

On super quasi Einstein manifolds

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Abstract. The notion of a super quasi Einstein manifold is introduced and some properties of such a manifold are obtained.

Introduction

The notion of a quasi Einstein manifold was introduced in a recent paper [1] by the author and R. K. MAITY. According to them a non-flat Riemannian manifold (M^n, g) ($n \geq 3$) is called quasi Einstein if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) \quad (1)$$

where a, b are scalars of which $b \neq 0$, A is a non-zero 1-form such that

$$g(X, U) = A(X)\forall X \text{ and } U \text{ is a unit vector field.} \quad (2)$$

In such a case a, b are called the associated scalars. A is called the associated 1-form and U is called the generator of the manifold. Such an n -dimensional manifold is denoted by the symbol $(QE)_n$.

Subsequently, the author introduced in another recent paper [2] a generalization of a $(QE)_n$, called a generalized quasi Einstein manifold which

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was defined as follows:

A non-flat Riemannian manifold (M^n, g) ($n \geq 3$) is called generalized quasi Einstein if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] \quad (3)$$

where a, b, c are scalars of which $b \neq 0, c \neq 0$ and A, B are two non-zero 1-forms such that

$$g(X, U) = A(X), \quad g(X, V) = B(X) \forall X \quad (4)$$

and U, V are two unit vector fields perpendicular to each other. In such a case a, b, c are called the associated scalars, A, B are called the associated main and auxiliary 1-forms and U, V are called the main and auxiliary generators of the manifold. Such an n -dimensional manifold was denoted by the symbol $G(QE)_n$.

This paper deals with super quasi Einstein manifolds which are defined as follows:

A non-flat Riemannian manifold (M^n, g) ($n \geq 3$) is called super quasi Einstein if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] + dD(X, Y) \quad (5)$$

where a, b, c, d are scalars of which $b \neq 0, c \neq 0, d \neq 0$, A, B are two non-zero 1-forms such that

$$g(X, U) = A(X), \quad g(X, V) = B(X) \forall X \quad (6)$$

U, V being mutually orthogonal unit vector fields, D is a symmetric $(0, 2)$ tensor with zero trace which satisfies the condition

$$D(X, U) = 0 \forall X. \quad (7)$$

In such a case a, b, c, d are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, U, V are called the main

and the auxiliary generators and D is called the associated tensor of the manifold. Such an n -dimensional manifold shall be denoted by the symbol $S(QE)_n$.

In this paper it is shown that the scalars $a+b$ and $a+dD(V, V)$ are the Ricci curvatures in the directions of the vector fields U and V respectively and the scalar d is less than the ratio which the length of the Ricci tensor S bears to the length of the associated tensor D . Further, some interesting properties of the curvature tensor R of type $(1, 3)$ are obtained. Moreover, it is shown that a viscous fluid space time admitting heat flux and obeying Einstein's equation without cosmological constant is a 4-dimensional semi Riemannian super quasi Einstein manifold.

1. The associated scalars of a $S(QE)_n$, ($n \geq 3$)

In this section we consider a $S(QE)_n$, ($n \geq 3$) with associated scalars a, b, c, d associated main and auxiliary 1-forms A, B , main and auxiliary generators U, V and associated symmetric $(0, 2)$ tensor D .

Then (5), (6) and (7) will hold. Since U and V are mutually orthogonal unit vector fields, we have

$$g(U, U) = 1, \quad g(V, V) = 1 \quad \text{and} \quad g(U, V) = 0. \quad (1.1)$$

Further

$$\text{trace } D = 0 \quad (1.2)$$

$$D(X, U) = 0 \quad \forall X. \quad (1.3)$$

In virtue of (4), $g(U, V) = 0$ can be expressed as

$$A(V) = B(U) = 0. \quad (1.4)$$

Now contracting (5) over X and Y we get

$$r = na + b \quad (1.5)$$

where r is the scalar curvature. Again from (5) we get

$$S(U, U) = a + b, \quad (1.6)$$

$$S(V, V) = a + dD(V, V) \quad \text{and} \quad (1.7)$$

$$S(U, V) = c. \quad (1.8)$$

If X is a unit vector field, then $S(X, X)$ is the Ricci curvature in the direction of X . Hence from (1.6) and (1.7) we can state that $a + b$ and $a + dD(V, V)$ are the Ricci curvatures in the directions of U and V respectively. Let

$$g(LX, Y) = S(X, Y) \quad \text{and} \quad (1.9)$$

$$g(\ell X, Y) = D(X, Y). \quad (1.10)$$

Further, let d_1^2 , and d_2^2 denote the squares of the lengths of the Ricci tensor S and the associated tensor D . Then

$$d_1^2 = S(Le_i, e_i) \quad \text{and} \quad (1.11)$$

$$d_2^2 = D(\ell e_i, e_i) \quad (1.12)$$

where $\{e_i\}$ $i = 1, 2, \dots, n$ is an orthonormal basis of the tangent space at a point of $S(QE)n$. Now from (5) we get

$$S(Le_i, e_i) = (n-1)a^2 + (a+b)^2 + dS(\ell e_i, e_i). \quad (1.13)$$

Again from (5) we have

$$S(\ell e_i, e_i) = dD(\ell e_i, e_i). \quad (1.14)$$

From (1.13) and (1.14) it follows that

$$S(Le_i, e_i) = (n-1)a^2 + (a+b)^2 + d^2D(\ell e_i, e_i). \quad (1.15)$$

Hence

$$d_1^2 = (n-1)a^2 + (a+b)^2 + d^2(d_2)^2. \quad (1.16)$$

From (1.16) we can write $d_1^2 - d^2d_2^2 > 0$. Hence

$$d < \frac{d_1}{d_2}. \quad (1.17)$$

Summing up we can state the following theorem:

Theorem 1. *In a $S(QE)n$ ($n \geq 3$) the scalars $a+b$ and $a+d$ $D(V, V)$ are the Ricci curvatures in the directions of the generators U and V respectively and the associated scalar d is less than the ratio which the length of the Ricci tensor S bears to the length of the associated tensor D .*

2. Conformally flat $S(QE)n$ ($n > 3$)

Let R be the curvature tensor of type $(1, 3)$ of a conformally flat $S(QE)n$ ($n > 3$). Then

$$\begin{aligned} {}'R(X, Y, Z, W) &= \frac{1}{n-2} [g(Y, Z)S(X, W) - g(X, Z)S(Y, W) \\ &\quad + g(X, W)S(Y, Z) - g(Y, W)S(X, Z)] \\ &\quad - \frac{r}{(n-1)(n-2)} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \end{aligned} \quad (2.1)$$

where $'R(X, Y, Z, W) = g[R(X, Y, Z), W]$. Using (5) we can express (2.1) as follows:

$$\begin{aligned} {}'R(X, Y, Z, W) &= a'[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ &\quad + b'[g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W) \\ &\quad + g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)] \\ &\quad + c'[g(Y, Z)\{A(X)B(W) + A(W)B(X)\} \\ &\quad - g(X, Z)\{A(Y)B(W) + A(W)B(Y)\} \\ &\quad + g(X, W)\{A(Y)B(Z) + A(Z)B(Y)\} \\ &\quad - g(Y, W)\{A(X)B(Z) + A(Z)B(X)\}] \\ &\quad + d'[g(Y, Z)D(X, W) - g(X, Z)D(Y, W) \\ &\quad + g(X, W)D(Y, Z) - g(Y, W)D(X, Z)] \end{aligned} \quad (2.2)$$

where

$$a' = \frac{2a(n-1) - r}{(n-1)(n-2)}, \quad b' = \frac{b}{n-2}, \quad c' = \frac{c}{n-2}, \quad d' = \frac{d}{n-2}. \quad (2.3)$$

Let $U^\perp$ denote the $(n-1)$ -dimensional distribution orthogonal to U in a conformally flat $S(QE)n$. Then $g(X, U) = 0$ if $X \in U^\perp$. Again if $g(X, U) = 0$, then $X \in U^\perp$. Hence from (2.2) we get the following properties of R :

$$\begin{aligned} R(X, Y, Z) &= \lambda[g(Y, Z)X - g(X, Z)Y] \\ &+ c'[g(Y, Z)B(X) - g(X, Z)B(Y)]U \\ &+ d'[D(Y, Z)X - D(X, Z)Y + g(Y, Z)\ell X - g(X, Z)\ell Y] \end{aligned} \quad (2.4)$$

when $X, Y, Z \in U^\perp$ and

$$R(X, U, U) = \mu X + d'\ell X \quad \text{when } X \in U^\perp \quad (2.5)$$

where $\lambda = \frac{(n-2)a-b}{(n-1)(n-2)}$, $\mu = \frac{a+b}{n-1}$ and c', d' have values given by (2.3). We can therefore state as follows:

Theorem 2. *In a conformally flat $S(QE)n$ ($n > 3$) the curvature tensor R of type $(1, 3)$ satisfies the properties given by (2.4) and (2.5).*

3. General relativistic viscous fluid space time admitting heat flux

Let (M^4, g) be a viscous fluid space time admitting heat flux and satisfying Einstein's equation without cosmological constant. Further, let U be the unit timelike velocity vector field of the fluid, V be the unit heat flux vector field and D be the anisotropic pressure tensor of the fluid. Then

$$g(U, U) = -1, \quad g(V, V) = 1, \quad g(U, V) = 0 \quad (3.1)$$

$$D(X, Y) = D(Y, X), \quad \text{trace } D = 0 \quad \text{and} \quad (3.2)$$

$$D(X, U) = 0 \quad \forall X. \quad (3.3)$$

Let

$$g(X, U) = A(X), \quad g(X, V) = B(X) \quad \forall X. \quad (3.4)$$

Further, let T be the $(0, 2)$ type of energy momentum tensor describing the matter distribution of such a fluid. Then [3]

$$T(X, Y) = (\sigma + p)A(X)A(Y) + pg(X, Y)$$

$$+ [A(X)B(Y) + A(Y)B(X)] + D(X, Y) \quad (3.5)$$

where σ , p denote the density and isotropic pressure and D denotes the anisotropic pressure tensor of the fluid.

It is known [4] that Einstein's equation without cosmological constant can be written as follows:

$$S(X, Y) - \frac{1}{2}rg(X, Y) = kT(X, Y) \quad (3.6)$$

where k is the gravitational constant and T is the energy momentum tensor of type $(0, 2)$.

In the present case (3.6) can be written as follows:

$$\begin{aligned} S(X, Y) - \frac{1}{2}rg(X, Y) = & k[(\sigma + p)A(X)A(Y) + pg(X, Y) \\ & + A(X)B(Y) + A(Y)B(X) + D(X, Y)]. \end{aligned}$$

Hence

$$\begin{aligned} S(X, Y) = & \left(kp + \frac{1}{2}r\right)g(X, Y) + k(\sigma + p)A(X)A(Y) \\ & + k[A(X)B(Y) + A(Y)B(X)] + kD(X, Y). \end{aligned} \quad (3.7)$$

From (3.7) it follows that the space time under consideration is a super quasi Einstein manifold with $kp + \frac{1}{2}r$, $k(\sigma + p)$, k , k as associated scalars, A and B as associated 1-forms, U , V as generators and D as the associated symmetric $(0, 2)$ tensor.

Hence we can state the following theorem

Theorem 3. *A viscous fluid spacetime admitting heat flux and satisfying Einstein's equation without cosmological constant is a 4-dimensional semi Riemannian super quasi Einstein manifold.*

References

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