}$$

where (c_k) is a sequence of real numbers increasing to ∞ . Choose (c_k) so that $(\frac{a_{i,k}}{c_k})_k$ tends to zero for all i . For this purpose, it is enough to replace those c_k 's with $k \geq i$ by bigger ones, for example by $k \max_{1 \leq i \leq k} a_{i,k}$. Here $\|\cdot\|$, $|\cdot|$ respectively denote the monotonous and the original norm in l .

We set $\gamma_i = \sum_k \frac{a_{i,k}}{2^k c_k}$. Then, for any $x \in A$

$$\begin{aligned} \left| \sum_i \alpha_i \gamma_i x_i e_i \right| &= \left| \sum_i \alpha_i x_i \left(\sum_k \frac{a_{i,k}}{2^k c_k} \right) e_i \right| \\ &= \left| \sum_k \frac{1}{2^k} \left(\sum_i \alpha_i \frac{a_{i,k}}{c_k} x_i e_i \right) \right| \leq 1 \end{aligned}$$

or $\|(x_i \gamma_i)\| \leq 1$ holds.

Fix $\epsilon > 0$, $k_0 \in \mathbb{N}$ and set $B = [e_i : \epsilon \gamma_i \leq a_{i,k_0}]$, $E = [e_i : \epsilon \gamma_i > a_{i,k_0}]$ where the square brackets denote the closed linear span of the corresponding vectors. For $x \in B$ and $k \in \mathbb{N}$, since

$$|x_i a_{i,k_0}| \geq |x_i \gamma_i \epsilon| \geq \left| x_i \epsilon \frac{a_{i,k}}{2^k c_k} \right|$$

we have

$$\begin{aligned} \frac{1}{\epsilon} 2^k c_k \|(x_i a_{i,k_0})\| &\geq \frac{1}{\epsilon} 2^k c_k \|(x_i \gamma_i \epsilon)\| \\ &\geq \frac{1}{\epsilon} 2^k c_k \left\| \left(x_i \epsilon \frac{a_{i,k}}{2^k c_k} \right) \right\| = |x|_k, \end{aligned}$$

that is B is a basic Banach subspace.

If $x \in A \cap E$, since $|x_i a_{i,k_0}| \leq |x_i \gamma_i \epsilon|$ we have

$$|x|_{k_0} = \|(x_i a_{i,k_0})\| < \|(\epsilon x_i \gamma_i)\| \leq \epsilon$$

which means that X has the *SCBS* property. $\square$

The following condition should be compared with the weak quasi-normability condition of Grothendieck (see for example [4]).

Definition. A Fréchet space X satisfies *the condition (QN)* if there is a map $\pi \in \mathbb{N}^{\mathbb{N}}$ such that for every $p \in \mathbb{N}$ and for every $\epsilon > 0$ there is a complemented Banach subspace B of X on which the topology given by p -th norm and the original topology coincide and

$$U_{\pi(p)} \subset B + \epsilon U_p \tag{1}$$

holds.

Proposition 2. *Every quasi-normable l -Köthe space X satisfies the condition (QN).*

PROOF. $|\cdot|_p^*$ denotes the gauge functional of U_p° i.e.

$$|x'|_p^* := \sup \{ |x'(x)| : x \in X, |x|_p \leq 1 \}, \quad x' \in X'.$$

Assume $a_{j,p} \neq 0$. Since $\|\cdot\|$ is monotonous, for any $x \in U_p$ one has

$$|e'_j(x)| = |x_j| = \|x_j e_j\| \leq \left\| \left(\frac{1}{a_{j,p}} x_i a_{i,p} \right)_i \right\| = \frac{1}{a_{j,p}} |x|_p \leq \frac{1}{a_{j,p}}.$$

Since the coefficient functionals e'_i are continuous, we have

$$1 = |e'_i(e_i)| \leq |e'_i|_p^* a_{i,p}$$

from which $|e'_j|_p^* = \frac{1}{a_{j,p}}$ follows.

By [10], Theorem 7, since X is quasi-normable, there exists a strictly increasing function ϕ such that for all $p \in \mathbb{N}$ there is $q \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists $c > 0$ such that

$$|e'_i|_q^* \leq c\phi\left(\frac{|e'_i|_p^*}{|e'_i|_q^*}\right) |e'_i|_k^*$$

or $\frac{a_{i,k}}{a_{i,q}} \leq c\phi\left(\frac{a_{i,q}}{a_{i,p}}\right)$ holds for each i .

Fix any $p \in \mathbb{N}$. Then there exists $q = \pi(p) \in \mathbb{N}$ such that the above property holds. For the given $\epsilon > 0$, we set

$$I_1 = \{i : \epsilon a_{i,q} \leq a_{i,p}\}, \quad B = [e_i : i \in I_1],$$

where the square bracket denotes the closed linear span of the corresponding vectors. Then obviously $\epsilon|x|_q = \epsilon\|(x_i a_{i,q})\| \leq \|(x_i a_{i,p})\| = |x|_p$ for any $x \in B$, since $|\epsilon x_i a_{i,q}| \leq |x_i a_{i,p}|$ and the norm is monotonous. On the other hand for any $k > q$

$$\frac{a_{i,k}}{a_{i,q}} \leq c\phi\left(\frac{a_{i,q}}{a_{i,p}}\right) \leq c\phi\left(\frac{1}{\epsilon}\right)$$

for every $i \in I_1$, so we have $|x|_k \leq c\phi\left(\frac{1}{\epsilon}\right)|x|_q$ for any $x \in B$. Hence B is a Banach space whose topology is given by the p -th norm.

Let $E = [e_i : i \notin I_1]$; then $X = B \oplus E$ and $\epsilon|x|_q > |x|_p$ for $x \in E$, therefore the condition (QN) holds. $\square$

Further, the condition (QN) characterizes quasinormability in terms of basic Banach subspaces in l -Köthe spaces (see [2], Proposition 3.3 and [4], Corollary 8). Indeed:

Proposition 3. *If X satisfies the condition (QN) then X is a quasinormable Fréchet space.*

PROOF. Since $\pi(p) > p$, $U_{\pi(p)} \subset U_p$ and condition 1 implies that $U_{\pi(p)} \subset A + \epsilon U_p$ where A is a ball in B with radius $c(1 + \epsilon)$. In fact, when $x \in U_{\pi(p)}$ we can write $x = b + \epsilon u$ for some $b \in B$ and $u \in U_p$. Let $|\cdot|_B$ denote the norm of B . Then $|b|_B \leq c|b|_p \leq c(|x|_p + |\epsilon u|_p) \leq c(1 + \epsilon)$ since p -th norm and the space X induce the same topology on the Banach space B . Hence X is a quasi-normable Fréchet space. $\square$

Proposition 4. *Every Fréchet space X with the condition (QN) has the SCBS property.*

PROOF. Let A be a bounded subset in X and $\pi(p)$ be as in condition 1. Find $\rho > 0$ such that $A \subset \rho U_{\pi(p)}$. Then by assumption it follows that $A \subset \rho U_{\pi(p)} \subset B + \rho \epsilon U_p \cap E$. $\square$

Definition. A Fréchet space X is said to satisfy the condition (AN) if for every $q \in \mathbb{N}$ there is $r \in \mathbb{N}$ so that for each $\epsilon > 0$ there exist two subspaces B and E of X (B is Banach and $X = B \oplus E$) such that

$$U_q^\circ \subset B' + \epsilon U_r^\circ \cap E'. \quad (2)$$

Here B' denotes the dual of B .

Proposition 5. *If X is a Fréchet space satisfying the condition (AN) , then X is an asymptotically normable Fréchet space.*

PROOF. Suppose X has the condition (AN) . Since $U_q^\circ \subset U_r^\circ$ for $r > q$, condition 2 implies that $U_q^\circ \subset B' \cap (1 + \epsilon)U_r^\circ + \epsilon U_r^\circ$. Since U_r° is weakly relatively compact, U_r° is strongly bounded. Hence $B' \cap (1 + \epsilon)U_r^\circ$ is strongly bounded or bounded in the norm topology of B' . So $U_q^\circ \subset MB_{B'} + \epsilon U_r^\circ$ holds for some $M > 0$, where $B_{B'}$ denotes the ball of B' . Of course this implies that X has DN_ϕ (in polarized form) i.e. X is asymptotically normable [13]. $\square$

Proposition 6. *The strong dual of a Fréchet space with the condition (AN) has the SCBS property.*

PROOF. Let X be a Fréchet space with the condition (AN) and let A' be a bounded subset of X' . Then there exists a $q > 0$ such that $A' \subset U_q^\circ$. By assumption, for this q , we find an $r > 0$ such that condition 2 holds. Also for an arbitrary zero-neighborhood C_α° of X' (where C_α is a bounded subset of X) there exists $s_\alpha > 0$ such that $C_\alpha \subset s_\alpha U_r$ and thus $U_r^\circ \subset s_\alpha C_\alpha^\circ$. Now apply condition 2 for $\frac{\epsilon}{s_\alpha}$ to obtain B' and E' depending on ϵ and α such that $U_q^\circ \subset B' + \frac{\epsilon}{s_\alpha} U_r^\circ \cap E'$ or $A' \subset U_q^\circ \subset B' + \epsilon C_\alpha^\circ \cap E'$. Hence X' has the *SCBS* property. $\square$

The following example shows that the *SCBS* property is not a necessary condition for the stability of a bounded perturbation of an automorphism up to some Banach subspace. Moreover it is not inherited by subspaces or quotients:

Example [12]. For $1 \leq p < \infty$, let $l_{p+} = \cap_{q>p} l_q = \cap_k l_{p_k}$, where $p_k \downarrow p$ (projective limit of l_p spaces). The topology of the Fréchet space l_{p+} may be then represented by means of the p_k -norms:

$$\|(x_n)\|_k = \left(\sum_n |x_n|^{p_k} \right)^{\frac{1}{p_k}}.$$

This space has the following properties:

- (i) It is a reflexive quasi-normable Fréchet space.
- (ii) Since $l_p \subset l_{p+} \subset l_q$ for all $q > p$, with continuous inclusions that are not compact, l_{p+} is not a Montel space.
- (iii) Since the canonical inclusions from $l_{p_{k+1}}$ into l_{p_k} are strictly singular and strictly cosingular, l_{p+} can have no infinite dimensional Banach subspace or quotient.

Since the canonical linking maps are strictly singular, we observe that any linear bounded operator T from l_{p+} to l_{p+} is also super strictly singular and hence $I + T$ is Fréholm ([15], [14]).

If l_{p+} had the *SCBS* property, then for a given bounded subset A , $k > 0$ and $\epsilon > 0$, by assumption and (iii), there would exist a finite dimensional subspace B of l_{p+} such that:

$$A \subset B + \epsilon U_k,$$

that is A is precompact. Hence l_{p+} would be Montel, which contradicts (ii). Therefore, l_{p+} does not have the *SCBS* property and consequently it can not satisfy the condition (QN) .

Proposition 7. *The SCBS property does not pass to quotients.*

PROOF. l_{p+} is a quasi-normable Fréchet space, and thus by MEISE–VOGT’s characterization [10] of quasinormability in Fréchet spaces, it is isomorphic to a quotient of an l_1 -valued generalized Köthe space which, by [5], has the *SCBS* property, but l_{p+} does not. $\square$

Proposition 8. *The SCBS property does not pass to subspaces.*

PROOF. l_{p+} is a countable projective limit of l_p -spaces and thus by Remark 24.5 in [11], it is isomorphic to a subspace of some countable product of Banach spaces. This product can be understood as a Banach-valued Köthe space which, by [5], has the *SCBS* property. However, l_{p+} does not have. $\square$

One can follow the steps of the proof of Theorem 1 in [5] to get the following:

Theorem 9. *If X is a complete barrelled locally convex space with the SCBS property and T is a linear bounded (respectively, compact) operator on X into X , then there exist complementary subspaces B and E of X such that:*

- (i) B is a Banach (respectively, finite dimensional) space; and
- (ii) if π_E and i_E are the canonical projection onto E and embedding into X , respectively, then $1_E + \pi_E T i_E$ is an automorphism on E .

PROOF. Since T is a bounded operator there exists $U_0 \in U(X)$ such that $T(U_0)$ is a bounded set in X , therefore

$$\forall U \in U(X) \exists C_U > 0 : p_U(Tx) \leq C_U p_{U_0}(x).$$

Since X has the *SCBS* property there exist complementary subspaces B and E of X such that B is a Banach (respectively finite dimensional) space and $T(U_0) \subset B + \frac{1}{2}U_0 \cap E$. Therefore, setting $T_1 = \pi_E T i_E : E \rightarrow E$, we obtain that

$$p_{U_0}(T_1x) \leq \frac{1}{2}p_{U_0}(x) \quad \forall x \in E.$$

Now it is easy to see that the operator $1_E - T_1$ is an automorphism of E . Indeed, consider for any $x \in E$ the series

$$Sx = x + T_1x + T_1^2x + \dots + T_1^m x + \dots \quad (3)$$

It is convergent in E because, for any $U \in U(X)$, we have

$$p_U(T_1^m x) \leq C_U p_{U_0}(T_1^{m-1}x) \leq C_U \left(\frac{1}{2}\right)^{m-1} p_{U_0}(x), \quad m = 1, 2, \dots,$$

so by Banach–Steinhaus theorem, since X is barrelled and complete, the formula (3) defines a linear continuous operator $S : E \rightarrow E$.

Since $(1_E - T_1)Sx = S(1_E - T_1)x = x$, the operator S is inverse to the operator $1_E - T_1$. $\square$

Exactly in the same way as in ([5], Theorem 2), Theorem 9 enables us to have the following modification of the generalized Douady lemma in ([16], Section 6):

Theorem 10. *Suppose X_1 is a complete barrelled locally convex space with the SCBS property and X_2, Y_1, Y_2 are topological vector spaces. If $X_1 \times X_2 \simeq Y_1 \times Y_2$ and $(X_1, Y_2) \in \mathcal{BF}$, then there exists complementary subspaces E and B in X_1 and complementary subspaces of G and F in Y_1 such that B is a Banach space, $F \simeq E$ and $B \times X_2 \simeq G \times Y_2$.*

If, in addition, $(Y_1, X_2) \in \mathcal{BF}$, then G is a Banach space.

Proposition 11. *Since the space ϕ of finite sequences with the direct sum topology is a complete barrelled DF space, then in the above theorem X_1 can be taken to be ϕ and Y_2 to be either a Fréchet space or a locally convex space with the individual countable boundedness condition (see [3]).*

Since, by Proposition 1, c_0 -Köthe spaces have the SCBS property, we get:

Proposition 12. *Let X be a c_0 -Köthe space, and F, G be complementary subspaces in X , i.e. $X = F \oplus G$. If G is an infinite dimensional Banach space then $G \simeq c_0$, and moreover, F and G are isomorphic to some basic subspaces of X .*

PROOF. We have $X \times \{0\} \simeq F \times G$ and by Theorem 10 there exist complementary basic subspaces E and B in X and complementary subspaces

F_1 and G_1 in F such that B is a Banach space and

$$F_1 \simeq E, \quad B \simeq G_1 \times G.$$

Since every infinite dimensional basic Banach subspace of a c_0 -Köthe space is isomorphic to c_0 we obtain that $B \simeq c_0$. On the other hand, each infinite dimensional complemented subspace of c_0 is isomorphic to c_0 (see [9]), so G is isomorphic to c_0 . Finally, since $B \simeq c_0$, its complemented subspace G_1 is isomorphic to some basic subspace of B and $F \simeq E \oplus G_1$ is isomorphic to some basic subspace of X . $\square$

It would be interesting to have this result for l -Köthe spaces (see, for example [6], Proposition 2.2.1).

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(Received February 26, 2003; revised October 13, 2003)