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Necessary and sufficient Tauberian conditions in the case of weighted mean sumable integrals over $\mathbb{R}_+$, II

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Abstract. We prove necessary and sufficient Tauberian conditions for locally integrable functions (in Lebesgue's sense) over $\mathbb{R}_+$, under which convergence follows from summability by weighted mean methods. The main results of this paper apply to all weighted mean methods and unify the results known in the literature for particular methods. Among others, the conditions in our theorems are easy consequences of the slow decrease condition for real-valued functions, or the slow oscillation condition for complex-valued functions. Therefore, practically all classical one-sided as well as two-sided Tauberian conditions for weighted mean methods are corollaries of our two main theorems.

1. Summability of integrals over $\mathbb{R}_+$ by weighted mean methods

Let P be a function defined on $\mathbb{R}_+ := [0, \infty)$ such that

$$P \quad \text{is nondecreasing on} \quad \mathbb{R}_+, \; P(0) = 0 \quad \text{and} \quad \lim_{t\to\infty} P(t) = \infty. \tag{1.1}$$

P is called a *weight function*, due to the fact that it induces a positive Borel measure on $\mathbb{R}_+$.

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For any complex-valued function $f : \mathbb{R}_+ \to \mathbb{C}$ which is integrable in Lebesgue's sense over every finite interval $(0, t)$ for $0 < t < \infty$, in symbol: $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, we set

$$s(x) := \int_0^x f(y)dy \text{ and } \sigma(t) := \frac{1}{P(t)} \int_0^t s(x)dP(x), \quad t > 0, \tag{1.2}$$

provided that $P(t) > 0$. The integral in the definition of $\sigma(t)$ exists as a Riemann–Stieltjes integral.

Now, σ is called the weighted mean of s and the formal integral

$$\int_0^\infty f(x)dx \tag{1.3}$$

is called *summable by the weighted mean method* determined by the weight function P, shortly: *summable* (W, P), if the following finite limit exists:

$$\lim_{t\to\infty} \sigma(t) = L. \tag{1.4}$$

It is easy to check that if the finite limit

$$\lim_{x\to\infty} s(x) = L \tag{1.5}$$

exists (in other words: if the improper integral $\int_0^{\to\infty} f(x)dx$ is convergent), then the limit in (1.4) also exists with the same L. The reverse implication is not true in general.

However, if a real-valued function $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is of constant sign on $\mathbb{R}_+$, then the existence of the limits in (1.4) and (1.5) are equivalent. Indeed, this claim follows from the equality

$$\sigma(t) := \frac{1}{P(t)} \int_0^t \left\{ \int_0^x f(y)dy \right\} dP(x) = \int_0^t f(y) \left\{ 1 - \frac{P(y)}{P(t)} \right\} dy, \tag{1.6}$$

where we used (1.2) and applied Fubini's theorem. In particular, in case $f(t) \ge 0$ we have $\sigma(t) \le s(t)$ for all $t > 0$. Assume that $\lim_{t\to\infty} s(t) = \infty$, then by (1.1) and (1.6), we have necessarily $\lim_{t\to\infty} \sigma(t) = \infty$. This proves the claimed equivalence of (1.4) and (1.5).

It is also clear that summability (W, P) of integral (1.3) does not depend on the values of $f(x)$ assumed on any finite interval $(0, x_0)$, where

$0 < x_0 < \infty$ is fixed. However, the value of the limit L in (1.4) (if it exists) does depend on the values of f assumed on the whol semi-axis $\mathbb{R}_+$.

The particular case of the *Cesàro method* of first order; briefly: summability $(C, 1)$ corresponding to the special choice $P(x) := x$, was studied by Hardy [2, on p. 11] and Titchmarsh [10, on p. 26]. The case when P is a (strictly) increasing and continuous function with (1.1) was studied by Karamata [4, see especially on p. 28 and 36]. The so-called Tauberian conditions obtained by them were only sufficient (and not necessary) for the validity of the implication (1.4) $\Longrightarrow$ (1.5); that is, those were only sufficient to conclude the convergence of integral (1.3) from its summability (W, P).

2. Main results

Our goal is to obtain necessary and sufficient conditions under which the convergence of integral (1.3) follows from its summability by the given weighted mean method.

To this effect, we introduce the notions of upper and lower allowed functions with respect to the given weight function P. Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing, continuous function such that $\rho(t) \to \infty$ as $t \to \infty$. We say that ρ is an *upper allowed function* with respect to P if

$$\liminf_{t\to\infty} \frac{P(\rho(t))}{P(t)} > 1. \tag{2.1}$$

Similarly, we say that ρ is a *lower allowed function* with respect to P if

$$\liminf_{t\to\infty} \frac{P(t)}{P(\rho(t))} > 1. \tag{2.2}$$

We denote by Λ_u and Λ_ℓ the classes of all upper and lower allowed functions, respectively.

For real-valued functions f we shall prove the following one-sided Tauberian theorem.

Theorem 1. *Assume that P satisfies* (1.1), $f : \mathbb{R}_+ \to \mathbb{R}$ *and* $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. *The convergence of integral* (1.3) *follows from its summability*

(W, P) to the same limit if and only if both of the following two conditions are satisfied:

$$\sup_{\rho \in \Lambda_u} \liminf_{t \to \infty} \frac{1}{P(\rho(t)) - P(t)} \int_t^{\rho(t)} \{s(x) - s(t)\} dP(x) \geq 0 \tag{2.3}$$

and

$$\sup_{\rho \in \Lambda_\ell} \liminf_{t \to \infty} \frac{1}{P(t) - P(\rho(t))} \int_{\rho(t)}^{t} \{s(t) - s(x)\} dP(x) \geq 0. \tag{2.4}$$

It is plain that it is enough to verify conditions (2.3) and (2.4) for some appropriate subclasses $\widetilde{\Lambda}_u \subset \Lambda_u$ and $\widetilde{\Lambda}_\ell \subset \Lambda_\ell$, respectively. In the sequel, we present appropriate subclasses in three important special cases.

(i) Following KARAMATA [4], assume that

$$P \text{ is strictly increasing and continuous on some interval } [t_0, \infty], \tag{2.5}$$

where $P(t_0) = 0$, $0 \leq t_0 < \infty$. Then the inverse function of P, denoted by P^{-1}, exists, and it is also increasing and continuous on $\mathbb{R}_+$.

This time, the subclasses

$$\widetilde{\Lambda}_u := \left\{\rho_\lambda(t) := P^{-1}(\lambda P(t)) : \lambda > 1\right\} \qquad \text{and}$$
$$\widetilde{\Lambda}_\ell := \left\{\rho_\lambda(t) := P^{-1}(\lambda P(t)) : 0 < \lambda < 1\right\}$$

are appropriate.

We recall that a function $s : \mathbb{R}_+ \to \mathbb{R}$ is said to be *slowly decreasing* with respect to P if

$$\lim_{\lambda \to 1+} \liminf_{t \to \infty} \min_{t \leq x \leq T} \{s(x) - s(t)\} \geq 0, \tag{2.6}$$

where

$$T := P^{-1}(\lambda P(t)), \quad t > 0. \tag{2.7}$$

The term “slow decrease” was introduced by SCHMIDT [9] for sequences of real numbers. For real-valued functions, definition (2.6) is due to KARAMATA [4, Theorem 5 on p. 36]. Clearly, condition (2.3) is a trivial consequence of (2.6).

The remarkable fact is that (2.6) is equivalent to the condition

$$\lim_{\lambda \to 1-} \liminf_{t \to \infty} \min_{T \leq x \leq t} \{s(t) - s(x)\} \geq 0, \tag{2.8}$$

where T is defined in (2.7). This claim will be proved in Section 3. (See Lemma 1 there.)

On the other hand, conditions (2.3) and (2.4) are independent of one another in general. An example will be given in Section 3 after Lemma 2.

Now, Theorem 1 together with Lemma 1 in Section 3 yield the theorem of Karamata [4, p. 36], where he indicated that it could be proved along the same lines as a corresponding theorem was proved in [3] by him.

Corollary 1. *Assume that* P *satisfies* (2.5), $f : \mathbb{R}_+ \to \mathbb{R}$ *and* $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. *If* s *defined in* (1.2) *is slowly decreasing with respect to* P, *then the convergence of integral* (1.3) *follows from its summability* (W, P) *to the same limit.*

(ii) If the weight function P satisfies (1.1) and is such that

$$\liminf_{t\to\infty} \frac{P(\lambda t)}{P(t)} > 1 \quad \text{for all } \lambda > 1, \tag{2.9}$$

then the subclasses

$$\widetilde{\Lambda}_u := \{\rho_\lambda(t) := \lambda t : \lambda > 1\} \quad \text{and} \quad \widetilde{\Lambda}_\ell := \{\rho_\lambda(t) := \lambda t : 0 < \lambda < 1\}$$

are appropriate. This particular case was studied in [7] in details.

(iii) A third example is that when P satisfies (1.1) and is such that

$$\liminf_{t\to\infty} \frac{P(t^\lambda)}{P(t)} > 1 \quad \text{for all } \lambda > 1. \tag{2.10}$$

This time the subclasses

$$\widetilde{\Lambda}_u := \left\{\rho_\lambda(t) := t^\lambda : \lambda > 1\right\} \quad \text{and} \quad \widetilde{\Lambda}_\ell := \left\{\rho_\lambda(t) := t^\lambda : 0 < \lambda < 1\right\}$$

are appropriate.

Analysing the proof of Theorem 1 in Section 3 reveals that Theorem 1 remains valid if conditions (2.3) and (2.4) are replaced by their symmetric counterparts

$$\inf_{\rho\in\Lambda_u} \limsup_{t\to\infty} \frac{1}{P(\rho(t)) - P(t)} \int_t^{\rho(t)} \{s(x) - s(t)\} dP(x) \le 0$$

and

$$\inf_{\rho\in\Lambda_\ell} \limsup_{t\to\infty} \frac{1}{P(t)-P(\rho(t))} \int_{\rho(t)}^{t} \{s(t)-s(x)\}dP(x) \le 0.$$

In the special case when the weight function P satisfies (2.1), we claim that both conditions just above are implied immediately by the property

$$\lim_{\lambda\to 1+} \limsup_{t\to\infty} \max_{t\le x\le T} \{s(x)-s(t)\} \le 0, \tag{2.11}$$

where T is defined in (2.7). We may say that the function $s : \mathbb{R}_+ \to \mathbb{R}$ with property (2.11) is *slowly increasing* with respect to P.

This claim follows easily from the fact that condition (2.11) is satisfied if and only if $(-s)$ is slowly decreasing; and therefore, from this observation and Lemma 1 it follows that (2.11) is equivalent to the condition

$$\lim_{\lambda\to 1-} \limsup_{t\to\infty} \max_{T\le x\le t} \{s(t)-s(x)\} \le 0.$$

For complex-valued functions f we shall prove the following two-sided Tauberian theorem.

Theorem 2. *Assume that P satisfies* (1.1), *$f : \mathbb{R}_+ \to \mathbb{C}$ and $f \in L^1_{\text{loc}}(\mathbb{R}_+)$. Then the convergence of integral* (1.3) *follows from its summability (W,P) to the same limit if and only if one of the following two conditions is satisfied:*

$$\inf_{\rho\in\Lambda_u} \limsup_{t\to\infty} \left| \frac{1}{P(\rho(t))-P(t)} \int_{t}^{\rho(t)} \{s(x)-s(t)\}dP(x) \right| = 0 \tag{2.12}$$

or

$$\inf_{\rho\in\Lambda_\ell} \limsup_{t\to\infty} \left| \frac{1}{P(t)-P(\rho(t))} \int_{\rho(t)}^{t} \{s(t)-s(x)\}dP(x) \right| = 0. \tag{2.13}$$

As in the case of Theorem 1, it is enough to verify conditions (2.12) or (2.13) for some appropriate subclasses $\widetilde{\Lambda}_u \subset \Lambda_u$ or $\widetilde{\Lambda}_\ell \subset \Lambda_\ell$, respectively.

For example, assume that the weight function P is such that condition (2.5) is satisfied. We recall that in this case a function $s : \mathbb{R}_+ \to \mathbb{C}$ is said to be *slowly oscillating* with respect to P if

$$\lim_{\lambda\to 1+} \limsup_{t\to\infty} \max_{t\le x\le T} |s(x)-s(t)| = 0, \tag{2.14}$$

where T is defined in (2.7). The term "slow oscillation" was introduced by Hardy [1] for sequences of numbers. For functions, (2.14) occurs in [4, Theorem 3 on p. 28] by KARAMATA.

Now, the following theorem of Karamata is an immediate consequence of Theorem 2.

Corollary 2. *Assume that P satisfies* (2.5), $f : \mathbb{R}_+ \to \mathbb{C}$ *and* $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. *If s defined in* (1.2) *is slowly oscillating with respect to P, then the convergence of integral* (1.3) *follows from its summability* (W, P) *to the same limit.*

We note that (2.14) is equivalent to the condition

$$\lim_{\lambda\to 1-} \limsup_{t\to\infty} \max_{T\le x\le t} |s(t) - s(x)| = 0,$$

which can be proved in the same way as Lemma 1 is proved in Section 3.

3. Proofs

The following lemma plays an important role in the proof of Corollary 1.

Lemma 1. *For any function $s : \mathbb{R}_+ \to \mathbb{R}$, conditions* (2.6) *and* (2.8) *are equivalent.*

PROOF. First, assume that (2.6) is satisfied. This means that for every $\varepsilon > 0$ there exist some $\lambda_1 = \lambda_1(\varepsilon) > 1$ and $t_1 = t_1(\varepsilon) > 0$ such that

$$s(x) - s(t) \ge -\varepsilon \quad \text{whenever} \quad t_1 \le t \le x \le P^{-1}(\lambda_1 P(t)). \tag{3.1}$$

Our aim is to estimate the minimum of the difference $s(t) - s(x)$ from below under the conditions

$$P^{-1}(\lambda_1^{-1} P(t)) \le x \le t \quad \text{and} \quad t \ge t_2, \tag{3.2}$$

where t_2 is chosen so large that $P^{-1}(\lambda_1^{-1} P(t_2)) \ge t_1$. Clearly, from (3.2) it follows that

$$t_1 \le x \quad \text{and} \quad t \le P^{-1}(\lambda_1 P(x)).$$

Thus, by (3.1) we have

$$s(t)-s(x)\geq \min_{x\leq\tau\leq t}\{s(\tau)-s(x)\}\geq \min_{x\leq\tau\leq P^{-1}(\lambda_1 P(x))}\{s(\tau)-s(x)\}\geq -\varepsilon.$$

This is true for all t and x in (3.2). Consequently, we have

$$\min_{P^{-1}(\lambda_1^{-1}P(t))\leq x\leq t}\{s(t)-s(x)\}\geq -\varepsilon \quad \text{whenever} \quad t\geq t_2.$$

Since $\varepsilon>0$ is arbitrarily small, (2.8) follows.

Second, assume that (2.8) is satisfied. An argument analogous to the above one yields (2.6). □

Next, we prove that if integral (1.3) is summable (W,P) to a finite limit, then the so-called *moving weighted averages* with respect to P also converge to the same limit. More precisely, the following lemma is valid.

Lemma 2. *Assume that P satisfies* (1.1), $f:\mathbb{R}_+\to\mathbb{C}$ *and* $f\in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ *is such that the finite limit L exists in* (1.4). *If $\rho\in\Lambda_u$, then*

$$\lim_{t\to\infty}\frac{1}{P(\rho(t))-P(t)}\int_t^{\rho(t)} s(x)dP(x)=L; \tag{3.3}$$

while if $\rho\in\Lambda_\ell$, then

$$\lim_{t\to\infty}\frac{1}{P(t)-P(\rho(t))}\int_{\rho(t)}^{t} s(x)dP(x)=L. \tag{3.4}$$

Proof. (i) Assume that $\rho\in\Lambda_u$. By (1.2),

$$\begin{aligned}\frac{1}{P(\rho(t))-P(t)}&\int_t^{\rho(t)} s(x)dP(x)\\ &=\frac{1}{P(\rho(t))-P(t)}\{P(\rho(t))\sigma(\rho(t))-P(t)\sigma(t)\}\\ &=\sigma(\rho(t))+\frac{P(t)}{P(\rho(t))-P(t)}\{\sigma(\rho(t))-\sigma(t)\}.\end{aligned} \tag{3.5}$$

By (2.1),

$$0<\limsup_{t\to\infty}\frac{P(t)}{P(\rho(t))-P(t)}=\left\{\liminf_{t\to\infty}\frac{P(\rho(t))}{P(t)}-1\right\}^{-1}<\infty.$$

Now, (3.3) follows from (1.1) and (3.5).

(ii) Assume that $\rho \in \Lambda_\ell$. By (1.2),

$$\begin{aligned}\frac{1}{P(t)-P(\rho(t))}\int_{\rho(t)}^{t} s(x)dP(x)\\ = \sigma(t) + \frac{P(\rho(t))}{P(t)-P(\rho(t))}\{\sigma(t)-\sigma(\rho(t))\}.\end{aligned} \tag{3.6}$$

By (2.2),

$$0 < \limsup_{t\to\infty} \frac{P(\rho(t))}{P(t)-P(\rho(t))} = \left\{\liminf_{t\to\infty} \frac{P(t)}{P(\rho(t))} - 1\right\}^{-1} < \infty,$$

and (3.4) follows from (1.1) and (3.6). □

Example. We present a simple example in the case of summability $(C,1)$ which shows that conditions (2.3) and (2.4) in Theorem 1 are independent of one another. We consider the function

$$f(x) := \begin{cases} 1 & \text{for } \ 2^n \le x < 2^n + 1, \\ 0 & \text{for } \ 2^n + 1 \le x < 2^n + 2, \\ -1 & \text{for } \ 2^n + 2 \le x < 2^n + 3; \ n = 2, 3, \dots; \\ 0 & \text{otherwise on } \ \mathbb{R}_+. \end{cases}$$

An elementary computation shows that the limit in (1.5) does not exist, but the limit in (1.4) does exist and equals 0. Therefore, Lemma 2 applies. This time $P(x) := x$, so condition (2.9) is satisfied, $\rho_\lambda(t) := \lambda t$ is an appropriate choice for $1 \ne \lambda > 0$. Consequently, for every $\lambda > 1$, we have

$$\begin{aligned}\liminf_{t\to\infty} \frac{1}{(\lambda-1)t}\int_t^{\lambda t} \{s(x)-s(t)\}dx\\ = \lim_{t\to\infty} \frac{1}{(\lambda-1)t}\int_t^{\lambda t} s(x)dx - \limsup_{t\to\infty} s(t) = 0 - 1 = -1;\end{aligned} \tag{2.3'}$$

while for every $0 < \lambda < 1$, we have

$$\begin{aligned}\liminf_{t\to\infty} \frac{1}{(1-\lambda)t}\int_{\lambda t}^{t} \{s(t)-s(x)\}dx\\ = \liminf_{t\to\infty} s(t) - \lim_{t\to\infty} \frac{1}{(1-\lambda)t}\int_{\lambda t}^{t} s(x)dx = 0 - 0 = 0.\end{aligned} \tag{2.4'}$$

Proof of Theorem 1. *Necessity.* Assume that integral (1.3) is convergent, that is, condition (1.5) is satisfied. Then (1.4) is also satisfied. Let $\rho \in \Lambda_u$ be arbitrary. By Lemma 2, we have

$$\begin{aligned}&\lim_{t\to\infty} \frac{1}{P(\rho(t)) - P(t)} \int_t^{\rho(t)} \{s(x) - s(t)\} dP(x) \\ &= \lim_{t\to\infty} \frac{1}{P(\rho(t)) - P(t)} \int_t^{\rho(t)} s(x) dP(x) - \lim_{t\to\infty} s(t) = L - L = 0. \qquad (3.7)\end{aligned}$$

This proves (2.3) even in a stronger form.

In the case when $\rho \in \Lambda_\ell$ is arbitrary, we obtain in an analogous way that

$$\lim_{t\to\infty} \frac{1}{P(t) - P(\rho(t))} \int_{\rho(t)}^{t} \{s(t) - s(x)\} dP(x) = 0, \qquad (3.8)$$

which is stronger than (2.4).

Sufficiency. Assume that integral (1.3) is summable (W, P), that is, condition (1.4) is satisfied, and that conditions (2.3) and (2.4) are also satisfied. We have to prove (1.5).

To this effect, let $\varepsilon > 0$ be given. By (2.3) and (2.4), there exist some $\rho_1 \in \Lambda_u$ and $\rho_2 \in \Lambda_\ell$ such that

$$\liminf_{t\to\infty} \frac{1}{P(\rho_1(t)) - P(t)} \int_t^{\rho_1(t)} \{s(x) - s(t)\} dP(x) \geq -\varepsilon \qquad (3.9)$$

and

$$\liminf_{t\to\infty} \frac{1}{P(t) - P(\rho_2(t))} \int_{\rho_2(t)}^{t} \{s(t) - s(x)\} dP(x) \geq -\varepsilon. \qquad (3.10)$$

By (1.4), (3.9) and Lemma 2, we conclude that

$$-\varepsilon \leq \lim_{t\to\infty} \frac{1}{P(\rho_1(t)) - P(t)} \int_t^{\rho_1(t)} s(x) dP(x) - \limsup_{t\to\infty} s(t) = L - \limsup_{t\to\infty} s(t);$$

while by (1.4), (3.10) and Lemma 2, we conclude that

$$-\varepsilon \leq \liminf_{t\to\infty} s(t) - \lim_{t\to\infty} \frac{1}{P(t) - P(\rho_2(t))} \int_{\rho_2(t)}^{t} s(x) dP(x) = \liminf_{t\to\infty} s(t) - L.$$

Combining the last two inequalities yields

$$L-\varepsilon \le \liminf_{t\to\infty} s(t) \le \limsup_{t\to\infty} s(t) \le L+\varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily small, (1.5) follows. □

Proof of Theorem 2. *Necessity.* Assume that integral (1.3) is convergent. In the same way as in the proof of the Necessity part of Theorem 1, we conclude (3.7) if $\rho \in \Lambda_u$ and (3.8) if $\rho \in \Lambda_\ell$.

Sufficiency. (i) Assume that integral (1.3) is summable (W,P) to a finite limit L, and that condition (2.12) is satisfied. We have to prove (1.5).

To this end, let $\varepsilon > 0$ be given. By (2.12), there exists $\rho_1 \in \Lambda_u$ such that

$$L_1 := \limsup_{t\to\infty} \left| \frac{1}{P(\rho_1(t)) - P(t)} \int_t^{\rho_1(t)} \{s(x) - s(t)\} dP(x) \right| \le \varepsilon. \tag{3.11}$$

By (1.4), (3.11) and Lemma 2, we can estimate as follows:

$$\limsup_{t\to\infty} |L - s(t)| \le \lim_{t\to\infty} \left| L - \frac{1}{P(\rho_1(t)) - P(t)} \int_t^{\rho_1(t)} s(x) dP(x) \right| + L_1 \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily small, (1.5) follows.

(ii) In the case when condition (2.13) is satisfied, the proof of (1.5) is similar to the case (i) just above. □

4. Particular choices of the weight function

(i) If $P(x) := x$ for all $x \in \mathbb{R}_+$, then the weighted mean method (W,P) is the summability method $(C,1)$. This case and even the more general case (2.9) were discussed in [7].

We remind the reader that in the real case, condition (2.6) of slow decrease is certainly satisfied if there exist constants $H \ge 0$ and $x_0 \ge 0$ such that

$$xf(x) \ge -H \quad \text{for almost every} \quad x > x_0.$$

For sequences of real numbers, an analogous condition was introduced by Landau [6]. In the complex case, the classical Tauberian condition is that

$$|xf(x)| \le H \quad \text{for almost every} \quad x > x_0,$$

yielding condition (2.14) of slow oscillation. For sequences of real numbers, an analogous condition was introduced by Hardy [1]. As to details, we refer to [7].

(ii) If

$$P(x) := \begin{cases} 0 & \text{for } 0 \le x < 1, \\ \log x & \text{for } x \ge 1, \end{cases}$$

then the weighted mean method (W, P) is called the *harmonic mean method* (of first order). We observe that the results in [7] are not applicable, because in this case condition (2.9) is not satisfied. But this is the typical case of (2.10).

Now, condition (2.6) of slow decrease is of the form

$$\lim_{\lambda \to 1+} \liminf_{t \to \infty} \min_{\log t \le \log x \le \lambda \log t} \{s(x) - s(t)\} \ge 0. \tag{4.1}$$

Furthermore, condition (2.14) of slow oscillation is of the form

$$\lim_{\lambda \to 1+} \limsup_{t \to \infty} \max_{\log t \le \log x \le \lambda \log t} |s(x) - s(t)| = 0. \tag{4.2}$$

The last two conditions are implied by the local conditions

$$(x \log x) f(x) \ge -H \quad \text{for almost every } x > x_0 \tag{4.3}$$

and

$$(x \log x)|f(x)| \le H \quad \text{for almost every } x > x_0, \tag{4.4}$$

respectively, where $H \ge 0$ and $x_0 \ge 1$ are constants.

To justify (4.1), let $\lambda > 1$ and $1 < t \le x \le t^\lambda$. By (4.3), we have

$$s(x) - s(t) = \int_t^x f(y) dy \ge -H \int_t^x \frac{dy}{y \log y} = -H \log \left(\frac{\log x}{\log t} \right) \ge -H \log \lambda.$$

Letting $\lambda \to 1+$ gives inequality (4.1). We remark that for sequences of real numbers, an analogous condition was introduced by Kwee [5].

The implication (4.4) $\Rightarrow$ (4.2) can be justified in a similar way.

(iii) If

$$P(x) := \begin{cases} 0 & \text{for } 0 \le x < e, \\ \log \log x & \text{for } x \ge e, \end{cases}$$

then the weighted mean method (W, P) is the harmonic mean method of second order. This time we may consider the subclasses

$$\widetilde{\Lambda}_u := \{\rho_\lambda(t) := \exp(\log t)^\lambda : t \geq e \text{ and } \lambda > 1\}$$

and

$$\widetilde{\Lambda}_\ell := \{\rho_\lambda(t) := \exp(\log t)^\lambda : t \geq e \text{ and } 0 < \lambda < 1\}$$

of Λ_u and Λ_ℓ, respectively.

Now, condition (2.6) of slow decrease is given by

$$\lim_{\lambda \to 1+} \liminf_{t \to \infty} \min_{\log\log t \leq \log\log x \leq \lambda \log\log t} \{s(x) - s(t)\} \geq 0,$$

and this follows from the local condition

$$x(\log x)(\log\log x) f(x) \geq -H \quad \text{for almost every } x > x_0,$$

where $H \geq 0$ and $x_0 \geq e$ are constants. Furthermore, condition (2.14) of slow oscillation is given by

$$\lim_{\lambda \to 1+} \limsup_{t \to \infty} \max_{\log\log t \leq \log\log x \leq \lambda \log\log t} |s(x) - s(t)| = 0,$$

which is certainly implied by the local condition

$$x(\log x)(\log\log x)|f(x)| \leq H \quad \text{for almost every } x > x_0.$$

(iv) The harmonic mean method of third order defined by means of the weight function

$$P(x) := \begin{cases} 0 & \text{for } 0 \leq x < e^e, \\ \log\log\log x & \text{for } x \geq e^e, \end{cases}$$

and those of mth order, where $m = 4, 5, \ldots$, can be treated analogously to the cases presented just above.

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