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# Varieties of involution semilattices of Archimedean semigroups

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**Abstract.** We give an indicator characterization of involution semigroup varieties consisting of involution semilattices of Archimedean semigroups. As a consequence, we describe identities which induce such a structure on an involution semigroup. This description is made explicit for the one-variable case.

The principal source of motivation for the present note is the paper [1] by Ćirić and Bogdanović. It is concerned with some special aspects of a general problem of great importance in the theory of semigroup varieties: given a family of identities (with certain syntactical properties), what can be said about the structure of semigroups satisfying such identities? And conversely: given a prescribed structural feature of semigroups, which identities 'force' the semigroups satisfying them to have the required feature? Namely, the most significant result of [1] is the solution of Problem 7.1 posed in the survey of Shevrin and Sukhanov [8], which asked for characterizations of semigroup varieties consisting entirely of semilattices of Archimedean semigroups. A sufficiently complete solution to this problem was known earlier only for periodic varieties, see Sapir and Sukhanov [7].

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It is the purpose of this note to sketch the main points of [1] and [2] for varieties of involution semigroups.

Recall that by an *involution semigroup* we mean a structure $(S, \cdot, ^*)$, where $(S, \cdot)$ is a semigroup, while the following identities: $(x^*)^* = x$ and $(xy)^* = y^*x^*$ are satisfied (i.e. $^*$ is an antiautomorphism of $S$ of order 2). Since these laws suffice to transform each expression involving multiplication and star into a form in which $^*$ acts only on variables, we find it convenient to introduce the notion of an *involution semigroup word* over an alphabet $X$, which is just an ordinary word over the 'double' alphabet $X \cup X^*$, where $X^* = \{x^* : x \in X\}$. Then, clearly, the free involution semigroup on $X$, $F_X^*$, consists of all nonempty involution semigroup words over $X$, the involution being defined by

$$(y_1 \dots y_n)^* = y_n^* \dots y_1^*$$

for all $y_1, \dots, y_n \in X \cup X^*$, where $y_i^* = x^*$ if $y_i = x \in X$, while $y_i^* = x$ for $y_i = x^* \in X^*$.

If $W$ is an involution semigroup word, the following functions may be useful. First, we have $c(W)$, the *content* of $W$, which is the set of all variables from $X$ occurring in $W$. However, one may consider the elements of $X^*$ as irreducible symbols of the alphabet, so that we obtain $c^*(W)$ the $^*$-*content* of $W$ (for example, $c^*(x^*yz^*x) = \{x, x^*, y, z^*\}$, while $c(x^*yz^*x) = \{x, y, z\}$). Finally, we define the set of *paired* variables of $W$ as

$$\pi(W) = \{x \in X : x, x^* \in c^*(W)\}.$$

One of the easiest ways to embed any semigroup into an involutorial one is the following construction. Let $S$ be the given semigroup, and let $S^\partial$ stand for its dual semigroup ($S^\partial = \{\overline{a} : a \in S\}$ and $\overline{a} \cdot \overline{b} = \overline{ba}$). Construct a semigroup on the set $S \cup S^\partial \cup \{\mathbf{0}\}$ (where $\mathbf{0} \notin S \cup S^\partial$) such that the multiplication $\circ$ is given by $a \circ b = ab$ for $a, b \in S$, $\overline{a} \circ \overline{b} = \overline{ba}$ for $\overline{a}, \overline{b} \in S^\partial$ and $a \circ b = \mathbf{0}$ otherwise. The involution is defined by $\mathbf{0}^* = \mathbf{0}$ and $a^* = \overline{a}$, $\overline{a}^* = a$ for all $a \in S$. In this way, it is easily checked that we obtain an involution semigroup, which we denote by $I_0^*(S)$. This is just the 0-direct union (or orthogonal sum) of $S$ and its dual. In particular, if $E$ is the trivial semigroup, then $I_0^*(E)$ is a three-element involution semilattice, which generates the variety $\mathcal{SL}^0$ (it is determined by the identities $xy = yx$, $xx^*y = xx^*$, cf. [4]).

The identities of involution semigroups of the above form were investigated in [3]. The results of [3] will be used here in the form of the following

**Lemma 1** ([3])**.** *Let $U$, $V$ be involution semigroup words and let $S$ be a semigroup. Then the identity $U = V$ holds in $I_0^*(S)$ if and only if either*

(i) *$\pi(U) \neq \emptyset$ and $\pi(V) \neq \emptyset$, or*

(ii) *$\pi(U) = \pi(V) = \emptyset$, and $U = V$ is obtained from a homotypical semigroup identity satisfied by $S$ whose reverse is also true in $S$, by replacing some of the variables by their stars.*

*In particular, $U = V$ holds in $\mathcal{SL}^0$ if and only if it is either of type* (i)*, or $\pi(U) = \pi(V) = \emptyset$ and $c^*(U) = c^*(V)$.*

Of course, by a *reverse* of a word $W = a_1a_2 \dots a_{m-1}a_m$ we mean $\overline{W} = a_na_{n-1} \dots a_2a_1$, while the reverse of the (semigroup) identity $U = V$ is just $\overline{U} = \overline{V}$. The following remark will be also useful in the sequel.

**Lemma 2.** *If a semigroup $S$ admits an involutorial antiautomorphism, then the set of its identities is closed for taking reverses.*

Proof. Assume that $U = V$ holds in $S$ and let $a_1, \dots, a_n \in S$ be arbitrary. Then we have $U(a_1^*, \dots, a_n^*) = V(a_1^*, \dots, a_n^*)$, implying $(U(a_1^*, \dots, a_n^*))^* = (V(a_1^*, \dots, a_n^*))^*$. The lemma now follows immediately by noting that from the involution axioms we have

$$\overline{W} = (W(x_1^*, \dots, x_n^*))^*$$

for any word $W$. □

For a semigroup $S$ and $a, b \in S$ we write $a \mid b$ ($a$ *divides* $b$) if $b = xay$ for some $x, y \in S^1$. If there exists $n \in \mathbb{N}$ such that $a \mid b^n$, we write $a \longrightarrow b$. A semigroup $S$ is *Archimedean* if $a \longrightarrow b$ for all $a, b \in S$. On the other hand, $S$ is a semilattice of Archimedean semigroups if $S$ has a congruence $\theta$ such that $S/\theta$ is a semilattice, and each of the $\theta$-classes is an Archimedean subsemigroup of $S$. We refer to [1] for an extensive list of papers dealing with such semigroups.

However, if $S$ is an involution semigroup, then the congruence $\theta$ with the above properties is easily shown to be compatible with $^*$, so that it is a $^*$-congruence, and $S/\theta$ is an involution semilattice (namely, such $\theta$, if it exists, is unique [6], and it is given by $a\ \theta\ b$ if and only if $\Sigma(a) = \Sigma(b)$, where $\Sigma(a) = \{x \in S : a \longrightarrow x\}$, whence it suffices to see that we have $\Sigma(a^*) = (\Sigma(a))^*$). In such a case, we use the term *involution semilattice* of Archimedean semigroups.

A classical result is that $S$ is a semilattice of Archimedean semigroups if and only if $a^2 \longrightarrow ab$ for all $a, b \in S$ (see [1, Theorem 1]). The lemma below gives a somewhat modified sufficient condition for an involution semigroup to have such a decomposition.

**Lemma 3.** *Let $S$ be an involution semigroup such that we have $a^2 \longrightarrow aa^* \longrightarrow ab$ ($a^2 \longrightarrow a^*a \longrightarrow ab$) for all $a, b \in S$. Then $S$ is an involution semilattice of Archimedean semigroups.*

Proof. Let us consider only the first case, the other one being analogous. We have that for each $a, b \in S$ there are $n, k \in \mathbb{N}$ such that $a^2 \mid (aa^*)^n$ and $aa^* \mid (ab)^k$, i.e. $(aa^*)^n = xa^2y$ and $(ab)^k = uaa^*v$ for some $x, y, u, v \in S^1$. We claim that for arbitrary $a, b \in S$ and any $\ell \in \mathbb{N}$ we have $(aa^*)^\ell \longrightarrow ab$. For this, it suffices to show that $(aa^*)^{2^r} \longrightarrow ab$ for all $r \geq 0$. For $r = 0$ the claim is true by assumption. So, assume that $(aa^*)^{2^r} \longrightarrow ab$ for some $r$. Then

$$(ab)^{k_1} = x_1(aa^*)^{2^r}y_1$$

for some $k_1 \in \mathbb{N}$ and $x_1, y_1 \in S^1$. But we also have that

$$((aa^*)^{2^r}y_1x_1)^{k_2} = x_2(aa^*)^{2^{r+1}}y_2$$

for some $k_2 \in \mathbb{N}$ and $x_2, y_2 \in S_1$. Hence,

$$\begin{aligned}(ab)^{k_1(k_2+1)} &= (x_1(aa^*)^{2^r}y_1)^{k_2+1} = x_1((aa^*)^{2^r}y_1x_1)^{k_2}(aa^*)^{2^r}y_1 \\ &= (x_1x_2)(aa^*)^{2^{r+1}}\left(y_2(aa^*)^{2^r}y_1\right),\end{aligned}$$

proving that $(aa^*)^{2^{r+1}} \longrightarrow ab$. Thus, our claim follows by induction. Finally, by setting $\ell = n$, we obtain $a^2 \longrightarrow ab$, and $S$ is a semilattice of Archimedean semigroups. $\square$

As it is usual, we denote by $B_2$ the five-element combinatorial Brandt semigroup, given by the presentation $\langle a, b \mid a^2 = b^2 = 0,\ aba = a,\ bab = b\rangle$.

However, we shall be much more interested in considering $B_2$ as an inverse semigroup, which is presented in the variety of involution semigroups by

$$\langle a \mid a^2 = 0,\ aa^*a = a\rangle.$$

So, $B_2$ consists of elements $0$, $a$, $a^*$, $aa^*$ and $a^*a$.

**Lemma 4.** *Let $\mathcal{V}$ be a variety of involution semigroups not containing $B_2$. Then for any $S \in \mathcal{V}$ and $a \in S$ we have $a^2 \longrightarrow aa^*$ and $a^2 \longrightarrow a^*a$.*

Proof. First of all, consider the following partition of the set of all 1-letter involution semigroup words:

$$\mathsf{W}_a = \{(xx^*)^n x : n \geq 0\},\quad \mathsf{W}_{a^*} = \{(x^*x)^n x^* : n \geq 0\},$$

$$\mathsf{W}_{aa^*} = \{(xx^*)^n : n \geq 0\},\quad \mathsf{W}_{a^*a} = \{(x^*x)^n : n \geq 0\},$$

$$\mathsf{W}_0 = F^*_{\{x\}} \setminus (\mathsf{W}_a \cup \mathsf{W}_{a^*} \cup \mathsf{W}_{aa^*} \cup \mathsf{W}_{a^*a}).$$

Clearly, $\mathsf{W}_0$ consists of all words from $F^*_{\{x\}}$ which contain $x^2$ or $(x^*)^2$ as a subword. The above partition defines an equivalence $\theta$ on $F^*_{\{x\}}$, which is easily seen to be a congruence, and $F^*_{\{x\}}/\theta \cong B_2$.

Now, if $B_2 \notin \mathcal{V}$, then $\mathcal{V}$ satisfies an identity $U = V$ which fails in $B_2$. In fact, there are then $a_1, \ldots, a_n \in B_2$ such that $U(a_1, \ldots, a_n)$ and $V(a_1, \ldots, a_n)$ represent different elements of $B_2$ (considered as an involution semigroup generated by $a$). Define a sequence of 1-letter involution semigroup words $W_1, \ldots, W_n$ such that for all $1 \leq i \leq n$ we have

$$W_i = \begin{cases} x & \text{if } a_i = a, \\ x^* & \text{if } a_i = a^*, \\ xx^* & \text{if } a_i = aa^*, \\ x^*x & \text{if } a_i = a^*a, \\ x^2 & \text{if } a_i = 0. \end{cases}$$

Then $U_1 = U(W_1, \ldots, W_n)$ and $V_1 = V(W_1, \ldots, W_n)$ are 1-letter involution semigroup words, and the identity $U_1(x) = V_1(x)$ holds in $\mathcal{V}$, but it still fails in $B_2$, namely, $U_1(a) \neq V_1(a)$.

Our goal is now to show that one of $U_1, V_1$ can be assumed to be from $\mathsf{W}_{aa^*}$, while the other belongs to $\mathsf{W}_0$. Indeed, one of these words (say, $U_1$)

does not belong to $\mathsf{W}_0$. But then, depending on the $\theta$-class of $U_1$, transform $U_1$ to $U_1' \in \mathsf{W}_{aa^*}$ by defining

$$U_1' = \begin{cases} U_1 x^* & \text{if } U_1 \in \mathsf{W}_a, \\ xU_1 & \text{if } U_1 \in \mathsf{W}_{a^*}, \\ U_1 & \text{if } U_1 \in \mathsf{W}_{aa^*}, \\ xU_1x^* & \text{if } U_1 \in \mathsf{W}_{a^*a}. \end{cases}$$

By applying this transformation to $V_1$, we obtain the word $V_1' \notin \mathsf{W}_{aa^*}$. If $V_1' \notin \mathsf{W}_0$ then depending on the $\theta$-class of $V_1$, we can similarly transform the identity $U_1' = V_1'$ to the identity $U_1'' = V_1''$, where $V_1'' \in \mathsf{W}_{aa^*}$. As $U_1' \in \mathsf{W}_{aa^*}$, it is a matter of a direct verification that $U_1'' \in \mathsf{W}_0$.

In any case, $\mathcal{V}$ satisfies an identity $U = V$ such that $U \in \mathsf{W}_{aa^*}$ and $V \in \mathsf{W}_0$, or, in more detail, an identity which is either of the form

$$(xx^*)^{n_1} = Px^2Q,$$

or of the form

$$(xx^*)^{n_2} = P'(x^*)^2Q',$$

for some involution semigroup words $P(x,x^*), Q(x,x^*), P'(x,x^*), Q'(x,x^*)$ and $n_1, n_2 \in \mathbb{N}$. By applying $^*$ to each of the equations above, we see that $\mathcal{V}$ must satisfy identities of both kinds just described. But the second one implies $(x^*x)^{n_2} = P''x^2Q''$, where $P'' = P'(x^*,x)$ and $Q'' = Q'(x^*,x)$, so that for every $S \in \mathcal{V}$ and $a \in S$ we have that $a^2$ divides both $(aa^*)^{n_1}$ and $(a^*a)^{n_2}$, as required. □

**Theorem 5.** *Let $\mathcal{V}$ be an involution semigroup variety. Then the following conditions are equivalent:*

(i) *any member of $\mathcal{V}$ can be decomposed (as a semigroup) into a semilattice of Archimedean semgiroups;*

(ii) *$\mathcal{V}$ does not contain $B_2$ and $I_0^*(B_2)$.*

Proof. (i)⇒(ii) This is immediate, as the semigroup reducts of both $B_2$ and $I_0^*(B_2)$ are not semilattices of Archimedean semigroups (see, for example, [1, Theorem 1]).

(ii)⇒(i) *Case 1:* $\mathcal{V}$ contains $\mathcal{SL}^0$. Therefore, for every identity $U = V$ which holds in $\mathcal{V}$ we have (by Lemma 1) either $\pi(U) \neq \emptyset$ and $\pi(V) \neq \emptyset$, or that $U = V$ is equivalent to a semigroup identity (some of the variables are replaced by their stars). Now, $I_0^*(B_2) \notin \mathcal{V}$, so $\mathcal{V}$ satisfies an identity which is false in $I_0^*(B_2)$. Such an identity clearly cannot be of the first type above. It follows that $\mathcal{V}$ satisfies a semigroup identity which is not true in $I_0^*(B_2)$. Since $B_2$ has a zero and admits an involution, this semigroup identity is false in $B_2$ (since $I_0^*(S)$ satisfies precisely those semigroup identities which are homotypical and, together with their reverses, hold in $S$). Thus, the semigroup reducts of members of $\mathcal{V}$ generate a (semigroup) variety which cannot contain $B_2$. By [1, Corollary 1], this variety (and, *a fortiori*, the class of reducts above) consists entirely of semilattices of Archimedean semigroups.

*Case 2:* $\mathcal{SL}^0$ is not contained in $\mathcal{V}$. Since any identity $U = V$ such that $\pi(U) = \pi(V) = \emptyset$ and $c^*(U) \neq c^*(V)$ implies an identity $U' = V'$ such that $\pi(U') = \emptyset$, $\pi(V') \neq \emptyset$ (if, for example, $x \in c^*(U)$, $x^* \in c^*(V)$, it suffices to take $U' = xU$, $V' = xV$), by Lemma 1 we have that $\mathcal{V}$ satisfies an identity of the latter type. Of course (by a suitable substitution), we may further assume that $U'$ is a semigroup word. By identifying all variables in $U' = V'$ we obtain an identity which is either of the form $x^k = W(x, x^*)xx^*Z(x, x^*)$, or of the form $x^k = W(x, x^*)x^*xZ(x, x^*)$. In the first case, for each $S \in \mathcal{V}$ and any $a, b \in S$, we have

$$(ab)^{k+1} = a(ba)^k b = (aW(ba, (ba)^*)b)aa^*(b^*Z(ba, (ba)^*)b),$$

that is, $aa^* \longrightarrow ab$. In the second case, we similarly obtain $a^*a \longrightarrow ab$. By Lemmas 3 and 4, these facts imply that every $S \in \mathcal{V}$ is a semilattice of Archimedean semigroups. □

Let $U = V$ be an involution semigroup identity. We call it an $\mathcal{SA}^*$-*identity* if the satisfaction of $U = V$ in an involution semigroup $S$ forces $S$ to be an involution semilattice of Archimedean semigroups (note that semigroup identities with an analogous property for ordinary semigroups were described in [1, Theorem 2]). As a consequence of the above results, we obtain the following

**Corollary 6.** *The following conditions are equivalent for an involution semigroup identity $U = V$:*

(i) *$U = V$ is a $\mathcal{SA}^*$-identity;*

(ii) *$U = V$ is not satisfied in $B_2$ and $I_0^*(B_2)$;*

(iii) *$U = V$ is not satisfied in $I_0^*(B_2)$, and there exist one-letter involution semigroup words $W_1, \dots, W_n$ such that*

$$U' = U(W_1, \dots, W_n) = V(W_1, \dots, W_n) = V'$$

*fails in $B_2$ (or equivalently, $U'$, $V'$ belong to different classes of words defined in the proof of Lemma 4);*

(iv) *$U = V$ is not satisfied in $I_0^*(B_2)$ and has consequences of the form $(xx^*)^n = W$, where $W$ contains either $x^2$, or $(x^*)^2$ as a subword.*

We omit the proof, as it can be easily reconstructed from the material already presented. Namely, (i)⇒(ii) is immediate; the proof of (ii)⇒(iii)⇒(iv) is contained in the proof of Lemma 3, while (iv)⇒(i) follows from Theorem 5 and the fact that an identity of the form $(xx^*)^n = W$, with $W$ as described above, fails in $B_2$.

As an application, we finish the note by explicitly listing all one-letter $\mathcal{SA}^*$-identities. Recall that all two-letter semigroup identities implying a semilattice decomposition into Archimedean components were described in Theorem 1 of [2], and the following result is basically its involutorial counterpart.

**Theorem 7.** *An involution semigroup identity in one variable is an $\mathcal{SA}^*$-identity if and only if it has one of the following forms:*

(1) *$U = V$, where $U \notin \mathsf{W}_0 \cup \{x, x^*\}$ (cf. the proof of Lemma 4) and $V \in \{x^n : n \geq 1\} \cup \{(x^*)^n : n \geq 1\}$;*

(2) *$x = V$, where $V$ is not of the form $(xx^*)^n x$ for any $n \geq 0$;*

(3) *$x^* = V$, where $V$ is not of the form $x^*(xx^*)^n$ for any $n \geq 0$.*

Proof. Let $U(x) = V(x)$ be a one-letter $\mathcal{SA}^*$-identity. Then, by the above corollary, there is a one-letter involution semigroup word $W$ such that $U' = U(W)$ and $V' = V(W)$ belong to different classes of $\theta$ (cf. Lemma 4). Without any loss of generality, let $U' \notin \mathsf{W}_0$, so that $U'$ contains neither $x^2$, nor $(x^*)^2$ as a subword. Then $U$ has the same property. Indeed, if $U \equiv U_1 x^2 U_2$, then we must have that $W$ belongs either to $\mathsf{W}_{aa^*}$, or to $\mathsf{W}_{a^*a}$, as $U' \notin \mathsf{W}_0$. But then either $U', V' \in \mathsf{W}_{aa^*}$, or $U', V' \in \mathsf{W}_{a^*a}$, contradicting the above assumption on the words $U'$, $V'$ (a similar conclusion follows if $U \equiv U_1(x^*)^2 U_2$). Hence, $U \notin \mathsf{W}_0$.

Now, if $U \notin \{x, x^*\}$, then $U$ contains occurrences of both $x$ and $x^*$. Thus, $V$ cannot contain both of $x, x^*$, for $U = V$ must fail in $I_0^*(B_2)$ (see Lemma 1). It follows that $U = V$ is just of the form (1). On the other hand, assume $U \in \{x, x^*\}$. If $U \equiv x$, then, clearly, $V$ is not of the form $(xx^*)^n x$, $n \geq 0$, for otherwise $U = V$ holds in $B_2$. Analogously, if $U \equiv x^*$, then $V$ is not of the form $x^*(xx^*)^n$, $n \geq 0$. So, $U = V$ has one of the forms (2), (3).

Finally, it remains to perform the routine check that all the identities listed in (1)–(3) are indeed false both in $B_2$ and $I_0^*(B_2)$, which confirms them as $\mathcal{SA}^*$-identities. □

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