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Intertwined basins of attraction for flows on a smooth manifold

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Abstract. In this article, we give some sufficient conditions to guarantee the existence of intertwined basins of attraction for flows on a smooth manifold.

1. Introduction

One of the objectives of science is prediction. For a dynamical system with multiple attractors, the main prediction problem then is to determine which basin of attraction a given point is in, that is, which attractor the trajectory through that point will be attracted to. If it happens that different basins of attraction have mutual boundaries, the basin boundaries can be much complicated (see [3], [5], [6] and references therein). So near such boundaries a small uncertainty in the position of the initial point may yield a large uncertainty as to which attractor the trajectory will go to. Several recent papers [2], [9]–[12] discuss the property of intertwined basins of attraction, in some extent it also leads to the obstruction to predictability. In [9] intertwined basins of attraction are elucidated by examples without much theoretical analysis. According to the work of [9], the author of [10] hopes to give a definition of intertwined basins, however his formula is self-contradictory (see [2]). After that, in [11] the same

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author tries to give a rigorous proof of the result in [10], he uses a new definition ([11, p. 656, Def. 2.1]). Also he use that new definition to discuss basin boundaries for OED's on the two-dimensional sphere S^2 ([12, p. 148, Def. 3.1]). However, we think that his new definition of intertwined basins is still unsuitable. We restate his definition word for word from [11], [12]:

"Suppose that ϕ_t is a flow for an ODE on R^2 (or S^2). Two basins of attraction are said to be intertwined, if they have a common boundary and that common boundary, call it ∂B, has the following property: There are points x and y in ∂B such that for every $\epsilon > 0$, there exists $t_1 > 0$ such that the point $\phi(y, t_1)$ is contained in the intersection of the ϵ-disc $D(x, \epsilon)$ centered at x and a line L_x transversal to the vector field generated by the flow at the point x, or equivalently $D(x, \epsilon) \cap \{\phi(y, t_1)\} \cap L_x \neq \emptyset$."

Consider a system on S^2 defined as follows: The equator is an unstable limit cycle. The trajectories through points in the southern hemisphere and in the northern hemisphere respectively spiral to two polar points. Using the stereogaphic projection, we get such a system from the planar system in polar coordinates: $\dot{r} = r(r-1)$ and $\dot{\theta} = 1$, by adding an infinity point to R^2 as a sink. It is easy to see that the system on S^2 has two sinks whose basins of attraction are respectively the southern hemisphere and the northern hemisphere with a common boundary the equator. Obviously, all the conditions of above definition are satisfied, but there exist no intertwined basins of attraction. So the definition is unsuitable. Also it is easy to give a similar example for planar systems.

The goal of this article is to describe the phenomenon of intertwined basins of attraction by a simple mathematical definition, and then to give some sufficient conditions to guarantee the existence of intertwined basins.

2. Preliminaries

The general information of flows on a surface may be found in [1]. For the convenience of reading, here we recall some basic notions. Let M be a smooth two-dimensional manifold with a metric ρ, on which there is a flow $f : M \times R \to M$ defined by the vector field:

$$\dot{x} = V(x), \quad x \in M. \tag{2.1}$$

Write $x{\cdot}t = f(x,t)$ and let $A{\cdot}J = \{x{\cdot}t \mid x \in A,\ t \in J\}$ for $A \subset M$ and $J \subset R$. So $x \cdot R = \{x\} \cdot R$ and $x \cdot R^+ = \{x\} \cdot R^+$ are respectively the trajectory and the positive semi-trajectory of a point $x \in M$. A set S is invariant under the flow f if $S \cdot R = S$ holds. Thus an invariant set is composed of whole trajectories. Throughout the paper for $A \subset M$, $\overline{A}$, Int A and ∂A denote respectively the closure, interior and boundary of A. The ω-limit set of $x \in M$ is the set $\omega(x) = \{y \in M \mid$ there is a sequence $t_n \in R^+$ such that $t_n \to +\infty$ and $x \cdot t_n \to y\}$, equivalently $\omega(x) = \bigcap_{t \geq 0} \overline{x \cdot [t, +\infty)}$. Similarly we define the α-limit set $\alpha(x)$ of x by reversing the direction of time. A compact and invariant set A is called an *attractor* for the flow f provided that A has a shrinking neighborhood, i.e., there is an open neighborhood U of A such that $\overline{U \cdot t} \subset U$ for $t > 0$ and $A = \bigcap_{t>0} U \cdot t$. If A is an attractor, its *basin of attraction* $B(A)$ is defined to be the set of initial points x such that $\omega(x) \subset A$, i.e., $\rho(x \cdot t, A) \to 0$ $(t \to +\infty)$, where $\rho(x \cdot t, A) = \inf\{\rho(x \cdot t, a) \mid a \in A\}$, with no confusion we also use ρ for the distance between a point and a set. Observe that the basin of attraction $B(A)$ can be expressed as $\bigcup_{t<0} U \cdot t$ for a shrinking neighborhood U of A. Thus $B(A)$ is an open set. By a *sink* we mean an equilibrium that is also an attractor. For a point $p \in M$, let $D(p, \epsilon)$ denote the open disc with radius $\epsilon > 0$ and center p.

Denote O to be a saddle point of System (2.1), then the *stable manifold* $W^s(O)$ and *unstable manifold* $W^u(O)$ are defined to be the following sets:

$$W^s(O) = \{x \in M \mid x \cdot t \to O \text{ as } t \to +\infty\},$$

$$W^u(O) = \{x \in M \mid x \cdot t \to O \text{ as } t \to -\infty\}.$$

The existence of a saddle point with its two branches of unstable manifold approaching different attractors plays an essential role in occurrence of intertwined basins of attraction.

Definition 1. Let p be a regular point of System (2.1), and L is a transversal at p. We call that the system has intertwined basins of attraction beside p, if there exists an arc $L_1 \subset L$ such that p is an endpoint of L_1, and for any $\epsilon > 0$ both

$$L_1 \cap B(A_1) \cap D(p, \epsilon) \neq \emptyset \quad \text{and} \quad L_1 \cap B(A_2) \cap D(p, \epsilon) \neq \emptyset \tag{2.2}$$

hold for two different attractors A_1 and A_2.

Since L is a transversal at p, all the trajectories crosses L in the same direction. Thus, on one side of the trajectory $p \cdot R$ the basins $B(A_1)$ and $B(A_2)$ approach to $p \cdot R$ alternately, of course they become narrower and narrower. Apparently both the basin of attraction $B(A_1)$ and its boundary $\partial B(A_1)$ are invariant sets, it follows that $p \cdot R \subset \partial B(A_1)$. Then by the continuity of dependence on initial conditions we see that the basins $B(A_1)$ and $B(A_2)$ intertwine together along the trajectory $p \cdot R$, it is a common segment of boundaries $\partial B(A_1)$ and $\partial B(A_2)$. In this situation the basins $B(A_1)$ and $B(A_2)$ are called to be of *intertwining property*, or the System (2.1) is said to be of intertwining property. According to the above argument, the prediction of which final attractor will be attained by the system is constrained by the intertwined basins near the solution arc $p \cdot R$.

3. Main results

In this section we assume that $O \in M$ is a saddle point, A_1 and A_2 are two attractors of the System (2.1). Let $B(A_1)$ and $B(A_2)$ be respectively the basins of A_1 and A_2. Denote by $W_1^s(O)$ and $W_2^s(O)$ the two branches of the stable manifold $W^s(O)$, similarly $W_1^u(O)$ and $W_2^u(O)$ respectively denote the two branches of the unstable manifold $W^u(O)$.

Theorem 3.1. *Suppose that $W_1^u(O) \subset B(A_1)$ and $W_2^u(O) \subset B(A_2)$. If $\alpha(W_1^s(O)) \cup \alpha(W_2^s(O))$ has a regular point p, then the System* (2.1) *has the intertwining property.*

Proof. Suppose that $p \in \alpha(W_1^s(O)) \cup \alpha(W_2^s(O))$ is a regular point. Let L be a transversal at p, then all the trajectories crosses L in the same direction. Thus for a point $q \in W_1^s(O) \cup W_2^s(O)$, the negative semi-trajectory $q \cdot R^-$ crosses L successively at t_i with $0 > t_1 > t_2 > \cdots$ ($t_i \to -\infty$) and $q \cdot t_i$ tends monotonously to p along L (see [4, Chap. 7]). Now for any $\epsilon > 0$ we have $q \cdot t_k \in D(p, \epsilon/2)$ for a sufficiently large $|t_k|$. Take a sufficiently small $\lambda > 0$ such that $D(q \cdot t_k, \lambda) \subset D(p, \epsilon)$ holds. Consider the diffeomorphism $F = f(\star, 1) : M \to M$, by the Inclination Lemma ([7, p. 82]), it is easy to see that both $F^n(D(q \cdot t_k, \lambda)) \cap B(A_1) \neq \emptyset$ and $F^n(D(q \cdot t_k, \lambda)) \cap B(A_2) \neq \emptyset$ hold for a sufficiently large n. Thus we

obtain that $f(D(q\cdot t_k,\lambda),n)\cap B(A_1)\neq\emptyset$ and $f(D(q\cdot t_k,\lambda),n)\cap B(A_2)\neq\emptyset$. It follows that (2.2) in Definition 2.1 is true, so the proof is completed. □

Corollary 3.2. *Suppose that the system* (2.1) *just has a finite number of sinks and a saddle point O whose unstable manifold connects two sinks p_1 and p_2. If the manifold M is compact, then the system* (2.1) *has the intertwining property.*

Proof. Since the manifold of M is compact, we have $\alpha(W_1^s(O))\cup\alpha(W_2^s(O))\neq\emptyset$. Certainly, any sinks are not contained in $\alpha(W_1^s(O))\cup\alpha(W_2^s(O))$. Now the corollary follows easily from the Theorem 3.1. □

A Problem. If the basins $B(A_1)$ and $B(A_2)$ have the intertwining property, is it possible that they intertwine elsewhere, not along $W^s(O)$ or $\alpha(W_1^s(O))\cup\alpha(W_2^s(O))$? Equivalently, is the converse of Theorem 3.1 also true?

In the following, we give a partial answer to the problem if M is a simply connected manifold.

Theorem 3.3. *Let M be a simply connected two-dimensional manifold. Suppose that $W_1^u(O)\subset B(A_1)$ and $W_2^u(O)\subset B(A_2)$, and futhermore we assume that there exist no equilibria in $M\setminus(A_1\cup A_2\cup\{O\})$. Then the System* (2.1) *has the intertwining property if and only if $\alpha(W_1^s(O))\cup\alpha(W_2^s(O))$ has a regular point p.*

Proof. By Theorem 3.1 we only need to prove the necessity. If the system (2.1) has the intertwining property, by Definition 2.1 there exist a regular point p and a transversal L at p such that for a subarc $L_1\subset L$ with the endpoint p and for any $\epsilon>0$ both $L_1\cap B(A_1)\cap D(p,\epsilon)\neq\emptyset$ and $L_1\cap B(A_2)\cap D(p,\epsilon)\neq\emptyset$ hold. To prove $p\in\alpha(W_1^s(O))\cup\alpha(W_2^s(O))$, we only need to show that for any $\epsilon>0$ there is a point of $W_1^s(O)\cup W_2^s(O)$ in the disc $D(p,\epsilon)$. Now choose two points $p_1\in L_1\cap B(A_1)\cap D(p,\epsilon)$ and $p_2\in L_1\cap B(A_2)\cap D(p,\epsilon)$ respectively such that the segment $L_1'=p_1p_2$ of L_1 lies in $D(p,\epsilon)$. Apparently, $p_1\cdot t\to A_1$ and $p_2\cdot t\to A_2$ hold as $t\to+\infty$. By the definition of attractor, there exists an open neighborhood U_i of A_i $(i=1,2)$ such that U_i and ∂U_i are homeomorphic to the open unit disc and its boundary S^1 respectively, further each point on ∂U_i $(i=1,2)$ goes into $\operatorname{Int}U_i$ under the flow f. Assume that $p_1\cdot R^+$ and $p_2\cdot R^+$

respectively intersect ∂U_1 and ∂U_2 at q_1 and q_2, also $W_1^u(O)$ and $W_2^u(O)$ respectively intersect ∂U_1 and ∂U_2 at q_1' and q_2'. Thus we can choose two suitable arcs q_1q_1' on ∂U_1 and $q_2'q_2$ on ∂U_2 to constitute a Jordan curve $C = p_1q_1q_1'Oq_2'q_2p_2p_1$ along four solution arcs such that C surrounds a bounded region G containing one branch of $W^s(O)$ but not containing A_1 and A_2. Since M is simply connected, C is a contractible loop. Then G is homeomorphic to the disc. Without loss of generality, we suppose $W_1^s(O) \subset G$. Observe that $O \notin \alpha(W_1^s(O))$ because O is a saddle point. Since there exist no equilibria in G, $W_1^s(O) \cap L_1' \neq \emptyset$ holds, otherwise by the Poincaré–Bendixson Theorem there is at least an equilibrium in G. It follows that $D(p, \epsilon) \cap W_1^s(O) \neq \emptyset$. Now we complete the proof. □

Remark. The conclusion of [12, Proposition 2.1] is contained in the famous Peixoto's Theorem [8]. By the Bendixson Formula [4, p. 166] and the index of an equilibrium, there exists no a system just with two sinks and a saddle point on S^2, so the result of [12, Theorem 3.6] is invalid.

References

[1] S. Aranson, G. Belitsky and E. Zhuzhoma, Introduction to qualitative theory of dynamical systems on surfaces, *AMS, Math. Monogr.*, 1996.

[2] C. Ding, On the Intertwined Basins of Attraction for Planar Flows, *Applied Mathematics and Computation* **48**(3) (2004), 801–805.

[3] C. Grebogi, E. Ott and J. Yorke, Basin boundary metamorphoses: changes in accessible boundary orbits, *Physica D* **24** (1987), 243–262.

[4] P. Hartman, Ordinary differential equations, 2nd edn., *Birkhäuser, Boston*, 1985.

[5] J. Kennedy and J. Yorke, Basins of Wada, *Physica D* **51** (1991), 213–225.

[6] S. McDonald, C. Grebogi, E. Ott and J. Yorke, Fractal basin boundaries, *Physica D* **17** (1985), 125–153.

[7] J. Palis and W. de Melo, Geometric theory of dynamical systems, An introduction, *Springer-Verlag*, 1982.

[8] M. M. Peixoto, Structural stability on two-dimensional manifolds, *Topology* **1** (1962), 101–120.

[9] L. Tedeschini-Lalli, Smoothly intertwined basins of attraction in a prey-predator model, *Acta applicandae Mathematicae* **38** (1995), 139–147.

[10] X. Yang, Intertwined basins of attraction in systems of O.D.E., *Chaos, Solitons and Fractals* **10**(9) (1999), 1453–1456.

[11] X. Yang, A proof for a theorem on intertwining property of attraction basin boundaries in planar dynamical systems, *Chaos, Solitons and Fractals* **15**(4) (2003), 655–657.

[12] X. Yang, Structure of basin boundaries of attractors of ODE's on S^2, *Chaos, Solitons and Fractals* **16**(1) (2003), 147–150.

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