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Experiments on the *abc*-conjecture

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Dedicated to the memory of Béla Brindza

Abstract. Computations based on tables maintained by A. Nitaj and publicly available on the Internet are described that lend support to refinements to the *abc*-conjecture of Oesterlé and Masser as proposed previously by the author.

1. Introduction

At a conference in Eger in 1996, some refinements were proposed to the *abc*-conjecture which appeared to relate well to the theory of logarithmic forms and to other aspects of Diophantine approximation [1]. In this note, we shall describe some computations which provide support to the earlier suggestions. We begin with a brief discussion of the *abc*-conjecture and its remarkable mathematical significance.

Let a, b, c be integers with no common factor and satisfying

$$a + b + c = 0.$$

We define N as the 'conductor' or 'radical' of abc, that is the product of all the distinct prime factors of abc. Motivated by a conjecture of Szpiro comparing the discriminant and conductor of an elliptic curve, Oesterlé

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formulated a simple assertion about the sizes of a, b, c and this was refined by MASSER [9] to give the following:

Conjecture 1. *For any $\epsilon > 0$, we have*

$$\max(|a|, |b|, |c|) \ll N^{1+\epsilon},$$

where the implied constant depends only on ϵ.

The conjecture would not hold with $\epsilon = 0$ as one verifies, for instance, by taking $a = 1$, $b = -3^{2^n}$ and noting that then 2^n divides c (see [7]). Masser was influenced by a theorem of MASON [8] that arose from studies on logarithmic forms and had already established the analogue of the conjecture for function fields; STOTHERS [12] had independently come upon the same result by way of the theory of Riemann surfaces. Various generalisations of the *abc*-conjecture in the context of algebraic number fields have been given by VOJTA [14][1].

The conjecture has many striking consequences and it is now recognized as one of the key problems of mathematics. In particular, it shows at once that, for given positive integers a, b, c, the Fermat–Catalan equation

$$ax^r + by^s + cz^t = 0$$

has only finitely many solutions in relatively prime integers x, y, z and positive exponents r, s, t satisfying $\lambda < 1$, where $\lambda = (1/r) + (1/s) + (1/t)$; if the constant in Conjecture 1 were effective, then all solutions could be effectively determined. In fact by Conjecture 1 we have $\max(|x|^r, |y|^s, |z|^t) \ll |xyz|^{1+\epsilon}$, where the implied constant depends on a, b, c and ϵ, whence $|xyz| \ll |xyz|^{(1+\epsilon)\lambda}$ and so, on taking ϵ sufficiently small, it follows that $|x|$, $|y|$ and $|z|$ are bounded. In another direction, Elkies has demonstrated that the *abc*-conjecture furnishes readily the famous theorem of Faltings on the Mordell conjecture and, furthermore, Langevin and Bombieri have shown that it gives the Thue–Siegel–Roth theorem.

Even more remarkably, if one assumes a so-called uniform *abc*-conjecture for number fields in the style of Vojta, then, as GRANVILLE and STARK [6] have demonstrated, one can prove the non-existence of the Siegel zero

[1] I am grateful to Serge Lang for drawing my attention to the paper of VAN FRANKENHUYSEN [13] which contains an interesting discussion relating to Vojta's work.

for Dirichlet L-functions. An effective uniform *abc*-conjecture would make effective Siegel's theorem on the size of the class number of an imaginary quadratic field and then, in view of work of Baker and Schinzel on the genera of binary quadratic forms, it would enable one to resolve in principle the famous 'numeri idonei' problem of Euler (see [2]). In yet another direction Granville [5] has demonstrated that the *abc*-conjecture would imply various results on square-free numbers of a kind studied by Hooley and others. All the applications of Conjecture 1 are basically straightforward though in some instances a theorem of G. V. Belyĭ [3] is utilized implying that, for any complete non-singular algebraic curve X defined over the algebraic closure of $\mathbb{Q}$, there exists a covering $X \to \mathbb{P}^1$ with three ramification points.

2. Interplay with logarithmic forms

There is a close connexion between the *abc*-conjecture and the theory of logarithmic forms. In fact it provides the only approach to date that gives a non-trivial estimate for $\max(|a|, |b|, |c|)$, namely

$$\log \max(|a|, |b|, |c|) \ll N^{1/3} (\log N)^3,$$

where the implied constant is absolute. The result is due to Stewart and Yu Kunrui [10], refining earlier work of Stewart and Tijdeman [11]; the proof is based on an archimedean estimate for logarithmic forms and some long and detailed papers by Yu Kunrui establishing non-archimedean analogues.

Though, as remarked in [2], it seems unlikely that the latter approach in itself is powerful enough to furnish results approximating the strength of Conjecture 1, the theory of logarithmic forms gives grounds for optimism nonetheless. To explain this, we need to discuss the refinements to the *abc*-conjecture mentioned at the beginning which were proposed at the Eger conference [1]. Let $\omega(n)$ denote the number of distinct prime factors of an integer n and define $\omega = \omega(abc)$. Further let $\Theta(N)$ denote the number of positive integers up to N that are composed only of prime factors of N.

Conjecture 2. *We have*

$$\max(|a|, |b|, |c|) \ll N(\log N)^{\omega}/\omega!\,,$$

or, more simply, $\max(|a|,|b|,|c|) \ll N\Theta(N)$, *where the implied constants are absolute.*

Conjecture 3. *There is an absolute constant* κ *such that*

$$\max(|a|,|b|,|c|) \ll N((\log N)/\omega(ab))^{\kappa\omega(ab)},$$

where the implied constant is absolute.

Now consider the logarithmic form

$$\Lambda = u_1 \log v_1 + \ldots + u_n \log v_n,$$

where $v_1, \ldots, v_n$ are positive integers and $u_1, \ldots, u_n$ are integers, not all 0. Assuming that $\Lambda \neq 0$ we can write $\Lambda = \log(a/b)$ for unique positive integers a, b with $(a,b) = 1$. We put $c = a - b$ and we note that, if p is a prime dividing c, then $\log(a/b)$ exists in the p-adic sense and we have $|\Lambda|_p = |c|_p$; if p does not divide c we simply define $|\Lambda|_p = 1$. Then in the case that $a = b + c$ with a, b, c positive, Conjecture 3 is equivalent to an estimate for the expression

$$\Xi = \min(1, |\Lambda|) \prod \min(1, p|\Lambda|_p),$$

with the product taken over all primes p; in a slightly weaker form, this estimate is given by

$$\log \Xi \gg -(\log v_1 + \ldots + \log v_n) \log u,$$

where $u = \max |u_j|$. Thus we see that a result of the strength of the *abc*-conjecture sufficient for all its main applications amounts essentially to (i) replacing the archimedean valuation $|\Lambda|$ by Ξ in the Baker–Wüstholz estimate

$$\log |\Lambda| > -C(n,d)h'(\alpha_1)\ldots h'(\alpha_n)h'(L)$$

for the non-vanishing logarithmic form $L = b_1 z_1 + \ldots + b_n z_n$, where

$$\Lambda = L(\log \alpha_1, \ldots, \log \alpha_n) \neq 0,$$

together with (ii) replacing the product $h'(\alpha_1)\ldots h'(\alpha_n)$ of the heights of the α's by the sum $h'(\alpha_1) + \ldots + h'(\alpha_n)$. Hitherto the archimedean and non-archimedean valuations have been treated individually and it would seem that the way forward for future research is to seek to combine them in an analogous way to the product formula in algebraic number theory.

3. Computational results

We come now to the main point of this note which is to describe some experimental results based on tables available on the Internet that give extreme values of a, b, c satisfying the hypotheses at the beginning (see the site `http://www.unicaen.fr/~nitaj/abc.html` maintained by Abderrahmane Nitaj). I am indebted to Jörg Waldvogel at ETH Zürich for generously supplying an initial set of computational results in this context and to Tom Fisher at the CMS Cambridge for furnishing a more recent set which we describe here. I first learned of the tables from a Colloquium Lecture at the University of Zürich given in December 2002 by Gisbert Wüstholz; he presented certain graphical evidence on the Oesterlé–Masser conjecture and I acknowledge with gratitude discussions that he and I and Jörg Waldvogel had subsequently about this.

In view of the graphs described in §4 below, it would seem that we can now formulate a completely explicit version of the *abc*-conjecture, namely

Conjecture 4. *We have*

$$\max(|a|, |b|, |c|) < \tfrac{6}{5} N (\log N)^{\omega} / \omega! \,.$$

The corresponding result for $\Theta(N)$ is

$$\max(|a|, |b|, |c|) < 24 N \Theta(N).$$

The constant $\frac{6}{5}$ in Conjecture 4 seems relatively secure. The constant 24 above seems less so and it is conceivable that it may need to be increased if a particularly good *abc* example were to emerge. Nonetheless it is unlikely that the bound would go much beyond this. The difference in our confidence relating to these constants lies in the fact that the function $\Theta(N)$ would appear to be in tighter agreement with the *abc*-conjecture than its approximation $(\log N)^{\omega}/\omega!\,$.

Note that Conjecture 4 gives immediately a proof of Fermat's Last Theorem. For certainly we have $(\log N)^{\omega}/\omega! \leqslant N$ and so if $x^n + y^n = z^n$ then Conjecture 4 implies that $z^n < \frac{6}{5}(xyz)^2 \leqslant \frac{6}{5} z^6$. Alternatively we can use the associated conjecture for $\Theta(N)$ since obviously $\Theta(N) \leqslant N$. Note also that Conjecture 4 or its analogue for $\Theta(N)$ enables one to give an explicit expression for the implied constant depending on ϵ in the basic

Oesterlé–Masser conjecture. In fact the function $\Theta(N)$ is related to classical questions concerning the number of lattice points in n-dimensional tetrahedra and F. Beukers [4] has shown in this context that the number of non-negative integers $k_1, \ldots, k_n$ with $k_1\rho_1 + \ldots + k_n\rho_n \leqslant x$ is given by

$$\frac{x^n}{n!\rho_1 \ldots \rho_n} + \frac{1}{2(n-1)!} \frac{\rho_1 + \ldots + \rho_n}{\rho_1 \ldots \rho_n} x^{n-1} + o(x^{n-1}).$$

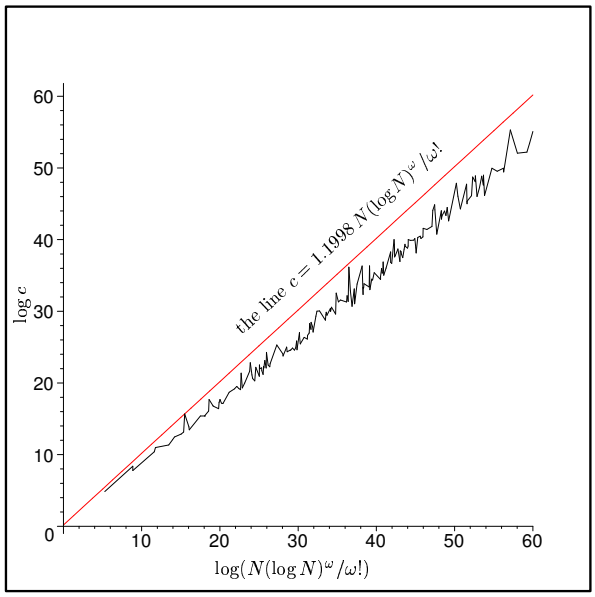
I would be interested to know if there is a recognized name for the function Θ; it was Granville who first pointed out that $\Theta(N)$ and the function $(\log N)^\omega/\omega!$ are of a similar order of magnitude, and indeed this follows readily from the result of Beukers above[2], but I have been unable to find explicit reference to Θ in the literature[3]. I would also be most interested if anyone were able to compute a counter-example to Conjecture 4.

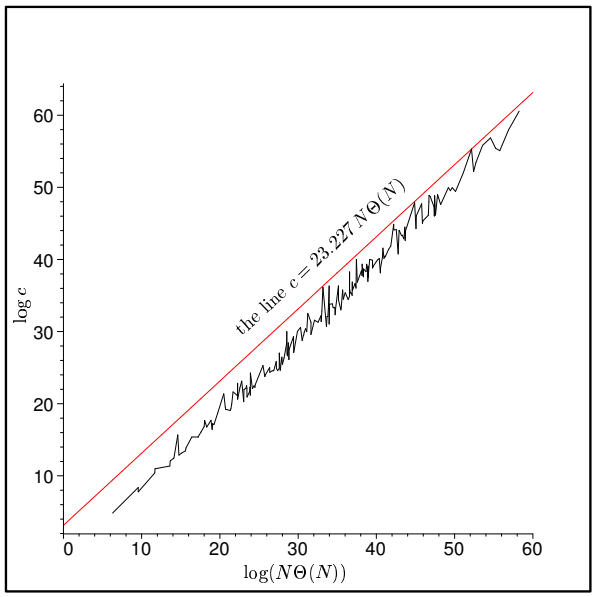
4. The graphs

We have computed graphs of $\log c$ against $\log(N(\log N)^\omega/\omega!)$ and $\log(N\Theta(N))$ respectively over the 196 extremal *abc*-examples, with $c = a + b$, listed in the tables referred to in §3 (as of June 2004). The data gives in each case an essentially linear array approximating in the first instance to the line $c = 1.1998N(\log N)^\omega/\omega!$ and in the second instance to the line $c = 23.227N\Theta(N)$. Note that these lines have slope 1 and their intersections with the vertical axes yield the values of the numerical factors.

In the first graph the points closest to the line correspond to $a = 2$, $b = 3^{10}.109$, $c = 23^5$ and to $a = 19.1307$, $b = 7.29^2.31^8$, $c = 2^8.3^{22}.5^4$. In the second graph the points closest to the line correspond to $a = 19.1307$, $b = 7.29^2.31^8$, $c = 2^8.3^{22}.5^4$, to $a = 7^2.41^2.311^3$, $b = 11^{16}.13^2.79$, $c = 2.3^3.5^{23}.953$ and to $a = 3^4.7^2.41$, $b = 2^{25}.227^7$, $c = 5^9.11^8.2489197589$.

[2]A deduction of this kind, which is attributed to Ennola, occurs in the work of Stewart and Tijdeman [11]; they prove there that there exist infinitely many a, b, c with $a+b = c$ and no common factor such that $\log(c/N) > (4-\delta)(\log N)^{1/2}/\log\log N$ for any $\delta > 0$.
[3]Serge Lang has recently informed me of an estimate by Valentin Bromer, namely $\log\Theta(N) \leqslant (\log 4 + o(1))\log N/\log\log N$.





I understand from Tom Fisher, and also from Jörg Waldvogel in connexion with the original versions, that producing these graphs from the

publicly available data was an essentially straightforward computational task. The only point requiring special attention was the calculation of the function $\Theta(N)$ and here a naive recursive method sufficed for the particular range of N and did not require any substantial computing time.

References

[1] A. Baker, Logarithmic forms and the *abc*-conjecture, Number theory, Eger, 1996; de Gruyter, Berlin, 1998, 37–44.

[2] A. Baker and G. Wüstholz, Number theory, transcendence and Diophantine geometry in the next millennium, Mathematics: frontiers and perspectives, *Amer. Math. Soc.* (2000), 1–12.

[3] G. V. Belyĭ, On the Galois extensions of a maximal cyclotomic field, *Izv. Akad. Nauk SSSR Ser. Mat.* **43** (1979), 267–276 (in *Russian*).

[4] F. Beukers, The lattice-points of n-dimensional tetrahedra, *Indag. Math.* **37** (1975), 365–372.

[5] A. Granville, *ABC* allows us to count squarefrees, *Internat. Math. Res. Notices* **19** (1998), 991–1009.

[6] A. Granville and H. M. Stark, *abc* implies no "Siegel zeros" for L-functions of characters with negative discriminant, *Invent. Math.* **139** (2000), 509–523.

[7] S. Lang, Die *abc*-Vermutung, *Elem. Math.* **48** (1993), 89–99.

[8] R. C. Mason, Diophantine equations over function fields, London Math. Soc. Lecture Notes **96**, *Camb. Univ. Press*, 1984.

[9] D. W. Masser, Open problems, Proc. Symp. Analytic Number Theory, (W. W. L. Chen, ed.), *Imperial Coll. London*, 1985.

[10] C. L. Stewart and Kunrui Yu, On the *abc*-conjecture I, II, *Math. Ann.* **291** (1991), 225–230; *Duke Math. J.* **108** (2001), 169–181.

[11] C. L. Stewart and R. Tijdeman, On the Oesterlé–Masser conjecture, *Monatshefte Math.* **102** (1986), 251–257.

[12] W. W. Stothers, Polynomial identities and Hauptmoduln, *Quart. J. Math. Oxford Ser.* **32** (1981), 349–370.

[13] M. van Frankenhuysen, The ABC conjecture implies Vojta's height inequality for curves, *J. Number Theory* **95** (2002), 289–302.

[14] P. Vojta, Diophantine approximations and value distribution theory, Lecture Notes in Math. **1239**, *Springer*, 1987.

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