Publ. Math. Debrecen
66/1-2 (2005), 121–128

On the distribution modulo 1 of the values of $F(n) + \alpha\sigma(n)$

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Abstract. Let $\sigma(n)$ denote the sum of the divisors of n and let α be a positive irrational number such that for each real number $\kappa > 1$ there exists a positive constant $c = c(\kappa, \alpha)$ for which the inequality $\|\alpha q\| > \frac{c}{q^\kappa}$ holds for every positive integer q, where $\|x\|$ stands for the distance between x and the closest integer. Then the function $F(n) + \alpha\sigma(n)$ is uniformly distributed modulo 1 for every additive function $F(n)$.

1. Introduction

According to a reformulated version of a well known theorem of H. Daboussi (see Daboussi and Delange [1], [2]), for every additive arithmetical function $F(n)$ and any irrational number α, the sequence $\ell_n := F(n) + \alpha n$ is uniformly distributed modulo 1. Kátai [4] proved that the same holds for $\ell_n := F(n) + Q(n)$, where $Q(x) := \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_k x^k \in \mathbb{R}[x]$, and at least one coefficient among $\alpha_1, \alpha_2, \ldots, \alpha_k$ is irrational. Repeating the definition given in [4], we say that $\mathcal{F}$ is the family of those sequences $t(n)$ for which $\ell_n := F(n) + t(n)$ is uniformly distributed modulo 1 for every additive function F. Most likely it is true that $t(n) := \alpha\sigma(n)$

Mathematics Subject Classification: 11K06.
Key words and phrases: arithmetical function, distribution modulo 1.
Research of the first author supported in part by a grant from NSERC.
Research of the second author supported by the Applied Number Theory Research Group of the Hungarian Academy of Science and by a grant from OTKA.

belongs to $\mathcal{F}$, where $\sigma(n)$ stands for the sum of the divisors of n. We can prove a somewhat weaker result, namely when α belongs to a certain class of positive irrational numbers. In fact, letting $\|x\|$ stand for the distance between x and the closest integer, we shall prove the following results.

Theorem 1. *Let α be a positive irrational number such that for each real number $\kappa > 1$ there exists a positive constant $c = c(\kappa, \alpha)$ for which the inequality*

$$\|\alpha q\| > \frac{c}{q^\kappa} \tag{1}$$

holds for every positive integer q. Then the function $t(n) = \alpha\sigma(n)$ belongs to $\mathcal{F}$.

Remark. Since the Lebesgue measure of those irrational α for which inequality (1) does not hold is zero, it follows that the set of irrational numbers for which the conclusion of Theorem 1 holds is indeed very large.

Theorem 2. *Let h be an integer valued multiplicative function such that $h(p) = Q(p)$ for every prime p and $h(p^a) = O(p^{ad})$ for some fixed number d for every prime p and every integer $a \geq 2$, where*

$$Q(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0, \quad k \geq 1,\ a_k > 0,\ a_j \in \mathbb{Z}.$$

Moreover, let α be a positive irrational number such that for each real number $\kappa > 1$ there exists a positive constant $c = c(\kappa, \alpha)$ for which the inequality

$$\|\alpha q\| > \frac{c}{q^\kappa}$$

holds for every positive integer q. Then the function $t(n) = \alpha h(n)$ belongs to $\mathcal{F}$.

2. Preliminary results

Given any real number ρ, we write $e(\rho)$ for $e^{2\pi i\rho}$. Then for each real number β, let

$$S_\beta(y) := \sum_{p \leq y} e(\beta p).$$

Lemma 1. *Let y be a large number and assume that*

$$R \le q \le y/R, \quad 1 \le R \le y^{1/4}, \quad (a,q) = 1, \quad \left|\beta - \frac{a}{q}\right| \le \frac{1}{q^2}.$$

Then

$$S_\beta(y) \ll \frac{Y}{\sqrt{R}}(\log y)^{16}.$$

Proof. This lemma is essentially due to I. M. Vinogradov. An explicit form and proof can be found in the book of Montgomery [5] (Corollary 16.3, page 142). □

Lemma 2. *Assume that α satisfies the conditions of Theorem 1. Then*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} e(\alpha\sigma(n)) = 0.$$

Proof. Let $\varepsilon > 0$ be fixed. Writing each integer $n \le x$ as $n = pm$, where $P(n) = p$ is the largest prime factor of n, we have that if $\mathcal{N}_1 = \mathcal{N}_1(x) := \{n \le x : P(n) \le x^\varepsilon\}$, then

$$\lim_{\varepsilon\to 0} \lim_{x\to\infty} \frac{1}{x} \#\mathcal{N}_1 = 0.$$

On the other hand the contribution of those integers n for which $P(n)^2 | n$ is negligeable. So let

$$\mathcal{N}_2 = \mathcal{N}_2(x) := \{n \le x : P(n) > x^\varepsilon,\ P(n)^2 \nmid n\},$$

so that

$$\sum_{n\in\mathcal{N}_2} e(\alpha\sigma(n)) = \sum_{m\le x^{1-\varepsilon}} \sum_{P(m)<p<x/m} e(\alpha(p+1)\sigma(m)) = \sum_{m\le x^{1-\varepsilon}} \Sigma_m,$$

say. Further write

$$\begin{aligned}\Sigma_m &= e(\alpha\sigma(m))\left(\sum_{p<x/m} e\big(\alpha\sigma(m)p\big) - \sum_{p<P(m)} e\big(\alpha\sigma(m)p\big)\right)\\ &= e\big(\alpha\sigma(m)\big)\left(\Sigma_m^{(1)} - \Sigma_m^{(2)}\right),\end{aligned}$$

say.

Let $\tau = x/(\log x)^{30}$. In order to estimate $\Sigma_m^{(1)}$, we shall approximate $\alpha\sigma(m)$ by a rational number a_m/q_m satisfying

$$\left|\alpha\sigma(m) - \frac{a_m}{q_m}\right| \le \frac{1}{q_m\tau}, \quad 1 \le q_m < \tau.$$

If $q_m > (\log x)^{40}$, we may apply Lemma 1 and get that

$$\Sigma_m^{(1)} \ll \frac{x/m}{\log^2(x/m)},$$

while if $q_m \le (\log x)^{40}$, we have

$$\left|\alpha - \frac{a_m}{q_m\sigma(m)}\right| \le \frac{1}{q_m\sigma(m)\tau},$$

so that setting $Q := q_m\sigma(m)$, we have that $Q^{1+\varepsilon/2} < \tau$, which does not hold if x is large.

In any event, it follows that

$$\sum_{m\le x^{1-\varepsilon}} \Sigma_m^{(1)} = o(x).$$

On the other hand, in order to estimate $\Sigma_m^{(2)}$, observe that

$$\begin{aligned}
\sum_{\substack{m\le x^{1-\varepsilon}\\ mP(m)\le x}} \frac{P(m)}{\log P(m)} &= \sum_{p\le x} \frac{p}{\log p} \sum_{\substack{pr\le x^{1-\varepsilon}\\ P(r)\le p\\ p^2r\le x}} 1 = \sum_{p\le x} \frac{p}{\log p} \sum_{\substack{r\le \min\left(\frac{x^{1-\varepsilon}}{p}, \frac{x}{p^2}\right)\\ P(r)\le p}} 1 \\
&= \sum_{p\le x^{\varepsilon}} \frac{p}{\log p} \sum_{\substack{r\le x^{1-\varepsilon}/p\\ P(r)\le p}} 1 + \sum_{x^{\varepsilon}<p\le x} \frac{p}{\log p} \sum_{\substack{r\le x/p^{\varepsilon}\\ P(r)\le p}} 1 \\
&\le \sum_{p\le x^{\varepsilon}} \frac{p}{\log p}\cdot\frac{x^{1-\varepsilon}}{p} + \sum_{x^{\varepsilon}<p\le x} \frac{p}{\log p}\cdot\frac{x}{p^2} \\
&= x^{1-\varepsilon} \sum_{p\le x^{\varepsilon}} \frac{1}{\log p} + x \sum_{x^{\varepsilon}<p\le x} \frac{1}{p\log p} \\
&\ll \frac{1}{\varepsilon^2}\frac{x}{\log^2 x} + \frac{1}{\varepsilon}\frac{x}{\log x} = o(x),
\end{aligned}$$

which implies that

$$\sum_{m \le x^{1-\varepsilon}} \Sigma_m^{(2)} = o(x),$$

thus completing the proof of Lemma 2. □

3. The proof of the theorems

We first prove Theorem 1.

Let F be an arbitrary additive function. We shall prove that, given any arbitrary positive integer k,

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} e(kF(n)) \cdot e(k\alpha\sigma(n)) = 0. \tag{2}$$

First observe that if $\|\alpha q\| > \frac{c(\kappa,\alpha)}{q^\kappa}$ holds for a certain irrational number α, then the same is true for $h\alpha$ (where h is an arbitrary fixed positive integer) with some other suitable constant $c(\kappa, h\alpha)$. Hence it is sufficient to prove (2) for $k = 1$. Now let $f(n) := e(F(n))$ and set

$$U(x) := \sum_{n\le x} f(n)e(\alpha\sigma(n)).$$

We will prove that

$$\limsup_{x\to\infty} \left| \frac{U(x)}{x} \right| = 0. \tag{3}$$

Let $\wp = \{p_1, p_2, \dots, p_R\}$ be a particular set of primes and set

$$A_\wp := \sum_{j=1}^{R} \frac{1}{p_j}, \qquad \omega_\wp(n) := \sum_{\substack{p\mid n \\ p\in\wp}} 1.$$

Assume that p_1 as well as $A_\wp$ are large numbers. Then, from the Turan–Kubilius inequality, we have

$$\sum_{n\le x} (\omega_\wp(n) - A_\wp)^2 \le cxA_\wp, \tag{4}$$

say. Now let

$$U_1(x) = \sum_{n\le x} f(n)e(\alpha\sigma(n))\omega_\wp(n) = \sum_{\substack{pm\le x\\ p\in\wp}} f(pm)e(\alpha\sigma(pm))$$

and

$$U_2(x) = \sum_{\substack{pm\le x\\ p\in\wp}} f(p)f(m)e\big(\alpha\sigma(p)\sigma(m)\big).$$

Using (4), we have that

$$\begin{aligned} |U_1(x) - A_\wp U(x)| &\le \sum_{n\le x} |\omega_\wp(n) - A_\wp| \\ \le \left(\sum_{n\le x} 1\right)^{1/2} &\left(\sum_{n\le x} |\omega_\wp(n) - A_\wp|^2\right)^{1/2} \le \sqrt{c}\cdot x\cdot\sqrt{A_\wp}. \end{aligned} \tag{5}$$

Moreover

$$|U_1(x) - U_2(x)| \le 2\sum_{\substack{p^2\nu\le x\\ p\in\wp}} 1 \le 2x\sum_{i=1}^{R}\frac{1}{p_i^2} \le 2x\frac{A_\wp}{p_1}. \tag{6}$$

We now estimate $U_2(x)$ as follows. First write

$$U_2(x) = \sum_{m\le x/p_1} f(m)\Sigma_m, \quad \text{with} \quad \Sigma_m = \sum_{p_j\le x/m} f(p_j)e(\alpha\sigma(p_j)\sigma(m)).$$

Thus

$$|U_2(x)|^2 \le \sum_{m\le x/p_1} |f(m)|^2 \sum_{m\le x/p_1} |\Sigma_m|^2 = S\cdot T,$$

say. It is clear that

$$S \ll \frac{x}{p_1}.$$

On the other hand, observe that

$$T = \sum_m \sum_{p_i,p_j\le x/m} f(p_i)\overline{f(p_j)}e\big(\alpha\big(\sigma(p_i) - \sigma(p_j)\big)\sigma(m)\big) = T_1 + T_2,$$

where in T_1 we sum for m and $p_i = p_j$, while

$$T_2 = \sum_{p_i \neq p_j} f(p_i)\overline{f(p_j)} \sum_{m \leq \min(\frac{x}{p_i}, \frac{x}{p_j})} e\big((\alpha\big(\sigma(p_i) - \sigma(p_j)\big)\sigma(m)\big).$$

Since condition (1) holds also for $\alpha(\sigma(p_i) - \sigma(p_j)) = \alpha(p_i - p_j)$, we can apply Lemma 2 and thus obtain that

$$T_2 = o(x).$$

Furthermore $T_1 \leq xA_\wp$. We may therefore conclude that $T \leq xA_\wp + o(x)$, which implies that

$$\limsup_{x\to\infty} \frac{|U_2(x)|}{x} \leq \frac{x}{\sqrt{p_1}}\sqrt{A_\wp}. \tag{7}$$

Thus, collecting (5), (6) and (7), we obtain that for x sufficiently large,

$$|A_\wp U(x)| \leq \sqrt{c} \cdot x\sqrt{A_\wp} + 2x\frac{A_\wp}{p_1} + 2\frac{x}{\sqrt{p_1}}\sqrt{A_\wp},$$

which implies that

$$\limsup_{x\to\infty} \frac{|U(x)|}{x} \leq \frac{\sqrt{c}}{\sqrt{A_\wp}} + \frac{2}{p_1} + \frac{2}{\sqrt{p_1}\sqrt{A_\wp}}. \tag{8}$$

Since p_1 and $A_\wp$ can be chosen arbitrarily large, then (3) holds and the proof of Theorem 1 is complete.

In order to prove Theorem 2, one can follow the same reasoning as that of Theorem 1, the only difference being that instead of using Lemma 1, one should use Theorem 10 of Hua [3].

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(Received July 21, 2003; revised September 9, 2003)