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Diophantine applications of Bennett's *abc* theorem

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This paper is dedicated to the memory of Professor Béla Brindza

Abstract. The purpose of this paper is to survey some Diophantine applications of a theorem of M. A. Bennett on systems of simultaneous Pell equations. One of the more notable consequences concerns Schäffer's equation $1^k+2^k+\cdots+x^k = y^n$, which was of great interest to Professor Brindza, as he proved some astonishing results concerning the solvability of this equation.

1. Introduction

A topic of great interest to Professor Brindza concerns a conjecture of Schäffer, which attempts to describe explicitly the set of integer solutions to the equation $1^k + 2^k + \cdots + x^k = y^n$. Seemingly unrelated to this is a rather innocent theorem of M. A. Bennett. Bennett [Bennett98] proved several remarkable results concerning the solvability of systems of simultaneous Pell equations, most notably being that any system of simultaneous Pell equations of the form

$$x^2 - ay^2 = 1, \quad z^2 - by^2 = 1,$$

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with a, b fixed positive integers, has at most three solutions in positive integers x, y, z. From the methods underlying this work, Bennett developed the tools to prove a very powerful, yet innocent looking result for a certain related type of system of simultaneous Pell equations. Before describing this result, we digress back in time to a beautiful theorem of Wilhelm Ljunggren.

In [Ljung42], LJUNGGREN proved that for any nonsquare positive integer d, the quartic equation $X^4 - dY^2 = 1$ has at most two solutions in positive integers X, Y. The argument Ljunggren gives uses an intricate construction involving units in quartic number fields, together with a remarkable application of Skolem's p-adic method. A close look at this proof shows that Ljunggren actually proved the following rather curious result in order to arrive at his theorem. The interested reader is referred to [Walsh00] for details of the proof.

Theorem 1.1. *Let a and c denote positive integers, then there is at most one triple of positive integers x,y, z with the property that ax^2, y^2, cz^2 are (ascending) consecutive integers.*

In the problem session of the Fifth Conference of the Canadian Number Theory Association, Herman J. J. te Riele posed the following problem:

When I became 49, I realized that this square is preceded by 3 times a square and followed by 2 times a square. Are there more (nontrivial) such squares?

Evidently, te Riele's question is answered in the negative by the formulation of Ljunggren's theorem given above. Nevertheless, te Riele's question seemed to provide the impetus for further work along these lines, and consequently BENNETT [Bennett99] proved the following generalization.

Theorem 1.2. *Let a, b and c denote positive integers, then there is at most one triple of positive integers x,y, z with the property that ax^2, by^2, cz^2 are (ascending) consecutive integers.*

This rather innocent looking statement provides the basis to improve upon a considerable number of results in the literature. We will endeavour to describe these applications in this article. Most of the applications we present are actually consequences of Theorem 2.1, given below, which is

a more refined statement than Theorem 1.2, and easily amenable to the applications we shall discuss.

2. A generalization of a theorem of Cohn and Ljunggren on $X^4 - dY^2 = 1$

In [Cohn97], COHN refined Ljunggren's theorem on the equation $X^4 - dY^2 = 1$. In particular, he employed a clever manipulation of Jacobi symbols evaluations to prove that for $d > 0$ squarefree, $d \neq 1785$, the quartic equation $X^4 - dY^2 = 1$ has at most one solution in positive integers X, Y, and if a solution exists, then it must come from the minimal unit greater than one in $\mathbb{Z}[\sqrt{d}\,]$ of norm 1, or its square.

Using Theorem 1.2, the following result was proved in [BenWal99]. We first require some notation. For a nonsquare positive integer d, we denote by $\epsilon_d = T + U\sqrt{d}$ the minimal unit greater than one in $\mathbb{Z}[\sqrt{d}\,]$ of norm 1, and for $k \geq 0$ we set $T_k + U_k\sqrt{d} = \epsilon_d^k$. For a positive integer b, we define the *rank of apparition*, $\beta(b)$, of b in $\{T_k\}$ to be the minimal index k for which b divides T_k, with the convention that $\beta(b) = \infty$ if no such index k exists.

Theorem 2.1. *Let $b > 1$ and $d > 1$ denote squarefree positive integers. If $T_k = bx^2$ for some integer x, then $k = \beta(b)$. Consequently, the Diophantine equation*

$$b^2X^4 - dY^2 = 1$$

has at most one solution in positive integers X, Y, and such a solution can be given explicitly in terms of ϵ_d.

One can immediately see the usefulness of this result by considering small values of b. Cohn's theorem states that if $T_k = X^2$ for some integer X, then $k = 1$ or $k = 2$. For $b \in \{2, 3, 5, 6\}$, it is not difficult to see that Theorem 2.1 implies that the equation $T_k = bX^2$ implies $k = 1$. In fact, pursuing this a little further one can deduce the following.

Corollary 2.1. *Let $b = 2^r3^s5^t7^u11^v$ for some integers $r, s, t, u, v \in \{0, 1\}$, not all zero. Then any solution of $T_k = bX^2$ with $X \in \mathbb{Z}$ implies*

$k = 1$, *unless*

(i) $b = 7$, *in which case* $k = 1$ *or* $k = 2$ *(but not both)*

(ii) $(b, d) = (11, 2)$, *in which case* $T_3 = 11 \cdot 3^2$

(iii) $(b, d) = (55, 1139)$, *in which case* $T_3 = 55 \cdot 423^2$.

Furthermore, given any squarefree positive integer $b > 1$ *there is an effective computable constant* $C = C(b)$*, depending only on* b*, such that if* $d > C$ *and* $T_k = bX^2$ *for some* $X \in \mathbb{Z}$*, then* $k = 1$ *or* $k = 2$.

Before proceeding to the next section, we discuss another immediate application of Theorem 2.1 in connection to integer points on a family of hyperelliptic curves determined by Tschebyscheff polynomials.

For $n \geq 1$, define a sequence of polynomials

$$P_n(x) = \frac{T_{2n+1}(x)}{x},$$

where $T_m(x)$ is the m-th Tchebyscheff polynomial satisfying

$$T_m(x) = x^m + \binom{m}{2} x^{m-2}(x^2-1) + \binom{m}{4} x^{m-4}(x^2-1)^2 + \cdots,$$

or alternatively,

$$T_m(x) = \frac{1}{2}\left((x + \sqrt{x^2-1}\,)^m + (x - \sqrt{x^2-1}\,)^m \right).$$

We consider the family of hyperelliptic curves C_n defined by

$$C_n : \ y^2 = P_n(x),$$

which for $n \geq 1$ is evidently a curve of genus $n-1$. A classical theorem of Siegel implies that for each $n \geq 2$, there are only a finite number of integer points (x, y) on C_n. Theorem 2.1 provides a much sharper result.

Corollary 2.2. *For* $n \geq 1$*, the only integer points on* C_n *are* $(x, y) = (1, \pm 1)$*, and also if* $n = (s^2-1)/2$ *for some integer* s*, the points* $(x, y) = (0, \pm s)$.

It is worth noting that each $P_n(x)$ is an even polynomial, i.e. that $P_n(\sqrt{x}\,)$ is a polynomial, and moreover, that the problem of determining all integer points on $y^2 = P_n(\sqrt{x}\,)$ remains unsolved, apart from certain special cases. The interested reader is referred to Section 2.3 of [Walsh00] for more on this problem, and its relation to quartic equations of the form $aX^4 - bY^2 = 1$, with $a > 1$ and nonsquare.

3. Generalizations of a theorem of Ljunggren

Ljunggren's work receives considerable coverage in the classical text of MORDELL [Mordell69] on Diophantine equations, but perhaps his most remarkable result covered in Mordell's book appears as Theorem 9 on p. 270 (for the original work, the reader is referred to [Ljung36]). The theorem is stated in [Mordell69] as follows.

Theorem 3.1. *The equation*

$$X^2 - dY^4 = 1,$$

where $d > 0$ and is not a perfect square, has at most two solutions in positive integers. Denote by α_d the fundamental unit in the ring of integers of the quadratic field $\mathbb{Q}(\sqrt{d}\,)$. If there are two solutions, these are given by either

$$X + Y^2\sqrt{d} = \alpha_d, \alpha_d^2 \quad \textit{or by} \quad X + Y^2\sqrt{d} = \alpha_d, \alpha_d^4,$$

the latter occurring for only a finite number of values of d.

There is evidently numerous ways in which one would wish to refine or generalize this beautiful theorem. In this section we discuss various such improvements, and their application to the resolution of certain parametric families of Thue equations.

We first state a result which refines Theorem 3.1, and whose proof makes use of Cohn's aforementioned result on the equation $X^4 - dY^2 = 1$. We retain the notation from the previous section.

Proposition 3.1. *Let b and $d > 1$ denote squarefree positive integers, and $T_k + U_k\sqrt{d} = \epsilon_d^k$ for $k \geq 1$. There is at most one index k for which $U_k = bX^2$ for some integer X, except in the following two cases:*

(i) *$T_1 = 2t^2$ and $U_1 = by^2$ for some integers t and y, in which case there is the second solution $U_2 = b(2ty)^2$.*

(ii) *$T = 169$, in which case U_1 and U_4 are both squares for $d = 1785$.*

This proposition suffers from the shortcoming that for a given integer b, it does not give explicit information on the index k for a solution of the equation $U_k = bX^2$, which was an important aspect of Theorem 2.1. Some progress along these lines was proved by MIGNOTTE and PETHŐ in [MigPet93]. In particular, they proved the following.

Theorem 3.2. *Let d denote a nonsquare positive integer such that $\epsilon_d = T + u^2\sqrt{d}$ for some integers T and u. A solution to $U_k = bX^2$ for $b \in \{1, 2, 3, 6\}$, with $x \in \mathbb{Z}$, implies $k \leq 3$, except only when $T = 169$, in which case $k = 4$.*

The following represents a refinement of Theorem 3.1, and was proved in [Walsh98] as a consequence of Theorem 2.1.

Theorem 3.3. *Let d denote a nonsquare positive integer such that $\epsilon_d = T + u^2\sqrt{d}$ for some integers T and u. Assume that $U_k = bX^2$ for some $b \in \{1, 2, 3, 5, 6, 10\}$, and $x \in \mathbb{Z}$. Then $k \leq 2$, except in the following cases.*

(i) *$T = 169$, in which case U_4 is a square.*

(ii) *3 divides u and $4T^2 - 1 = 3y^2$ for some integer y, in which case $U_3 = 3(uy)^2$.*

(iii) *$(b, d) = (5, 24)$, in which case $U_4 = 5 \cdot 14^2$.*

A general result on the index k for solutions to $U_k = bX^2$, which would be completely analogous to Theorem 2.1, remains elusive. Conjecture 1 in [Walsh98] provides what is almost certainly the correct statement, but Theorem 2.1 can only be used to prove the following partial result. For a given positive integer b, and sequence $\{U_k\}$, defined earlier, we let $\alpha(b)$ denote the minimal index k for which b divides U_k.

Theorem 3.4. *If $\alpha(b)$ is even, then the only possible solution to $U_k = bX^2$, for some integer X, is $k = \alpha(b)$, except in the case that $2T_1^2 - 1 = v^2$ and $TU = bu^2$ for some integers u, v, in which case $U_4 = b(2uv)^2 = U_{2\alpha(b)}$.*

We now turn our attention to some refinements of Theorem 3.1 which lead to the complete solution of certain quartic Thue equations that have been considered in the work of Cusick [Cusick92], Cohn [Cohn95], and Stroeker [Stroeker89]. Once again the results obtained are made possible by Theorem 2.1.

Let k denote a positive integer. In [Cusick92], Cusick determined all integer solutions to the family of Thue equations $x^4 - kx^2y^2 + y^4 = 1$. This was extended by Cohn [Cohn95] to equations of the form $x^2 - kxy^2 + y^4 = c$ for $c \in \{\pm 1, \pm 2, \pm 4\}$, but with the assumption that k is odd for $c = 1$ and $c = 4$. The following refinement of Theorem 3.1, which appeared in

[Walsh99], is proved using Theorem 2.1, and this refinement yields Cohn's result without the extra assumption that k must be odd.

Theorem 3.5. *Let d be a nonsquare positive integer. Then there are at most two positive indices k for which $U_k = 2^\delta X^2$, with X an integer and $\delta = 0$ or 1, and if two such indices $k_1 < k_2$ exist, then $k_1 = 1$ and $k_2 = 2$, unless $d \in \{1785, 7140, 28560\}$, in which case the only solutions to $U_k = 2^\delta X^2$ are $k = 1, 2, 4$.*

As a consequence of this theorem, we obtained in [Walsh99] the following complete solution to the cases left open by Cohn in [Cohn95].

Corollary 3.1. *Let k be an even positive integer.*

1. *The only solutions to $x^2 - kxy^2 + y^4 = 1$ in non-negative integers (x, y) are $(k, 1), (1, 0), (0, 1)$, unless either k is a perfect square, in which case there are also the solutions $(1, \sqrt{k}\,), (k^2 - 1, \sqrt{k}\,)$, or $k = 338$ in which there are the solutions $(x, y) = (114243, 6214), (13051348805, 6214)$.*

2. *The equation $x^2 - kxy^2 + y^4 = 4$ has only the solution $(x, y) = (2, 0)$, unless $k = 2v^2$ for some integer v, in which case there are also the solutions $(2, \sqrt{2k}\,), (2k^2 - 2, \sqrt{2k}\,)$.*

We finish off this section by exhibiting yet another application of Theorem 2.1 to quartic Thue equations, which provides an improved computational approach to solving a family of Thue equations considered by Stroeker [Stroeker89].

In [Stroeker89], Stroeker describes a method to solve Thue equations of the form

$$X^4 - 2rX^2Y^2 - sY^4 = 1.$$

This method is somewhat ad hoc, and in fact fails to work in many instances, as the method requires that the coefficients r and s satisfy certain congruence conditions, and moreover, the method requires that certain technical conditions hold during the course of the algorithm. Using Theorem 2.1, we can get around both of these issues completely, provide a much simpler algorithm to solve any equation in the above family, and moreover, the method will solve the more general family of equations

$$X^2 - 2rXY^2 - sY^4 = 1.$$

The main point to be raised here is that in Theorem 3.1, Ljunggren needs to make the assumption that two solutions exist in order to get information on the indices corresponding to these solutions. The main point of the following refinement of Theorem 3.1 is to pin down where a solution can occur in the case that only one solution exists. As mentioned earlier, the proof uses Theorem 2.1, but also more recent work of the F. Luca and author [LucaWal01]. The result appears in [Walsh02].

In what follows, let r, s be the integers, and $D = r^2 + s$.

Theorem 3.6. 1. *There are at most two positive integer solutions* (X, Y) *to* $X^2 - DY^4 = 1$. *If two solutions* $Y_1 < Y_2$ *exist, then* $Y_1^2 = U_1$ *and* $Y_2^2 = U_2$, *except only if* $D = 1785$ *or* $D = 16 \cdot 1785$, *in which case* $Y_1^2 = U_1$ *and* $Y_2^2 = U_4$.

2. *If only one positive integer solution* (X, Y) *to* $X^2 - DY^4 = 1$ *exists, then* $Y^2 = U_l$ *where* $U_1 = lv^2$ *for some squarefree integer* l, *and either* $l = 1$, $l = 2$, *or* $l = p$ *for some prime* $p \equiv 3 \pmod 4$.

Using this result, one can solve any Thue equation $X^2 - 2rXY^2 - sY^4 = 1$ as follows. First, put $D = r^2 + s$, and if $D > 0$ and not a square (or otherwise the problem is trivial), compute $\epsilon_D = T + U\sqrt{D}$, which is the minimal solution to $X^2 - DY^2 = 1$. Factor U into the form $U = l \cdot y^2$ with l squarefree, and then simply check whether any of $U = U_1, U_2, U_l$ are squares. To exhibit the usefulness of this result, Stroeker's approach breaks down in particular case $r = -1$, $s = 2$. Using our approach, we see that $D = 3$, $\epsilon_D = 2 + \sqrt{3}$, and the only squares are $U_1 = 1$ and $U_2 = 4$, which shows that $(X, Y) = (1, 1)$ is the only positive integer solution to $X^2 + 2XY^2 - 2Y^4 = 1$.

4. On the equation $X^3 - dY^2 = \pm 1$

In [Cohn91], Cohn investigated the solvability of the Diophantine equation

$$x^3 - Ny^2 = \pm 1.$$

Improving upon previous work of Stroeker [Stroeker76], Cohn proved the following theorem.

Theorem 4.1. *Let N denote a squarefree positive integer with no prime factor of the form $3k+1$. Then the equation $x^3 - Ny^2 = 1$ has no solutions in positive integers, and the equation $x^3 - Ny^2 = -1$ has no solutions in positive integers, unless $N \in \{1, 2\}$, in which case $(N, x, y) = (1, 2, 3)$ and $(N, x, y) = (2, 23, 78)$ are the only solutions.*

The interesting case in this theorem arises when the respective irreducible quadratic factors take on values of the form $3z^2$, for otherwise the result is an immediate consequence of quadratic reciprocity. Cohn deals with this case in a very clever manner by determining all of the integer solutions to the respective equations

$$x^2 + x + 1 = 3z^2, \quad x - 1 = 3Nw^2$$

and

$$x^2 - x + 1 = 3z^2, \quad x + 1 = 3Nw^2,$$

which are equivalent respectively to

$$3N^2w^4 + 3Nw^2 + 1 = z^2$$

and

$$3N^2w^4 - 3Nw^2 + 1 = z^2.$$

We reformulate Cohn's theorem in terms of these Diophantine equations as follows.

Theorem 4.2. *If N is a squarefree integer not divisible by any prime $p \equiv 1 \pmod 3$, then the equation $3N^2w^4 + 3Nw^2 + 1 = z^2$ has no positive integer solutions (w, z), and the equation $3N^2w^4 - 3Nw^2 + 1 = z^2$ has no solutions in positive integers, unless $N \in \{1, 2\}$, in which case $(N, w, z) = (1, 1, 1)$ and $(N, w, z) = (2, 2, 13)$ are the only solutions.*

In [LucaWalsh01] Luca and the author proved a more general result concerning integer points on a large class of elliptic curves, which includes the particular curves considered by Cohn. In particular, using Theorem 2.1, the following theorem was proved. If a positive integer n is of the form $n = ma^2$ for some squarefree positive integer m and an integer a, we refer to m as the *squarefree class* of n, and denote it by $m = \langle n \rangle$.

Theorem 4.3. *Let d be a positive integer with $d \equiv 3 \pmod 4$, and let $\epsilon_d = T + U\sqrt{d} > 1$ denote the minimal solution to $X^2 - dY^2 = 1$, with the assumption that T is even. For a squarefree positive integer N which is not divisible by any odd prime p with $(-d/p) = 1$, the Diophantine equation*

$$dN^2w^4 + dUNw^2 + (T/2)^2 = z^2$$

has no solutions in positive integers (w, z), and the Diophantine equation

$$dN^2w^4 - dUNw^2 + (T/2)^2 = z^2$$

has no solutions in positive integers (w, z), except only if $N = \langle U \rangle$, in which case

$$(w, z) = \left(\sqrt{\frac{U}{N}}, T/2 \right)$$

is the only solution, and $N = \langle 2U \rangle$, in which case

$$(w, z) = \left(T\sqrt{\frac{2U}{N}}, (T/2)(4T^2 - 3) \right)$$

is the only solution.

We remark that the special case of $d = 3$ in this theorem is precisely the case proved by Cohn. Also, if the value d in this theorem is prime, then T is even, thereby removed this condition from the hypothesis given in the theorem.

5. Near squares in linear recurrences

Let r, s, U_0, U_1 denote integers. The relation

$$U_{n+1} = rU_n - sU_{n-1}$$

defines a binary linear recurrence sequence $\{U_n\}$ for $n \geq 1$. For a polynomial $P(x)$ of degree at least two with integer coefficients, Nemes and Pethő described necessary conditions for the general equation

$$U_n = P(x)$$

to have infinitely many solutions in integers (n, x). In the particular case $P(x) = bx^2$, for $b \geq 1$, precise results on the solutions of this equation have been obtained by many individuals. In the more general case that $P(x)$ is an arbitrary quadratic polynomial, and the sequence in question is of Lucas–Lehmer type, there exist methods to determine all solutions. The method of Baker [Baker69] provides an explicit upper bound for the size of solutions to $U_n = P(x)$. In [Tzanakis96], Tzanakis describes an algorithmic approach to determine all solutions to $U_n = P(x)$. It is our interest to determine families of such equations for which one can make explicit statements of solvability. In the present paper we consider the particular case that $P(x) = cx^2 - 1$ for an even positive integer c, and for which the linear recurrence sequence above is given by $(r, s, U_0, U_1) = (2T, 1, 0, 1)$, for some positive integer $T > 1$. Similar problems were considered by Robbins in [Robbins81].

Let $T > 1$ denote a positive integer, and define $\alpha = T + \sqrt{T^2 - 1}$. For $n \geq 1$, define sequences $\{T_n\}$ and $\{U_n\}$ by

$$\alpha^n = T_n + U_n\sqrt{T^2 - 1}.$$

Also, for $i \geq 1$, define sequences $\{p_i\}$, $\{q_i\}$ by

$$p_i + q_i\sqrt{2} = (1 + \sqrt{2}\,)^i.$$

Employing a technique of Ljunggren's in [Ljung36], developed further in work of Cohn [Cohn98], together with Theorem 2.1, the following result was proved in [Walsh03].

Theorem 5.1. (i) *If $(T, c) = (q_{2i+1}, 2)$ for some $i \geq 1$, then the equation*

$$U_n = cx^2 \pm 1$$

has only the two positive integer solutions $(n, x) = (1, 1), (3, p_{2i+1})$.

(ii) *If (T, c) is any other pair of positive integers for which $T > 1$ and c is even, then the equation*

$$U_n = cx^2 \pm 1$$

has only one solution in positive integers (n, x), and if a solution exists, then $n < c$.

6. Schäffer's equation

É. Lucas [Lucas1877] studied the diophantine equation

$$y^2 = 1^2 + 2^2 + \cdots + x^2,$$

and it was G. N. Watson [Watson1918] who later correctly proved that the only positive integer solutions are $x = y = 1$ and $x = 24, y = 70$. Schäffer [Schäffer56] furthered this work by studying the more general equation

$$y^q = 1^k + 2^k + \cdots + x^k.$$

The main result of this work was a proof that the only positive integers $(k, q > 1)$ for which this equation has infinitely many solutions are $(k, q) \in \{(1, 2), (3, 2), (3, 4), (5, 2)\}$.

Schäffer also made the following conjecture.

Conjecture 6.1. *Let k and $q > 1$ be positive integers, with (k, q) not in the above list. Then apart from the solution $(x, y) = (24, 70)$ when $k = q = 2$, the only solution is the trivial solution $x = y = 1$.*

In recent years there have been numerous papers on this topic, most notably by Béla Brindza (see [Brindza84] or [Brindza90] for example). The interested reader may wish to refer to the notes at the end of Chapter 10 in [ShoreyTijdeman86].

In a recent paper [BrindzaPintér2000], Brindza and Pintér proved the following

Theorem 6.1. *For $k \geq 2$ even, the equation*

$$y^2 = 1^k + 2^k + \cdots + (x - 1)^k$$

has at most $\max\{c_1, 9^k\}$ solutions in integers x and y, where c_1 is an effectively computable absolute constant.

Although this theorem does not prove Schäffer's conjecture, its proof provides a methodology for finding all integer solutions to the equation $y^2 = 1^k + 2^k + \cdots + (x - 1)^k$ for fixed even integers k. In particular, this was the goal in [Pintér2001], wherein Pintér found all solutions to this equation for $k \in \{2, 4, 6, 8, 10, 14\}$.

Pintér's approach is to reduce the problem of finding all integer solutions to $y^2 = 1^k + 2^k + \cdots + x^k$, for a fixed even integer k, to the problem of determining all integer points on a small collection of elliptic curves of the form $Z^3 - Z = aW^2$, where a takes on finitely many values which depending entirely on k. This can effectively be done using the techniques in [Gebel94], but the problem that arises in this approach is the enormous amount of time it can take to determine explicitly a basis for the Mordell–Weil group of each curve.

An alternative approach is to represent each collection of elliptic curves $Z^3 - Z = aW^2$, for a fixed value k, as a collection of elliptic curves written in the form $b^2X^4 - dY^2 = 1$. It is straightforward to see that this can be done, and that integer points are preserved by this transformation. Using Theorem 2.1, one can then determine all integer points on each collection of curves in a fairly straightforward manner by computing the minimal solution to the Pell equation $X^2 - dY^2 = 1$ for a number of values of d. As a consequence of this reduction using Theorem 2.1, we improved upon the aforementioned result of PINTÉR in [Jacobson03] as follows.

Theorem 6.2. *For $2 \leq k \leq 58$ and k even, the only positive integer solution (x, y) to the equation $y^2 = 1^k + 2^k + \cdots + x^k$ is the trivial solution $(x, y) = (1, 1)$, except in the case $k = 2$, for which there is the solution $(x, y) = (24, 70)$.*

Under the assumption of the Extended Riemann Hypothesis, the result holds also for even values of k in the range $60 \leq k \leq 70$.

We note that some more recent progress on Schäffer's conjecture has been made by BENNETT, GYŐRY, and PINTÉR in [BGP04], who have shown that the conjecture is true in the case that $k \leq 11$ and q arbitrary. In [Pintér2004], PINTÉR has shown that the conjecture is true in the case that $k \leq 58$ is even and q arbitrary, and in [Pintér2004a], he has proved that the conjecture holds in the case that $k \leq 61$ is odd and $q > 4$ is even. A survey paper by GYŐRY and PINTÉR on this progress can be found in [GP03].

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