Publ. Math. Debrecen
65/3-4 (2004), 439–460

On the diophantine equation $S_m(x) = g(y)$

By CSABA RAKACZKI (Debrecen)

To the memory of Professor Béla Brindza

Abstract. In this paper we characterize those polynomials $g(y)$ with rational coefficients and positive integers m for which $1^m + 2^m + \cdots + x^m = g(y)$ has infinitely many integer solutions. As an application of this result we give an ineffective finiteness result concerning equation $S_m(x) = F(\binom{y}{n})$, where $F(y) \in \mathbb{Q}[y]$.

1. Introduction

In our paper we study the diophantine equation

$$S_m(x) = g(y) \quad \text{in positive integers } x, y, \tag{1}$$

where $g(y)$ is a polynomial with rational coefficients, m is a positive integer and

$$S_m(x) = 1^m + 2^m + \cdots + x^m.$$

In 1956, SCHÄFFER [18] established an ineffective finiteness theorem for the number of solutions of (1) in the special case when $g(y) = y^n$. An effective version of this result was proved by GYŐRY, TIJDEMAN and VOORHOEVE

Mathematics Subject Classification: 11D41.
Key words and phrases: higher degree equations.
This paper is supported in part by the Netherlands Organization for Scientific Research (NWO), the Hungarian Academy of Sciences, by grants N34001, T42985 and F34981 of the HNFSR and by the FKFP grant 3272-13/066/2001.

[11] who investigated SCHÄFFER's equation in the more general case when the exponent n is also unknown. Later, several generalizations, extensions and related results have been obtained see e.g. [6]–[9], [13], [16], [19]–[22] and the references given there. Recently, JACOBSON, PINTÉR and WALSH [12] and BENNETT, GYŐRY and PINTÉR [1] resolved Schäffer's equation for $n = 2$, m even with $m \leq 58$, and for arbitrary n and $m \leq 11$, respectively. For a survey of these results we refer to [10].

The main purpose of the present paper is to characterize those integers m and polynomials $g(y) \in \mathbb{Q}[y]$ for which (1) may have infinitely many integer solutions x, y. Further, we give for each type of these pairs $(m, g(y))$ an equation of the form (1) which has infinitely many solutions (cf. Theorem 1). In our Theorem 2 we give an application of Theorem 1, and we characterize those positive integers m and polynomials $F(x)$ with integer coefficients and with degree one or odd prime for which equation (2) has only finitely many integer solutions. Our results are ineffective because the proof is based upon an ineffective finiteness criterion of BILU and TICHY [4] on diophantine equations of the form $f(x) = g(y)$. We mention that it is complicated to apply the criterion of [4] to special equations. In our proofs, we shall also need some results on Bernoulli polynomials (cf. [3]) and Dickson polynomials (cf. [2]).

2. New results

To present our results we define special pairs $(m, g(x))$. In what follows, let $\delta(x) \in \mathbb{Q}[x]$ be a linear polynomial, and $q(x) \in \mathbb{Q}[x]$ a non-zero polynomial. As is known $S_m(x)$ is a polynomial from $\mathbb{Q}[x]$ with degree $m + 1$ (cf. (4) below). Further, for m odd, $S_m(x)$ can be written in the form $\psi_m\big((x + 1/2)^2\big)$ with an appropriate polynomial $\psi_m(x) \in \mathbb{Q}[x]$. Now define special pairs $(m, g(x))$ as follows:

- Special pair of type I: $(m, S_m(q(x)))$, where $q(x)$ is not constant.
- Special pair of type II: m is odd and $g(x) = \psi_m(\delta(x)q(x)^2)$.
- Special pair of type III: m is odd and $g(x) = \psi_m(c\delta(x)^t)$, where $c \in \mathbb{Q} \setminus \{0\}$, $t \geq 3$ is an odd integer.

- Special pair of type IV: m is odd and $g(x) = \psi_m((a\delta(x)^2 + b)q(x)^2)$, where $a, b \in \mathbb{Q} \setminus \{0\}$.
- Special pair of type V: m is odd and $g(x) = \psi_m(q(x)^2)$.
- Special pair of type VI: $m = 3$ and $g(x) = \delta(x)q(x)^2$.
- Special pair of type VII: $m = 3$ and $g(x) = q(x)^2$.

Theorem 1. *Let m be a positive integer and $g(x) \in \mathbb{Q}[x]$ be a polynomial of degree greater than 2. Then equation (1) has only finitely many integer solutions x, y, unless $(m, g(x))$ is a special pair. Further, for each type of special pairs, the polynomials $\delta(x)$, $q(x)$ and the numbers a, b, c, t can be chosen so that for the corresponding $g(x)$ and for every odd m, equation (1) has infinitely many integer solutions x, y.*

In 2002 Bilu *et al.* [3] proved that the equation $S_m(x) = y(y-1)\ldots(y-(n-1))$ has, at most, finitely many solutions in rational integers x, y for $m \geq 1$, $n \geq 2$ and $(m, n) \neq (1, 2)$. There proof is based upon the decomposition of Bernoulli polynomials and Theorem A. In our second theorem we extend their result and give a general finiteness statement for equation (2).

Theorem 2. *Let $F(x) \in \mathbb{Q}[x]$ be a polynomial of degree p with $p = 1$ or $p \geq 3$ prime. Then the equation*

$$S_m(x) = F\left(\binom{y}{n}\right) \quad \text{in integers } x \geq 1,\ y \geq n, \tag{2}$$

where $n > 2$ if $p = 1$, has only finitely many solutions apart from the cases when

- $\deg F(x) = 1$ *and* $(m, n) \in \{(1, 4), (2, 3), (3, 4)\}$,
- $F(x) = S_m(\delta(x))$, *where* $m = p - 1$ *and* $\delta(x) \in \mathbb{Q}[x]$ *is linear,*
- $F(x) = \psi_m(\delta(x))$ *and* $n = 1$, 2 *or* 4, *where* $\delta(x) \in \mathbb{Q}[x]$ *is linear,*
- $m = 3$, $F(x) = \delta(x)q(x)^2$ *and* $n = 1$, 2 *or* 4, *where* $\delta(x) \in \mathbb{Q}[x]$ *is linear.*

In the proof of Theorem 2 we shall give in each exceptional case, apart from the last one, a concrete equation which has infinitely many integer solutions x, y. It possible that in the last exceptional case there are equations with infinitely many integer solutions but we are not able to find such an equation.

3. Auxiliary results

To formulate Bilu and Tichy's explicit finiteness criterion for diophantine equations of the form $f(x) = g(y)$, we have to define five kinds of 'standard pairs' of polynomials. Let a, b be non-zero rational numbers.

A *standard pair of first kind* is $(x^t, (ax^r q(x)^t))$ or switched $((ax^r q(x)^t, x^t))$, where $0 \le r < t$, $(r,t) = 1$ and $r + \deg q(x) > 0$.

A *standard pair of second kind* is $\big(x^2, (ax^2+b)q(x)^2\big)$ (or switched).

Denote by $D_k(x,a)$ the k-th Dickson polynomial which is defined by the formula

$$D_k(x,a) = \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \frac{k}{k-i} \binom{k-i}{i} (-a)^i x^{k-2i}. \tag{3}$$

A *standard pair of third kind* is $\big(D_k(x,a^t), D_t(x,a^k)\big)$, where $(k,t) = 1$.

A *standard pair of fourth kind* is $\big(a^{-k/2} D_k(x,a), b^{-t/2} D_t(x,b)\big)$, where $(k,t) = 2$.

A *standard pair of fifth kind* is $\big((ax^2-1)^3, 3x^4 - 4x^3\big)$ (or switched).

The following theorem is the main result of Bilu and Tichy [4].

Theorem A. *Let $f(x)$, $g(x) \in \mathbb{Q}[x]$ be nonconstant polynomials such that the equation $f(x) = g(y)$ has infinitely many solutions in rational integers x, y. Then $f(x) = \varphi\big(f_1(\lambda(x))\big)$ and $g(x) = \varphi\big(g_1(\mu(x))\big)$, where $\lambda(x)$, $\mu(x) \in \mathbb{Q}[x]$ are linear polynomials, $\varphi(x) \in \mathbb{Q}[x]$, and $(f_1(x), g_1(x))$ is a standard pair.*

Using this theorem, Kulkarni and Sury [14] has recently obtained a finiteness result concerning equations of the form $x(x+1)\dots(x+(n-1)) = g(y)$, where $g(y) \in \mathbb{Q}[y]$ is of degree ≥ 2.

Theorem B (Kulkarni, Sury, [14])**.** *Let $g(x)$ be a polynomial of degree $n \ge 2$ in $\mathbb{Q}[x]$, and let $f(z) = z(z+1)(z+2)\dots(z+(m-1))$ with $m \ge 3$. If the equation*

$$g(x) = f(z)$$

has infinitely many integer solutions x, z then the pair $(m, g(x))$ is one of the following exceptional pairs:

- *Exceptional pair A: $\big(m, \psi(p(x))\big)$, where $p(x)$ is a nonconstant polynomial.*

- *Exceptional pair B: m even, $g(x) = \phi_m\big(\delta(x)p(x)^2\big)$, where $p(x)$ can be a constant polynomial.*
- *Exceptional pair C: m even, $g(x) = \phi_m\big(c\delta(x)^r\big)$ for some non-zero constant c and some odd integer $r \geq 3$.*
- *Exceptional pair D: m even, $g(x) = \phi_m\big((a\delta(x)^2 + b)p(x)^2\big)$, where $a, b \in \mathbb{Q}$.*
- *Exceptional pair E: m even, $g(x) = \phi_m\big(p(x)^2\big)$.*
- *Exceptional pair F: $m = 4$, $g(x) = b\delta(x)^2 + \frac{9}{16}$, where $b \in \mathbb{Q} \setminus \{0\}$,*

where $\delta(x)$ is a linear polynomial, $p(x)$ is a non-zero polynomial in $\mathbb{Q}[x]$ and $\phi_m(x) = \big(x - \frac{1}{4}\big)\big(x - \frac{9}{4}\big)\ldots\Big(x - \frac{(m-1)^2}{4}\Big)$, $\psi(x) = x(x+1)(x+2)\ldots(x+(m-1))$.

A *decomposition* of a polynomial $F(x) \in \mathbb{C}[x]$ is defined as $F(x) = G_1(G_2(x))$, where $G_1(x)$, $G_2(x) \in \mathbb{C}[x]$. The decomposition is nontrivial if $\deg G_1(x)$, $\deg G_2(x) > 1$. Two decompositions $F(x) = G_1(G_2(x))$ and $F(x) = H_1(H_2(x))$ are called equivalent if there exists a linear polynomial $t(x) \in \mathbb{C}[x]$ such that $G_1(x) = H_1(t(x))$ and $H_2(x) = t(G_2(x))$. The polynomial $F(x)$ is called *decomposable* if it has at least one nontrivial decomposition, and *indecomposable* otherwise.

The following two results are due to Bilu *et al.* [3]. The first is concerned with the decomposition of the n-th Bernoulli polynomial. The n-th Bernoulli polynomial $B_n(x)$ is defined by

$$te^{tx}/(e^t - 1) = \sum_{n=0}^{\infty} B_n(x)t^n/n!\,.$$

Set $B_n = B_n(0)$.

Theorem C. *The polynomial $B_n(x)$ is indecomposable for odd n. If $n = 2m$ is even, then any nontrivial decomposition of $B_n(x)$ is equivalent to $B_n(x) = \widetilde{B}_m\big((x - 1/2)^2\big)$, where $\widetilde{B}_m(x) \in \mathbb{Q}[x]$ is an indecomposable polynomial of degree m.*

The next result, which is also proved in [3], is a technical lemma which we will be needed in our proofs.

Lemma 1. *Let a_1, b_1, $c_1 \in \mathbb{Q} \setminus \{0\}$ and a_0, b_0, $c_0 \in \mathbb{Q}$. Then the polynomial $S_m(a_1x + a_0)$ is neither of the form $b_1x^q + b_0$ with $q \geq 3$, nor*

of the form $c_1 D_k(x,a)+c_0$, *where* $D_k(x,a)$ *is the* k*-th Dickson polynomial with* $k>4$ *and* $a \in \mathbb{Q}\setminus\{0\}$.

The following lemma is a simple consequence of Theorem C and the well-known relation

$$S_m(x) = (B_{m+1}(x+1) - B_{m+1})/(m+1). \tag{4}$$

Lemma 2. *The polynomial* $S_m(x)$ *is indecomposable for even* m. *If* $m=2k-1$ *is odd then any nontrivial decomposition of* $S_m(x)$ *is equivalent to*

$$S_m(x) = \psi_m\left(\left(x+\frac{1}{2}\right)^2\right), \tag{5}$$

where $\psi_m(x) = t(\widetilde{B}_k(x))$ *with* $t(x) = (x - B_{m+1})/(m+1)$ *and with the* $\widetilde{B}_k(x)$ *specified in Theorem C.*

In the next two lemmas we show that the polynomials $B_m(x)$ and $\psi_m(x)$ have at least one non-real zero if $m \geq 6$.

Lemma 3. *For* $m \geq 6$ *the* m*-th Bernoulli polynomial* $B_m(x)$ *has a non-real zero.*

Lemma 4. *If* $m > 6$ *is odd then both polynomials* $\psi'_m(x)$ *and* $\psi'_m(ax^2+b)$ *have at least one non-real zero, where* a, b *are rationals with* $a \neq 0$.

For $P(x) \in \mathbb{C}[x]$, a complex number c is said to be an extremum if $P(x)-c$ has multiple roots. The P-type of c is defined to be the tuple $(\alpha_1, \alpha_2, \ldots, \alpha_s)$ of the multiplicities of the distinct roots of $P(x)-c$. Theorem D is concerning Dickson polynomials. For a proof see, for instance [2, Proposition 3.3].

Theorem D. *For* $a \neq 0$ *and* $k \geq 3$, $D_k(x,a)$ *has exactly two extrema* $\pm 2a^{\frac{k}{2}}$. *If* k *is odd, then both are of* P*-type* $(1,2,2,\ldots,2)$. *If* k *is even, then* $2a^{\frac{k}{2}}$ *is of* P*-type* $(1,1,2,\ldots,2)$ *and* $-2a^{\frac{k}{2}}$ *is of* P*-type* $(2,2,\ldots,2)$.

The next theorem is due to Ping-Zhi [15].

Theorem E. *Let $a \neq 0$, $b \neq 0$, c and $n \geq 3$ be integers. Then apart from the case when $n = 4$, $c/a = -1/24$ or $3/128$, $r = 2$ and b/a is not a square, all rational integer solutions x, y, r of the equation*

$$a\binom{x}{n} = by^r + c \tag{6}$$

with x, $y > 1$, $r > 1$ satisfy

$$\max(|x|, y, r) < C_1,$$

where C_1 is an effectively computable constant depending only on a, b, c and n.

The last lemmas, which are concerned with some properties of the m-th Bernoulli polynomials $B_m(x)$, are due to Brillhart [5].

Lemma 5. *If m is odd then $B_m(x)$ has no multiple roots. For even m the only polynomial which can be a multiple factor of $B_m(x)$ over $\mathbb{Q}$ is $x^2 - x - b$, where b is a positive, odd integer.*

Lemma 6. *Let b be a real number. Then $B_m(x)$ has $\frac{1}{2} + \frac{bi}{2}$ as a root iff m is odd and $b = 0$.*

4. Proofs

We start with the proofs of Lemma 3 and Lemma 4.

Proof of Lemma 3. It is easy to check that the polynomial $B_6(x)$ has non-real zeros. Let $m \geq 6$ and assume that our statement is true for this m. We show that the lemma is true for $m + 1$, too. Suppose that $B_{m+1}(x)$ has only real zeros. Then, by Lemma 5, all the zeros of

$$B'_{m+1}(x) = (m + 1)B_m(x)$$

are real numbers, which is a contradiction. □

Proof of Lemma 4. From

$$S_m(x) = \frac{B_{m+1}(x + 1) - B_{m+1}}{m + 1} = \psi_m\left(\left(x + \frac{1}{2}\right)^2\right) \tag{7}$$

we deduce that

$$B_m\left(x+\frac{1}{2}\right) = S'_m\left(x-\frac{1}{2}\right) = 2x\psi'_m\left(x^2\right). \tag{8}$$

By Lemma 3 we can see that $B_m(x+1/2)$ has non-real zero, thus

$$\exists\, c,d\in\mathbb{R} : d\neq 0 \text{ such that } \psi'_m\big((c+di)^2\big) = \psi'_m\big(c^2-d^2+2cdi\big) = 0. \tag{9}$$

If $c = 0$ then $B_m(1/2 + di) = 0$ by (8) and (9). But this contradicts Lemma 6. So $c \neq 0$ and we can write

$$\psi'_m(x) = A\prod_{j=1}^{\frac{m-1}{2}}(x-\alpha_j), \tag{10}$$

where $\alpha_1 \in \mathbb{C}\setminus\mathbb{R}$ and $A \in \mathbb{Q}\setminus\{0\}$. It follows from now that

$$\psi'_m\big(ax^2+b\big) = A\prod_{j=1}^{\frac{m-1}{2}}\big(ax^2+b-\alpha_j\big) \tag{11}$$

and $x = \sqrt{(\alpha_1-b)/a}$ is a non-real zero of $\psi'_m(ax^2+b)$. □

4.1. Proof of Theorem 1.

Proof. Let $g(x) \in \mathbb{Q}[x]$ with $\deg g(x) > 2$. Suppose that equation (1) has infinitely many integer solutions x, y. Then by Theorem A, there exist polynomials $\lambda(x)$, $\mu(x)$, $\varphi(x)$, $f_1(x)$, $g_1(x) \in \mathbb{Q}[x]$ with $\deg\lambda(x) = \deg\mu(x) = 1$ such that

$$S_m(x) = \varphi(f_1(\lambda(x))), \quad \text{and} \quad g(x) = \varphi(g_1(\mu(x))), \tag{12}$$

where $(f_1(x), g_1(x))$ is a standard pair. Since $\deg S_m(x) = m+1$ we obtain from Lemma 2 and (12) that $\deg\varphi(x) = 1$ or $(m+1)/2$ or $m+1$.

4.1.1. *The case* $\deg\varphi(x) = m+1$. If we assume that $\deg\varphi(x) = m+1$, then we get from (12) that $\deg f_1(x) = 1$. Thus $S_m(x) = \varphi(t(x))$, where $t(x) \in \mathbb{Q}[x]$ is a linear polynomial. If

$$t(x) = t_1x + t_0 \text{ then we set } \quad t^{-1}(x) = \frac{1}{t_1}x - \frac{t_0}{t_1}.$$

We obtain $S_m(t^{-1}(x)) = \varphi(t(t^{-1}(x))) = \varphi(x)$. Hence we have

$$g(x) = \varphi(g_1(\mu(x))) = S_m(t^{-1}(g_1(\mu(x)))) = S_m(q(x)),$$

where $q(x) = t^{-1}(g_1(\mu(x))) \in \mathbb{Q}[x]$. So, if $\deg \varphi(x) = m + 1$, equation (1) may have infinitely many integer solutions x, y only if $g(x)$ is of the form $S_m(q(x))$, where $q(x) \in \mathbb{Q}[x]$. Obviously, if $q(y) \in \mathbb{Z}$ for infinitely many integers y, then for these integers y, $x = q(y)$, y are integer solutions of (1).

4.1.2. *The case* $\deg \varphi(x) = 1$. Let $\varphi(x) = \varphi_1 x + \varphi_0$, where φ_1, $\varphi_0 \in \mathbb{Q}$ and $\varphi_1 \neq 0$. We study now the five kinds of standard pairs.

First consider the case when, in (12), $(f_1(x), g_1(x))$ *is a standard pair of first kind.* From (12) we get then that either

i) $S_m(\lambda^{-1}(x)) = \varphi_1 x^t + \varphi_0$,

or

ii) $S_m(\lambda^{-1}(x)) = \varphi_1 a x^r q(x)^t + \varphi_0$, where $0 \leq r < t$, $(r, t) = 1$ and $r + \deg q(x) > 0$.

In the first case, by Lemma 1 we obtain a contradiction if $t = m + 1 \geq 3$. In the remaining case $m = 1$, a simple calculation gives from *i)* that $\varphi_0 = -1/8$ and

$$g(x) = \frac{1}{2}(2\varphi_1 a \mu(x) q(\mu(x))^2) - \frac{1}{8} = \psi_1(\delta(x) q_1(x)^2),$$

where $\delta(x) = 2\varphi_1 a \mu(x)$ and $q_1(x) = q(\mu(x))$. Hence we get that $(m, g(x))$ is special pair of type II with $m = 1$.

In the second case, $g(\mu^{-1}(x)) = \varphi_1 x^t + \varphi_0$. Thus, if $t = \deg g(x) > 3$ then $r \leq 3$ and the polynomial $q(x)$ must be constant. Otherwise $S_m\big(\lambda^{-1}(x)\big) - \varphi_0$ would have a zero with at least four multiplicities, which is impossible by a result of Brillhart [5] (see below where we study the case of standard pair of fifth kind). But then

$$S_m\big(\lambda^{-1}(x)\big) = \varphi x^r + \varphi_0 \quad \text{with} \quad \varphi \in \mathbb{Q} \setminus \{0\}. \tag{13}$$

From Lemma 1 we know that r can be only 2. Now, from the investigation of the case *i)* we see again that $\varphi_0 = -1/8$, whence

$$g(x) = \varphi_1 \mu(x)^t - \frac{1}{8} = \frac{1}{2}(2\varphi_1 \mu(x)^t) - \frac{1}{8} = \psi_1(c\delta(x)^t), \tag{14}$$

where $t > 3$ odd. That is $(m, g(x))$ is a special pair of type III.

For $t = 3$

$$S_m(x) = \varphi_1 a\lambda(x)^r q(\lambda(x))^3 + \varphi_0, \tag{15}$$

where $r = 1$ or 2. If $q(x)$ is a constant polynomial we get back (13), so we can assume that $q(x)$ is not a constant. Then from

$$B_m(x+1) = S'_m(x) = \varphi_1 a\lambda(x)^{r-1} q(\lambda(x))^2 \big(r\lambda'(x)q(\lambda(x)) + 3\lambda(x)q'(\lambda(x))\lambda'(x)\big)$$

we infer that $B_m(x)$ has a multiple factor over the rational numbers, namely $q(\lambda(x-1))$. Then Lemma 5 gives for us that m is even and $q(\lambda(x-1))$ can be only $x^2 - x - b$, where b is a positive, odd integer. Comparing the degrees in (15) we can see that m can be only 6 while $r = 1$. But in this case we have that $S_6(x) - \varphi_0$ has a root with three multiplicities and so $B_6(x)$ has a double root. However, the discriminant of $B_6(x)$ is 31/1815156 which is a contradiction.

Let now in (12) $(f_1(x), g_1(x))$ *be a standard pair of second kind.*

Then either

i) $S_m(\lambda^{-1}(x)) = \varphi_1 x^2 + \varphi_0$

or

ii) $S_m(\lambda^{-1}(x)) = \varphi_1(ax^2 + b)q(x)^2 + \varphi_0$.

In the first case it is easy to see that $m = 1$, $\varphi_0 = -1/8$. Hence

$$\begin{aligned} g(x) &= \varphi_1(a\mu(x)^2 + b)q(\mu(x))^2 - \frac{1}{8} \\ &= \psi_1\big((2\varphi_1 a\mu(x)^2 + 2\varphi_1 b)q(\mu(x))^2\big), \end{aligned}$$

that is $(m, g(x))$ is a special pair IV with $m = 1$. In the second case $\deg g(x) = 2$. But this contradicts the conditions of our theorem.

Next let $(f_1(x), g_1(x))$ *be a standard pair of third kind.*

Now from (12) we know that

$$S_m\big(\lambda^{-1}(x)\big) = \varphi(f_1(x)) = \varphi_1 D_k(x, a^t) + \varphi_0. \tag{16}$$

Since $\deg D_k(x, a^t) = k$ and $\deg S_m(\lambda^{-1}(x)) = m+1$, we get from Lemma 1 that (16) is not possible, provided that $m > 3$. In case $m = 1$ we can

deduce from (16) that

$$k = 2,\ \varphi_1 = \frac{1}{8u^2} \quad \text{and} \quad \varphi_0 = \frac{a^t}{4u^2} - \frac{1}{8},$$

where $u \in \mathbb{Q} \setminus \{0\}$. Hence

$$\begin{aligned} g(x) &= \varphi_1 D_t(\mu(x), a^2) + \varphi_0 \\ &= \frac{1}{2}\left(\frac{D_t(\mu(x), a^2) + 2a^t}{4u^2}\right) - \frac{1}{8}. \end{aligned}$$

It follows from Theorem D that the polynomial $D_t(\mu(x), a^2)$ has exactly two extrema: $\pm 2a^t$. Since t is odd, $-2a^t$ is of type $(1, 2, 2, \ldots, 2)$. From these we infer that $D_t(\mu(x), a^2) + 2a^t = \delta_1(x)q_1(x)^2$, where $\delta_1(x)$, $q_1(x) \in \mathbb{Q}[x]$ with $\deg \delta_1(x) = 1$ and so $(m, g(x))$ is a special pair II with $m = 1$, $\delta(x) = \delta_1(x)/4u^2$ and $q(x) = q_1(x)/2u$.

If $m = 2$ then $k = 3$, $(t, 3) = 1$ and $S_2(x) = \varphi_1 D_3(\lambda(x), a^t) + \varphi_0$. From this last equality we can deduce that $\varphi_0 = 0$ and

$$2x(2x + 1)(2x + 2) = 24\varphi_1 D_t(\mu(y), a^3). \tag{17}$$

From Theorem B we can infer that if (17) has infinitely many integer solutions then

$$\begin{gathered} 24\varphi_1 D_t(\mu(y), a^3) = p(y)(p(y) + 1)(p(y) + 2), \\ \text{where } p(y) \in \mathbb{Q}[y]. \end{gathered} \tag{18}$$

However, it follows from (18) that $t = 3 \deg p(y)$ which contradicts $(t, 3) = 1$.

Finally, if $m = 3$, after some computations we obtain from (16) that $m = 3$, $k = 4$, t is odd, $\varphi_0 = 2\varphi_1 a^{2t}$ and $g(x) = \varphi_1\big(D_t(\mu(x), a^4) + 2a^{2t}\big)$. Using again Theorem D we get that $(m, g(x) = \delta(x)q(x)^2)$ is a special pair of type VI.

Assume now that $(f_1(x), g_1(x))$ is a standard pair of fourth kind. In this case we have

$$\begin{gathered} S_m(\lambda^{-1}(x)) = \varphi(f_1(x)) = \varphi_1 a^{-\frac{k}{2}} D_k(x, a) + \varphi_0, \\ \text{where } k \geq 2 \text{ is even.} \end{gathered} \tag{19}$$

But, by Lemma 1, this is impossible in case $m > 3$. Since k is even we have to study only the cases $m = 1$ and $m = 3$. After comparing the appropriate

coefficients in (19) we get that, in case $m = 1$, $\varphi_0 = 2\varphi_1 - 1/8$, while, in case $m = 3$, $\varphi_0 = 2\varphi_1$. Thus using again Theorem D, in the first case

$$\begin{aligned} g(x) &= \varphi_1 b^{-t/2} D_t(\mu(x), b) + \varphi_0 \\ &= \varphi_1 b^{-t/2}\big(D_t(\mu(x), b) + 2b^{t/2}\big) - \frac{1}{8} = \psi_1\big(cq(x)^2\big), \end{aligned}$$

where $c = 2\varphi_1 b^{-t/2}$ and in the second case

$$\begin{aligned} g(x) &= \varphi_1 b^{-t/2} D_t(\mu(x), b) + \varphi_0 \\ &= \varphi_1 b^{-t/2}\big(D_t(\mu(x), b) + 2b^{t/2}\big) = \varphi_1 b^{-t/2} q(x)^2. \end{aligned}$$

It is easy to see that in both cases the equation (1) may have infinitely many solutions only if the rational numbers c and $\varphi_1 b^{-t/2}$ are squares. So we obtained the special pairs of type V and VII.

Finally, let $(f_1(x), g_1(x))$ *be a standard pair of fifth kind.*

From (12) we deduce that one of the two cases holds:

i) $S_m\big(\lambda^{-1}(x)\big) = \varphi\big(f_1(x)\big) = \varphi_1\big(ax^2 - 1\big)^3 + \varphi_0$,

ii) $S_m(\lambda^{-1}(x)) = \varphi(f_1(x)) = \varphi_1(3x^4 - 4x^3) + \varphi_0$.

In both cases we infer that the polynomial $S_m(\lambda^{-1}(x)) - \varphi_0$ has a zero with at least three multiplicities. But the number of the roots as well as their multiplicities of an algebraic equation remain unchanged if we replace x by a linear polynomial, so it follows that the polynomial $S_m(x) - \varphi_0$ has also a zero with at least three multiplicities. By virtue of $\deg S_m(x) = m + 1$ we get that, in case *i*), $m = 5$, while, in case *ii*), $m = 3$. From Lemma 5 we know that for n odd, the n-th Bernoulli polynomial $B_n(x)$ has no multiple roots. Since

$$\frac{d}{dx}\left(S_m(x) - \varphi_0\right) = \frac{d}{dx}\frac{B_{m+1}(x+1)}{m+1} = B_m(x+1)$$

we arrive at a contradiction in both cases.

4.1.3. *The case* $\deg \varphi(x) = (m+1)/2$. Obviously, in this case m is odd. Now from (12) we infer that $\deg f_1(x) = 2$. Hence $(f_1(x), g_1(x))$ cannot be a standard pair of fifth kind. Further, the polynomial $S_m(x)$ has a nontrivial decomposition. By Lemma 2 we know that any nontrivial decomposition of $S_m(x)$ is equivalent to the decomposition $S_m(x) = \psi_m\big((x+1/2)^2\big)$,

so there exists a linear polynomial $u(x) = u_1x + u_0$ such that

$$\varphi(x) = \psi_m(u(x)) \quad \text{and} \quad u(f_1(\lambda(x))) = \left(x + \frac{1}{2}\right)^2. \tag{20}$$

First consider the case when, in (12), $(f_1(x), g_1(x))$ *is a standard pair of first kind.*

Assume that $(f_1(x), g_1(x)) = (x^t, ax^r p(x)^t)$, where $0 \le r < t$, $(r, t) = 1$ and $r + \deg p(x) > 0$. Since $\deg f_1(x) = 2$, we have $(f_1(x), g_1(x)) = (x^2, axp(x)^2)$. If $\lambda(x) = \lambda_1 x + \lambda_0$ then from (20) we deduce that $u(x) = (1/\lambda_1^2)x$ and

$$g(x) = \psi_m(u(g_1(\mu(x)))) = \psi_m\left(\frac{a\mu(x)p(\mu(x))^2}{\lambda_1^2}\right). \tag{21}$$

If we define the polynomials $\delta(x)$ and $q(x)$ by $\delta(x) = a\mu(x)/\lambda_1^2$, $q(x) = p(\mu(x))$, then the pair $(m, g(x))$ is of special pair of type II.

We mention that we can choose a linear polynomial $\delta(x) \in \mathbb{Q}[x]$ and a polynomial $q(x) \in \mathbb{Q}[x]$ such that $(m, g(x))$ is a special pair of type II and equation (1) has infinitely many integer solutions x, y. Indeed, if $\delta(y)$ is the square of a rational number for infinitely many integer y and for these integers y, $\sqrt{\delta(y)}q(y) - 1/2 \in \mathbb{Z}$ then $x = \sqrt{\delta(y)}q(y) - 1/2$, y are solutions of (1). For example, if $\delta(x) = x$ and $q(x) = x^t + \cdots + x + 1/2$, then for every integer k the integers $x = (2k+1)q((2k+1)^2) - 1/2$ and $y = (2k+1)^2$ are solutions of (1).

In the switched case $(f_1(x), g_1(x)) = (ax^r p(x)^t, x^t)$, where $0 \le r < t$, $(r, t) = 1$ and $r + \deg p(x) > 0$. In view of $\deg f_1(x) = 2$ there are two possibilities:

i) $r = 0$, $t = 1$ and $\deg p(x) = 2$

or

ii) $r = 2$, $t > 2$ odd and $\deg p(x) = 0$.

If *i*) holds, we have $g_1(x) = x$ and thus

$$g(x) = \psi_m(u(g_1(\mu(x)))) = \psi_m(u(\mu(x))). \tag{22}$$

Then the pair $(m, g(x))$ is a special pair of type II with $\delta(x) = u(\mu(x))$ and $q(x) \equiv 1$. In case *ii*) we get from (20) that $f_1(x) = bx^2$ and $u(x) = x/(b\lambda_1^2)$,

where $b \in \mathbb{Q} \setminus \{0\}$. Thus

$$g(x) = \psi_m(u(g_1(\mu(x)))) = \psi_m\left(\frac{(\mu(x))^t}{b\lambda_1^2}\right). \tag{23}$$

It is easy to see that in this case the pair $(m, g(x))$ is a special pair of type III with $\delta(x) = \mu(x)$, $c = 1/(b\lambda_1^2)$ and $t \geq 3$ odd.

In the last case we can choose the linear polynomial $\delta(x) \in \mathbb{Q}[x]$, the $c \in \mathbb{Q}\setminus\{0\}$ and the odd integer $t \geq 3$ such that for infinitely many integer y, $c\delta(y)^t$ is a rational square and $\sqrt{c\delta(y)^t} - 1/2 \in \mathbb{Z}$. Then $x = \sqrt{c\delta(y)^t} - 1/2$, y are solutions of (1). For example, if we choose $c = 1/4$, $t = 29$, $\delta(x) = x$, then we obtain that $x = -1/2 + (2k+1)^{29}/2$, $y = (2k+1)^2$ are solutions of (1) for every integer k.

Next assume that in (12) $(f_1(x), g_1(x))$ *is a standard pair of second kind.*

Now $(f_1(x), g_1(x)) = (x^2, (ax^2 + b)p(x)^2)$ or switched. If $f_1(x) = (ax^2 + b)p(x)^2$ then we have $g_1(x) = x^2$ and $p(x)$ is a constant polynomial. Hence

$$g(x) = \psi_m(u(g_1(\mu(x)))) = \psi_m\left(u_1\mu(x)^2 + u_0\right). \tag{24}$$

This means that $(m, g(x))$ is a special pair of type IV with $q(x) \equiv 1$. When $f_1(x) = x^2$, after some simple calculation it follows from (20) that $u(x) = x/(\lambda_1^2)$. So

$$g(x) = \psi_m(u(g_1(\mu(x)))) = \psi_m\left(\frac{\left(a\mu(x)^2 + b\right) p(\mu(x))^2}{\lambda_1^2}\right). \tag{25}$$

With the notation $\delta(x) = \mu(x)$ and $q(x) = p(\mu(x))/\lambda_1$ we get that

$$g(x) = \psi_m\big((a\delta(x)^2 + b)q(x)^2\big).$$

This implies that the pair $(m, g(x))$ is a special pair of type IV. Similarly, in case of special pairs of type II and III we can see that if $\big(a\delta(y)^2 + b\big)q(y)^2$ is a rational square and $\sqrt{\big(a\delta(x)^2 + b\big)q(y)^2} - 1/2 \in \mathbb{Z}$ for infinitely many integers y then (1) has infinitely many integer solutions x, y.

For example, consider the case when $a = 8$, $b = -119$, $\delta(x) = 2x - 13$ and $q(x) \in \mathbb{Q}[x]$ is a polynomial for which $q(x) - 1/2 \in \mathbb{Z}[x]$. In this case

(1) has infinitely many integer solutions, namely

$$x = \frac{2a_{2w+1}q(y) - 1}{2}, \quad y = \frac{b_{2w+1} + 26}{4} \quad \text{for } w = 0, 1, \ldots,$$

where a_w and b_w are defined by $(a_0, b_0) = (3, 8)$, $(a_{w+1}, b_{w+1}) = (3a_w+4b_w, 2a_w + 3b_w)$.

Let now $(f_1(x), g_1(x))$ be a standard pair of third kind.

In this case $(f_1(x), g_1(x)) = (D_2(x, a^t), D_t(x, a^2))$ with odd t. Substituting the value of $f_1(x) = x^2 - 2a^t$ into (20) we obtain that $u_1 = 1/\lambda_1^2$ and $u_0 = 2a^t/\lambda_1^2$. Further,

$$g(x) = \psi_m(u(g_1(\mu(x)))) = \psi_m\left(\frac{D_t(\mu(x), a^2) + 2a^t}{\lambda_1^2}\right). \tag{26}$$

From Theorem D we know that the polynomial $D_t(\mu(x), a^2)/\lambda_1^2$ has exactly two extrema: $\pm 2a^t/\lambda_1^2$. Since t is odd both extrema are of type $(1, 2, 2, \ldots, 2)$. From these we deduce that $g(x) = \psi_m(\delta(x)q(x)^2)$, where $\delta(x)$, $q(x) \in \mathbb{Q}[x]$ with $\deg \delta(x) = 1$. Hence $(m, g(x))$ is a special pair of type II.

Finally, let $(f_1(x), g_1(x))$ be a standard pair of fourth kind.

Then $(f_1(x), g_1(x)) = (a^{-1}D_2(x, a), b^{-t/2}D_t(x, b))$ with even t. From (20) it follows that $u_1 = a/\lambda_1^2$, $u_0 = 2a/\lambda_1^2$ and

$$g(x) = \psi_m(u(g_1(\mu(x)))) = \psi_m\left(\frac{ab^{-t/2}D_t(\mu(x), b) + 2a}{\lambda_1^2}\right). \tag{27}$$

Now, the extrema of the polynomial $ab^{-t/2}D_t(\mu(x), b)/\lambda_1^2$ are $\pm 2b^{t/2}ab^{-t/2}/\lambda_1^2 = \pm 2a/\lambda_1^2$, and the extremum $-2a/\lambda_1^2$ is of type $(2, 2, \ldots, 2)$ by Theorem D. Therefore $g(x) = \psi_m(q(x)^2)$ and $(m, g(x))$ is a special pair of type V. It is easy to see that if in the last case $q(x) - 1/2 \in \mathbb{Z}[x]$ then (1) has infinitely many integer solutions, for example $(x, y) = (q(k) - 1/2, k)$, where $k \in \mathbb{Z}$.

The proof of Theorem 1 is completed. □

4.2. Proof of Theorem 2.

Proof. In case $\deg F(x) = 1$ our equation (2) is of the form

$$S_m(x) = a\binom{y}{n} + b, \tag{28}$$

where $a \neq 0$, b are rational numbers. Assume that $n > 2$ and (28) has infinitely many integer solutions x, y, then from Theorem 1 and Theorem B we get that

$$\begin{aligned} &\Big(m, \frac{a}{n!}f_n(y) + b\Big) \text{ is a special pair and} \\ &\left(n, \frac{n!}{a}(S_m(x) - b)\right) \text{ is an exceptioanl pair,} \end{aligned} \tag{29}$$

where $f_n(x) = x(x-1)\dots(x-(n-1))$. If we compare the degrees of the polynomials in each possible (special pair, exceptional pair) case, we get Table 1, where C means a contradiction, p and q denote the degrees of the polynomials $p(x)$ of Theorem B and $q(x)$ of Theorem 1, respectively. The row index and the column index indicate the corresponding special pair $\big(m, \frac{a}{n!}f_n(y) + b\big)$ and exceptional pair $\big(n, \frac{n!}{a}(S_m(x) - b)\big)$, respectively.

For example, if $\big(m, \frac{a}{n!}f_n(y) + b\big)$ is a special pair II and $\big(n, \frac{n!}{a}(S_m(x) - b)\big)$ is an exceptional pair A, we get that

$$\frac{a}{n!}f_n(y) + b = \psi_m\big(\delta(x)q(x)^2\big) \quad \text{and} \quad \frac{n!}{a}(S_m(x) - b) = \psi(p(x)). \tag{30}$$

Comparing the degrees we have

$$n = \frac{m+1}{2}(2\deg q(x) + 1) \quad \text{and} \quad m + 1 = n \deg p(x). \tag{31}$$

It gives us that $2 = \deg p(x)(2\deg q(x) + 1)$, from which it follows that $\deg q(x) = 0$, $\deg p(x) = 2$ and $m + 1 = 2n$. In the case when $\Big(\big(m, \frac{a}{n!}f_n(y) + b\big), \big(n, \frac{n!}{a}(S_m(x) - b)\big)\Big)$ is a pair of $(\mathrm{II}, \mathrm{B})$ we obtain that

$$n = \frac{m+1}{2}(2\deg q(x) + 1) \quad \text{and} \quad m + 1 = \frac{n}{2}(2\deg p(x) + 1),$$

from which we get $4 = (2\deg q(x) + 1)(2\deg p(x) + 1)$ which is impossible.

We now study those cases in which we do not get a contradiction comparing the degrees. We see from the Table 1 and the definitions of special pairs and exceptional pairs that in cases $(\mathrm{II}, \mathrm{A})$, $(\mathrm{II}, \mathrm{D})$ and $(\mathrm{II}, \mathrm{E})$

$$\frac{ac}{n!}f_n'(cy + d) = \psi_m'(y). \tag{32}$$

Since every zeros of $f_n'(cy + d)$ are reals, we get $m < 6$ by Lemma 4. We infer from $m+1 = 2n$ that (m, n) can be only $(5, 3)$. However, if $m = 5$ and

	A	**B**	**C**	**D**	**E**	**F**
I	$p = q = 1$ $n = m + 1$	$p = 0$, $q = 2$ $n = 2(m + 1)$	C	$p = 0$, $q = 1$ $n = m + 1$	$p = q = 1$ $n = m + 1$	$m = 1$, $n = 4$
II	$p = 2$, $q = 0$ $m + 1 = 2n$	C	C	$p = 1$, $q = 0$ $m + 1 = 2n$	$p = 2$, $q = 0$ $m + 1 = 2n$	C
III	C	C	C	C	C	C
IV	$p = 1$, $q = 0$ $n = m + 1$	$p = 0$, $q = 1$ $n = 2(m + 1)$	C	$p = q = 0$ $n = m + 1$	$p = 1$, $q = 0$ $n = m + 1$	$m = 1$, $n = 4$ $q = 1$
V	$p = q = 1$ $n = m + 1$	$p = 0$, $q = 2$ $n = 2(m + 1)$	C	$p = 0$, $q = 1$ $n = m + 1$	$p = q = 1$ $n = m + 1$	$m = 1$, $n = 4$ $q = 2$
VI	C	C	C	C	C	C
VII	$m = 3$, $n = 4$	$m = 3$, $n = 8$ $p = 0$	C	$m = 3$, $n = 4$	$m = 3$, $n = 4$	C

Table 1: Exceptional pairs

$n = 3$, (32) is not possible, because the polynomial $\psi_5'(y)$ has a rational zero, while the polynomial $f_3'(y)$ has only irrational zeros.

In the following six cases $(\mathrm{IV}, \mathrm{A})$, $(\mathrm{IV}, \mathrm{D})$, $(\mathrm{IV}, \mathrm{E})$, (V, A), (V, D), (V, E) we can deduce that

$$\frac{ac}{n!} f_n'(cy + d) = \psi_m'(uy^2 + v)2uy, \tag{33}$$

where c, d, u, $v \in \mathbb{Q}$ and $cu \neq 0$. Hence, using again the above argument and Lemma 4, we get that (m, n) can be only $(3, 4)$ and $(5, 6)$. If $(m, n) = (5, 6)$ we have from (33) that

$$\frac{ac}{n!} f_6'(cy + d) = \frac{uy}{48}\big(4(uy^2 + v) - 1\big)\big(12(uy^2 + v) - 7\big). \tag{34}$$

If we substitute $y = 0$ into (34) we get that $d = 5/2$. Since $\Big(15 + \sqrt{105 + 24\sqrt{7}}\Big)/6$ is a zero of $f_6'(y)$, it is easy to compute from (34) that

$$\left(4\left(u\frac{105 + 24\sqrt{7}}{36c^2} + v\right) - 1\right)\left(12\left(u\frac{105 + 24\sqrt{7}}{36c^2} + v\right) - 7\right) = 0. \tag{35}$$

But (35) is impossible because $\sqrt{7}$ is not rational. Hence (m,n) can not be $(5,6)$.

Investigating the cases (I,A), (I,D) and (I,E), we get that

$$\frac{a}{n!}f_n(cy+d)+b=S_m(y). \tag{36}$$

It follows from this that

$$\frac{ac}{n!}f_n'(c(y-1)+d)=S_m'(y-1)=B_m(y). \tag{37}$$

By Lemma 3, (37) holds only if $m<6$. Thus in view of $n=m+1$ the pair (m,n) can be only $(2,3)$, $(3,4)$, $(4,5)$ or $(5,6)$. We know that 0, $1/2$ and 1 are zeros of the polynomials $B_3(y)$ and $B_5(y)$. Further, it is easy to check that there is only one rational zero of the polynomials $f_4'(y)$, $f_6'(y)$ and $f_5''(y)$. Hence from (37) we obtain that $-c+d=d=-c/2+d$ and so $c=0$, which is a contradiction. This implies that (m,n) can not be $(3,4)$, $(4,5)$ or $(5,6)$.

We now study the cases (I,B), (IV,B), (V,B), $(\mathrm{VII},\mathrm{B})$. It is easy to see that

$$\frac{n!}{a}(S_m(x)-b)=\phi_n(cx+d). \tag{38}$$

If we differentiate in (38) we get

$$B_m(x)=S_m'(x-1)=\frac{ac}{n!}\phi_n'(c(x-1)+d). \tag{39}$$

But from Lemma 3 we know that this is possible only if $m<6$, because all the zeros of $\phi_n'(x)$ are real. Using $n=2(m+1)$ we find that (m,n) can be only $(1,4)$, $(2,6)$, $(3,8)$, $(4,10)$ or $(5,12)$. Since $\phi_8'(x)$, $\phi_{12}'(x)$ and $\phi_{10}''(x)$ have not any rational zero, but $B_{2k+1}(1/2)=0$ for $k\geq 0$, by (39), (m,n) can not be $(3,8)$, $(4,10)$ or $(5,12)$. In case $m=2$ and $n=6$, comparing the zeros of the polynomials $B_2(x)$ and $\phi_6'(c(x-1)+d)$ in (39), we get that $c=\pm 4\sqrt{21}/3$. But in (39) c is rational thus $(m,n)\neq(2,6)$.

Now, in each of the remaining cases $(m,n)=(1,4)$, $(2,3)$, $(3,4)$ we give a concrete equation which has infinitely many integer solutions x, y.

(m,n)	Equation	Solutions
$(1,4)$	$S_1(x) = 3\binom{y}{4}+1$	$x = \frac{y^2-3y}{2}$
$(2,3)$	$S_2(x) = \frac{1}{4}\binom{y}{3}$	$y = 2x+2$
$(3,4)$	$S_3(x) = 24\binom{y}{4}+1$	$x = \frac{a_w-1}{2}, \quad y = \frac{b_w+3}{2}$

Here, in the last row a_w and b_w are defined by $(a_0, b_0) = (21, 15)$ and $(a_{w+1}, b_{w+1}) = (3a_w + 4b_w, 2a_w + 3b_w)$ for $w = 0, 1, \ldots$.

Next, suppose that $p = \deg F(x) \geq 3$ is a prime, and that equation (2) has infinitely many integer solutions x, y with $x \geq n$, $y \geq 1$. Then the equation

$$S_m(x) = F(y) \quad \text{in positive integers } x, y, \tag{40}$$

has also infinitely many solutions. It follows from Theorem 1 that the pair $(m, F(x))$ is a special pair. Since $\deg F(x) = p \geq 3$ is a prime we get from the definition of special pairs that $(m, F(x))$ can be special pair only if $m \in \{1, 3, p-1, 2p-1\}$.

If $m = 1$, then our equation takes the form

$$S_1(x) = 1 + 2 + \cdots + x = \binom{x+1}{2} = F\left(\binom{y}{n}\right). \tag{41}$$

Then it follows from Theorem 2 of our paper [17] that equation (41) may have infinitely many integer solutions only if $n = 1$, 2 or 4. In each of these cases we gave in [17] concrete examples for equations of the form (41) with infinitely many integer solutions.

In case $m = 3$ we have $F(x) = \delta(x)q(x)^2$, where $\delta(x)$, $q(x) \in \mathbb{Q}[x]$ with $\deg \delta(x) = 1$. In this case our equation is of the form

$$\big(x(x+1)\big)^2 = 4\delta\left(\binom{y}{n}\right) q\left(\binom{y}{n}\right)^2. \tag{42}$$

But by Theorem E, (42) has only finitely many integer solutions apart from $n = 1$, 2 and 4.

The case $m = p - 1$ occurs only if $F(x) = S_m(q(x))$, where $q(x)$ is a linear polynomial with rational coefficients. Then we have to study the

equation

$$S_m(x) = S_m\left(q\left(\binom{y}{n}\right)\right). \tag{43}$$

Obviously, if $q(x) \in \mathbb{Z}[x]$ then $x = q\left(\binom{y}{n}\right)$, $y \geq n$ are solutions of (43).

In the last case when $m = 2p - 1$, we have $F(x) = \psi_m(\delta(x))$, where $\delta(x)$ is a linear polynomial with rational coefficients. Now, from (2) we get the equation

$$\psi_m\left(\left(x + \frac{1}{2}\right)^2\right) = S_m(x) = F\left(\binom{y}{n}\right) = \psi_m\left(\delta\left(\binom{y}{n}\right)\right). \tag{44}$$

Of course, equation (44) has infinitely many integer solutions only if the equation

$$4\delta\left(\binom{y}{n}\right) = (2x+1)^2 \tag{45}$$

has also infinitely many solutions. But we know from Theorem E that equation (45) may have infinitely many integer solutions x, y only if $n = 1$, 2 or 4. It is easy to see that the equation

$$4\left(\frac{1}{2}y + \frac{1}{4}\right) = (2x+1)^2 \tag{46}$$

has infinitely many positive integer solutions. If $n = 2$, the equation

$$4\left(\frac{1}{4}\binom{y}{2} + \frac{15}{4}\right) = (2x+1)^2 \tag{47}$$

has infinitely many solutions in positive integers x, y, namely

$$x = \frac{b_{2w+1} - 2}{4}, \quad y = \frac{a_{2w+1} + 1}{2}, \quad \text{for } w = 0, 1, \ldots,$$

where a_w and b_w are defined by $(a_0, b_0) = (3, 8)$, $(a_{w+1}, b_{w+1}) = (3a_w + 4b_w, 2a_w + 3b_w)$.

Finally, when $n = 4$ it is easy to check that for each integer $y \geq 4$,

$$x = \frac{y^2 - 3y}{2}$$

is a solution of the equation

$$4\left(6\binom{y}{4} + \frac{1}{4}\right) = (2x+1)^2 \tag{48}$$

in positive integers. □

5. Acknowledgements

The author thanks Professors K. Győry and Á. Pintér for their useful and valuable advice and remarks.

References

[1] M. A. Bennett, K. Győry and Á. Pintér, On the diophantine equation $1^k + 2^k + \cdots + x^k = y^n$, *Compositio Math.* (*to appear*).

[2] Y. F. Bilu, Quadratic factors of $f(x) = g(y)$, *Acta Arith.* **90** (1999), 341–355.

[3] Y. Bilu, B. Brindza, P. Kirschenhofer, Á. Pintér and R. F. Tichy, Diophantine equations and Bernoulli polynomials, with appendix by A. Schinzel, *Compositio Math.* **131** (2002), 173–188.

[4] Y. F. Bilu and R. F. Tichy, The diophantine equation $f(x) = g(y)$, *Acta Arith.* **95** (2000), 261–288.

[5] J. Brillhart, On the Euler and Bernoulli polynomials, *J. Reine Angew. Math.* **234** (1969), 45–64.

[6] B. Brindza, On some generalizations of the diophantine equation $1^k + 2^k + \cdots + x^k = y^z$, *Acta Arith.* **44** (1984), 99–107.

[7] B. Brindza and Á. Pintér, On equal values of power sums, *Acta Arith.* **77** (1996), 97–101.

[8] B. Brindza and Á. Pintér, On the number of solutions of the equation $1^k+2^k+\cdots + x^k = y^n$, *Publ. Math. Debrecen* **56** (2000), 271–277.

[9] K. Dilcher, On a diophantine equation involving quadratic characters, *Compositio Math.* **57** (1986), 383–403.

[10] K. Győry and Á. Pintér, On the equation $1^k + 2^k + \cdots + x^k = y^n$, *Publ. Math. Debrecen* **62** (2003), 403–414.

[11] K. Győry, R. Tijdeman and M. Voorhoeve, On the equation $1^k + 2^k + \cdots + x^k = y^z$, *Acta Arith.* **37** (1980), 234–240.

[12] M. Jacobson, Á. Pintér and P. G. Walsh, A computational approach for solving $1^k + 2^k + \cdots + x^k = y^2$, *Math. Comp.* **72** (2003), 2099–2110.

[13] H. Kano, On the equation $s(1^k + 2^k + \cdots + x^k) + r = by^z$, *Tokyo J. Math.* **13** (1990), 441–448.

[14] M. Kulkarni and B. Sury, On the diophantine equation $x(x+1)\ldots(x+(m-1)) = g(y)$, *Indag. Math.* **14** (2003), 35–44.

[15] Y. Ping-Zhi, On a special diophantine equation $a\binom{x}{n} = by^r + c$, *Publ. Math. Debrecen* **44** (1994), 137–143.

[16] Á. Pintér, A note on the equation $1^k + 2^k + \cdots + (x-1)^k = y^m$, *Indag. Math. (N.S.)* **8** (1997), 119–123.

[17] Cs. Rakaczki, On the diophantine equation $F\left(\binom{x}{n}\right) = b\binom{y}{m}$, *Periodica Math. Hung.* (*to appear*).

[18] J. J. Schäffer, The equation $1^p + 2^p + \cdots + n^p = m^q$, *Acta Math.* **95** (1956), 155–189.

[19] J. Urbanowicz, On the equation $f(1)1^k + f(2)2^k + \cdots + f(x)x^k + R(x) = by^z$, *Acta Arith.* **51** (1988), 349–368.

[20] J. Urbanowicz, On diophantine equations involving sums of powers with quadratic characters as coefficients I., *Compositio Math.* **92** (1994), 249–271.

[21] J. Urbanowicz, On diophantine equations involving sums of powers with quadratic characters as coefficients II., *Compositio Math.* **102** (1996), 125–140.

[22] M. Voorhoeve, K. Győry and R. Tijdeman, On the diophantine equation $1^k + 2^k + \cdots + x^k + R(x) = y^z$, *Acta Math.* **143** (1979), 1–8, Corr. **159** (1987), 151–152.

csaba rakaczki
institute of mathematics
university of debrecen
h-4010 debrecen, p.o. box 12
hungary

E-mail: rcsaba@math.klte.hu

(Received April 19, 2004)