

Translations in hyperbolic geometry of finite or infinite dimension

By WALTER BENZ (Hamburg)

Abstract. Based on separable translation groups T , Euclidean and Hyperbolic Geometry of (finite or infinite) dimension ≥ 2 can be characterized ([2]). The separability assumption of T expresses the existence of a special factorization of its kernel. In a first result of the present note the possibility of this factorization will be characterized geometrically. Another result answers the question when exactly two arbitrary surjective hyperbolic isometries, written in the form $\alpha_1\tau_1\beta_1$ and $\alpha_2\tau_2\beta_2$, coincide, where α_i, β_i are surjective orthogonal mappings and τ_i translations with the same axis, $i = 1, 2$. Also a characterization of hyperbolic translations will be given.

1. Separability

Let X be a *real inner product space* of (finite or infinite) dimension ≥ 2 , $O(X)$ be its *orthogonal group*, and e be a fixed element of X satisfying $e^2 = 1$. Suppose that

$$T : \mathbb{R} \rightarrow \text{Perm } X$$

is a mapping of $\mathbb{R}$ into the group of all permutations of X . The mapping T is called a *translation group* of X ([2]) with axis e provided the following properties hold true.

(a) $T_{t+s} = T_t \cdot T_s$ for all $t, s \in \mathbb{R}$,

Mathematics Subject Classification: 39B22, 39B72, 51M05, 51M10.

Key words and phrases: translation equation, translation groups, separable translation groups, hyperbolic geometry over real inner product spaces.

- (b) For $x, y \in X$ satisfying $y - x \in \mathbb{R}e$ there exists exactly one $t \in \mathbb{R}$ with $T_t(x) = y$,
 (c) $T_t(x) - x \in \mathbb{R}e$ for all $x \in X$ and all $t \in \mathbb{R}$.

Here T_t designates the image of $t \in \mathbb{R}$ under T , and $T_t(x)$ the image of $x \in X$ under the permutation T_t of X . Property (a) is the so-called *translation equation* (J. ACZÉL [1, pp. 245–253], Z. MOSZNER and J. TABOR [5]). If $e^\perp := \{h \in X \mid he = 0\} =: H$,

$$\varrho(h, \xi) := [T_\xi(h) - h] \cdot e \quad (1)$$

with $h \in e^\perp$ and $\xi \in \mathbb{R}$ is called the *kernel* of T . It determines the structure of T ([2]).

The translation group T is called *separable* ([2]) provided the following property holds true.

- (d) $\varrho(h, \xi) = \varphi(\xi)\psi(h)$ for all $\xi \in \mathbb{R}$ and $h \in H$ with functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ and $\psi : H \rightarrow \mathbb{R}_{>0}$ satisfying $\varphi(0) = 0$ and $\varphi(t_1) \leq \varphi(t_2)$ for all reals $t_1 \leq t_2$.

$\mathbb{R}_{>0}$ designates the set of all positive, and $\mathbb{R}_{\geq 0}$ the set of all non-negative reals. $\|x\|$ stands for $\sqrt{x^2}$ for all $x \in X$.

Theorem 1. Suppose that $T : \mathbb{R} \rightarrow \text{Perm } X$ satisfies (a) and (b). Then T is a separable translation group if and only if

- (c') $T_t(x) - x \in \mathbb{R}_{\geq 0} \cdot e$ for all $x \in X$ and all $t \in \mathbb{R}_{\geq 0}$,
 (d') $\frac{\|T_\alpha(h) - h\|}{\|T_\beta(h) - h\|} = \frac{\|T_\alpha(0)\|}{\|T_\beta(0)\|}$ for all $h \in H$ and all $\alpha, \beta \in \mathbb{R} \setminus \{0\}$,
 hold true.

PROOF. A) (a), (b) and (c') imply (c).
 We show more:

$$T_t(x) - x \in \mathbb{R}_{\geq 0} \cdot (-e) \text{ for } x \in X \text{ and } t \leq 0. \quad (2)$$

Since $-t \geq 0$, (c') implies

$$T_{-t}(T_t(x)) - T_t(x) = \mu \cdot e$$

for a suitable $\mu \geq 0$. Hence $x - T_t(x) \in \mathbb{R}_{\geq 0} \cdot e$. Observe here $T_0(x) = x$.

B) (a), (b), (c') and (d') *imply* (d).

Mainly from (b) we obtain that $T_t(x) = x$ holds true if and only if $t = 0$. Hence (d') is well-defined, because $T_\beta(h) - h$ and $T_\beta(0)$ are both unequal to 0. By (c) and (1) we get

$$T_\xi(h) - h = \varrho(h, \xi) \cdot e \quad (3)$$

for all $h \in H$ and $\xi \in \mathbb{R}$. Hence, by (c'), (2),

$$\frac{\varrho(h, t)}{t} \geq 0 \quad \text{for all } h \in H \text{ and } t \neq 0. \quad (4)$$

From (c'), $T_{t_2-t_1}(T_{t_1}(h)) - T_{t_1}(h) \in \mathbb{R}_{\geq 0} \cdot e$, and (a) we obtain

$$\varrho(h, t_1) \leq \varrho(h, t_2) \quad \text{for } h \in H \text{ and } t_1 \leq t_2. \quad (5)$$

Given $h \in H$ and $\xi \in \mathbb{R}$ there exists exactly one $t \in \mathbb{R}$ with $\varrho(h, t) = \xi$; this follows from (b) by defining $x = h$ and $y = h + \xi e$. Hence the function

$$t \rightarrow \varrho(h, t) \quad (6)$$

must be for fixed $h \in H$ a monotonically increasing bijection of $\mathbb{R}$ with $\varrho(h, 0) = 0$.

By (3), (4), we obtain

$$\|T_\xi(h) - h\| = \operatorname{sgn} \xi \cdot \varrho(h, \xi)$$

for all $\xi \neq 0$ and $h \in H$. Hence, by (d'),

$$\varrho(h, \xi) = \varrho(0, \xi) \cdot \frac{\varrho(h, 1)}{\varrho(0, 1)} \quad (7)$$

for all $\xi \neq 0$ and $h \in H$. Because of $\varrho(h, 0) = 0$, formula (7) holds true for $\xi = 0$ as well. Define

$$\varphi(\xi) := \varrho(0, \xi) \quad \text{and} \quad \psi(h) := \frac{\varrho(h, 1)}{\varrho(0, 1)}.$$

Because of $\operatorname{sgn} 1 = 1$, we get $\psi(h) > 0$ for all $h \in H$, and also $\psi(0) = 1$. What we proved about function (6), implies that φ is a monotonically increasing bijection of $\mathbb{R}$ with $\varphi(0) = 0$.

C) (a), (b), (c) and (d) *imply* (c').

Observe, by (3), (a),

$$T_t(h + \varrho(h, \tau)e) = T_t(T_\tau(h)) = T_{\tau+t}(h) = h + \varrho(h, \tau + t)e. \quad (8)$$

Since $X = H \oplus \mathbb{R}e$, we get the uniquely determined decomposition

$$x = h + x_0e,$$

$h \in H, x_0 \in \mathbb{R}$, for a given $x \in X$. Writing $x_0 =: \varrho(h, \tau)$, we obtain, by (8),

$$T_t(x) = T_t(h + \varrho(h, \tau)e) = x + (\varrho(h, \tau + t) - \varrho(h, \tau))e. \quad (9)$$

(d) implies $\varrho(h, t_1) \leq \varrho(h, t_2)$ for all $h \in H$ and $t_1 \leq t_2$. If $t \geq 0$, then $\tau + t \geq \tau$. Hence, by (9), property (c') holds true.

D) (a), (b), (c) and (d) *imply* (d'). (d) and (3) *imply*

$$\|T_\xi(h) - h\| = |\varphi(\xi)| \cdot \psi(h)$$

for all $\xi \in \mathbb{R}$ and $h \in H$. Hence (d') holds true. $\square$

Remark. Theorem 1 remains true, if we replace there property (d') by the following

(d*) $\varrho(h, \xi) = \varphi(\xi)\psi(h)$ for all $\xi \in \mathbb{R}$ and $h \in H$ with functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ and $\psi : H \rightarrow \mathbb{R}$, which, of course, is weaker than (d).

PROOF. 1. (a), (b), (c), (d) *imply* (a), (b), (c'), (d*).

This is obvious as far as (a), (b), (d*) are concerned, and, with respect to (c'), it follows from step C of the previous proof.

2. (a), (b), (c'), (d*) *imply* (a), (b), (c), (d).

Clear for (c), in view of step A. In order to prove (d), let $\varphi_0 : \mathbb{R} \rightarrow \mathbb{R}$ and $\psi_0 : H \rightarrow \mathbb{R}$ be functions according to (d*), satisfying

$$\varrho(h, \xi) = \varphi_0(\xi)\psi_0(h)$$

for all $\xi \in \mathbb{R}$ and $h \in H$. We now will apply results of step B as far as they were derived without assumption (d'). If there existed $h_0 \in H$ with $\psi_0(h_0) = 0$, we would obtain $\varrho(h_0, \xi) = 0$ for all $\xi \in \mathbb{R}$, contradicting the structure of function (6). Define

$$\varphi(\xi) = \psi_0(0)\varphi_0(\xi), \quad \psi(h) = \frac{\psi_0(h)}{\psi_0(0)},$$

and observe $\varrho(h, \xi) = \varphi(\xi)\psi(h)$. By (5), $t_1 \leq t_2$ implies

$$\varphi(t_1) = \varrho(0, t_1) \leq \varrho(0, t_2) = \varphi(t_2).$$

Because of $T_0(x) = x$ for all $x \in X$ (see step A), we get, by (1),

$$\varphi(0) = \psi_0(0)\varphi_0(0) = \varrho(0, 0) = [T_0(0) - 0]e = 0.$$

Hence, by (5),

$$0 = \varphi(0)\psi(h) = \varrho(h, 0) \leq \varrho(h, 1) = \varphi(1)\psi(h).$$

Observe $0 = \varphi(0) \leq \varphi(1)$. If $\varphi(1)$ were 0, $\varrho(h, \xi) = 0$ would have distinct solutions $\xi = 0, \xi = 1$. In view of $\psi_0(h) \neq 0$, the inequality

$$0 \leq \varphi(1)\psi(h)$$

implies $\psi(h) > 0$. Hence ψ is a function from H into $\mathbb{R}_{>0}$. Hence (d) holds true. $\square$

2. Examples

Important examples of separable translation groups are the following. Let again X be a real inner product space of (finite or infinite) dimension ≥ 2 , and let $e \in X$ satisfy $e^2 = 1$. Define T by (9) on the basis of

$$(E) \quad \varrho(h, t) := t \quad (\text{Euclidean Geometry}),$$

$$(H) \quad \varrho(h, t) := \sinh t \cdot \sqrt{1 + h^2} \quad (\text{Hyperbolic Geometry}).$$

We proved in [2], based heavily on the theory of Functional Equations (J. ACZÉL [1], Z. DARÓCZY [4]), the

Theorem. *Let T be a separable translation group with axis e , and suppose that $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ is not identically 0, and satisfies*

- (i) $d(x, y) = d(y, x)$,
- (ii) $d(x, y) = d(\omega(x), \omega(y))$,
- (iii) $d(x, y) = d(T_t(x), T_t(y))$,

$$(iv) \quad d(0, \beta e) = d(0, \alpha e) + d(\alpha e, \beta e)$$

for all $x, y \in X$, $\omega \in O(X)$, $t, \alpha, \beta \in \mathbb{R}$ with $0 \leq \alpha \leq \beta$. Then, up to isomorphism, we obtain

$$(E) \quad \text{with } d(x, y) = \sqrt{(x - y)^2}$$

or

$$(H) \quad \text{with } \cosh d(x, y) = \sqrt{1 + x^2} \sqrt{1 + y^2} - xy$$

for all $x, y \in X$, $h \in e^\perp$, and $t \in \mathbb{R}$. Hence, (X, d) is the Euclidean Metric Space with classical translations (E), or (X, d) is the Hyperbolic Metric Space in the form of the Weierstrass model with hyperbolic translations (H).

Another separable translation group T is given by (9) with

$$\varrho(x - (xe)e, t) = t^3 \cdot (1 + x^2 - (xe)^2)$$

for all $x \in X$ and $t \in \mathbb{R}$.

The translation group with $\varrho(h, t) = \sinh(t \cdot 2^{h^2})$ is not separable.

3. A characterization of hyperbolic translations

Based on (9) and $\varrho(h, t) = \sinh t \cdot \sqrt{1 + h^2}$ we get the hyperbolic translations

$$T_t(x) = x + \left[(xe)(\cosh t - 1) + \sqrt{1 + x^2} \sinh t \right] e \quad (10)$$

of X with axis e , where the fixed element $e \in X$ satisfies $e^2 = 1$. The group $\{T_t \mid t \in \mathbb{R}\}$ will be denoted by T . As already mentioned in Section 2, the notion of distance in the Hyperbolic Metric Space (X, d) is given by $d(x, y) \geq 0$ and

$$\cosh d(x, y) = \sqrt{1 + x^2} \sqrt{1 + y^2} - xy$$

for $x, y \in X$. If $\varrho > 0$ is a fixed real number, $N > 1$ a fixed integer, and $f : X \rightarrow X$ a mapping satisfying

$$d(x, y) = \varrho \quad \text{implies} \quad d(f(x), f(y)) \leq \varrho,$$

$$d(x, y) = N\rho \quad \text{implies} \quad d(f(x), f(y)) \geq N\rho,$$

for all $x, y \in X$, then

$$d(x, y) = d(f(x), f(y)) \quad (11)$$

holds true for all $x, y \in X$, and, moreover,

$$f(x) = \alpha T_t \beta(x) \quad \text{for all } x \in X \quad (12)$$

for suitable T_t of the form (10), $\alpha \in O(X)$, and β linear, orthogonal ([3]).

The mapping (12) is surjective (and hence bijective) if and only if β is in $O(X)$ as well.

Lemma 1. Given $\alpha \in O(X)$ with $\alpha(e) = \varepsilon e$, $\varepsilon \in \mathbb{R}$. Then

$$\alpha T_t \alpha^{-1}(x) = T_{\varepsilon t}(x)$$

for all $x \in X$ and $t \in \mathbb{R}$.

PROOF. Because of $\alpha(e)\alpha(e) = ee$, we obtain $\varepsilon^2 = 1$. With $\alpha^{-1}(e) = \varepsilon e$ and

$$x = h + x_0 e, \quad h \in e^\perp, \quad x_0 \in \mathbb{R},$$

we get $\alpha^{-1}(h)\alpha^{-1}(e) = he = 0$, i.e. $\alpha^{-1}(h) \in e^\perp$, and hence, by $\alpha^{-1}(h)\alpha^{-1}(h) = h^2$ and (10),

$$\begin{aligned} \alpha T_t \alpha^{-1}(x) &= \alpha T_t (\alpha^{-1}(h) + x_0 \varepsilon e) \\ &= \alpha \left(\alpha^{-1}(h) + \left[x_0 \varepsilon \cosh t + \sqrt{1 + h^2 + x_0^2} \sinh t \right] e \right) \\ &= x + \left[(xe)(\cosh(\varepsilon t) - 1) + \sqrt{1 + x^2} \sinh(\varepsilon t) \right] e = T_{\varepsilon t}(x). \quad \square \end{aligned}$$

Corollary. Define for $x = h + x_0 e$, $h \in e^\perp$, $x_0 \in \mathbb{R}$,

$$\chi(x) = h - x_0 e.$$

Then $\chi T_t = T_{-t} \chi$ for all $t \in \mathbb{R}$.

PROOF. Observe $\chi \in O(X)$, $\chi(e) = -e$ and Lemma 1. $\square$

Theorem 2. *Let $f : X \rightarrow X$ satisfy (11) for all $x, y \in X$. If f is surjective, then*

$$f(x) - x \in \mathbb{R}e \quad \text{for all } x \in X \quad (13)$$

holds true if and only if $f \in T \cup T \cdot \chi$.

PROOF. 1. Obviously, $f \in T \cup T \cdot \chi$ satisfies (13).

2. If $f \in O(X)$ has property (13), then $f = \text{id}$ or $f = \chi$. In order to prove this statement, notice first $f(e) - e \in \mathbb{R}e$, i.e. $f(e) = \lambda e$ with a suitable $\lambda \in \mathbb{R}$. Hence, by $f \in O(X)$, $e^2 = (f(e))^2$, i.e. $1 = \lambda^2$. Because of

$$0 = he = f(h)f(e) = f(h) \cdot \lambda e$$

for $h \in e^\perp$, we obtain $f(h) \in e^\perp$, and thus

$$f(h + x_0e) = f(h) + x_0\lambda e, \quad f(h) \in e^\perp, \quad (14)$$

for $x = h + x_0e$, $h \in e^\perp$, $x_0 \in \mathbb{R}$. By (13),

$$f(h + x_0e) = h + x_0e + \mu e \quad (15)$$

with a suitable $\mu \in \mathbb{R}$. Hence, by (14), (15), $f(h) = h$, i.e., by (14),

$$f(h + x_0e) = h + x_0\lambda e.$$

Thus $f = \text{id}$ for $\lambda = 1$, and $f = \chi$ for $\lambda = -1$.

3. Assume now that $f : X \rightarrow X$ is surjective, and that it satisfies (13). Hence f is of form (12) with $\beta \in O(X)$, i.e.

$$f = \alpha T_t \beta \quad \text{with } \alpha, \beta \in O(X), \quad t \in \mathbb{R}.$$

If $t = 0$, then $f \in O(X)$, i.e., by step 2, $f \in T \cup T\chi$. Assume $t \neq 0$. Hence $T_t(0) \neq 0$. By (13), $\alpha T_t \beta(0) = \lambda e$, with a suitable $\lambda \in \mathbb{R}$. Hence

$$0 \neq T_t(0) = \lambda \alpha^{-1}(e),$$

and thus $\alpha^{-1}(e) = \varepsilon e$, $\varepsilon \in \mathbb{R}$, because of $T_t(0) \in \mathbb{R}e$. So we obtain $\alpha(e) = \varepsilon e$ with $\varepsilon^2 = 1$, i.e., by Lemma 1,

$$f = \alpha T_t \alpha^{-1} \cdot \alpha \beta = T_{\varepsilon t} \cdot \gamma$$

with $\gamma := \alpha \beta \in O(X)$. Since $T_{-\varepsilon t}$ and $T_{\varepsilon t} \cdot \gamma$ have property (13), hence also $T_{-\varepsilon t} \cdot T_{\varepsilon t} \gamma = \gamma$. This implies $\gamma = \text{id}$ or $\gamma = \chi$, by step 2. Thus $f \in T \cup T\chi$. $\square$

$T_t \in T$ has a fixpoint if and only if $t = 0$. On the other hand, every $T_t \cdot \chi$ has a fixpoint. This leads to the following characterization of hyperbolic translations, which is a corollary of Theorem 2.

A surjective and distance preserving mapping $f : X \rightarrow X$ is a hyperbolic translation $\neq \text{id}$ (with axis e) if and only if

$$0 \neq f(x) - x \in \mathbb{R}e$$

holds true for all $x \in X$.

Given two surjective hyperbolic isometries, i.e. mappings

$$f = \alpha T_t \beta \quad \text{and} \quad g = \gamma T_s \delta$$

with $\alpha, \beta, \gamma, \delta \in O(X)$, $t, s \in \mathbb{R}$, where T_t, T_s are translations with axis e . The question we now would like to answer is the following

when and only when is $f = g$?

Lemma 2. *Let ξ, η be elements of $O(X)$, and t, s be reals. Then*

$$\xi T_t = T_s \eta \tag{16}$$

holds true if and only if

$$\text{Case } ts = 0 : t = s = 0 \quad \text{and} \quad \xi = \eta,$$

$$\text{Case } ts \neq 0 : t = \varepsilon s, \varepsilon^2 = 1 \quad \text{and} \quad \xi = \eta, \xi(e) = \varepsilon e.$$

PROOF. $\xi T_t(0) = T_s \eta(0)$ implies

$$\xi(e) \cdot \sinh t = e \cdot \sinh s. \tag{17}$$

Since $\xi(e)\xi(e) = e \cdot e = 1$, we obtain $t = s = 0$, and hence $\xi = \eta$ from (16), in the case $ts = 0$. On the other hand, $t = s = 0$ and $\xi = \eta$ imply (16). In the case $ts \neq 0$, we get $\xi(e) = \varepsilon e, \varepsilon^2 = 1$, and $t = \varepsilon s$ from (17). Hence, by (16) and Lemma 1,

$$T_s \eta = \xi T_t \xi^{-1} \cdot \xi = T_s \cdot \xi,$$

i.e. $\xi = \eta$. On the other hand, $ts \neq 0, t = \varepsilon s, \varepsilon^2 = 1, \xi = \eta, \xi(e) = \varepsilon e$ imply $\xi T_t = T_s \eta$. $\square$

Theorem 3. *Let $\alpha, \beta, \gamma, \delta$ be elements of $O(X)$, and t, s be reals. Then*

$$\alpha T_t \beta = \gamma T_s \delta \quad (18)$$

holds true if and only if

$$\text{Case } ts = 0 : t = s = 0 \quad \text{and} \quad \alpha\beta = \gamma\delta,$$

$$\text{Case } ts \neq 0 : t = \varepsilon s, \varepsilon^2 = 1 \quad \text{and} \quad \alpha\beta = \gamma\delta, \alpha(e) = \varepsilon\gamma(e).$$

PROOF. Since (18) is equivalent with (16) by defining $\xi = \gamma^{-1}\alpha$, $\eta = \delta\beta^{-1}$, we may apply Lemma 2. Hence (16) is the same as $t = s = 0$ and $\alpha\beta = \gamma\delta$ in the case $ts = 0$, and the same as

$$t = \varepsilon s, \varepsilon^2 = 1, \gamma^{-1}\alpha = \delta\beta^{-1}, \gamma^{-1}\alpha(e) = \varepsilon e,$$

i.e. the same as $t = \varepsilon s, \varepsilon^2 = 1, \alpha\beta = \gamma\delta$ and $\alpha(e) = \varepsilon\gamma(e)$, in the case $ts \neq 0$. $\square$

References

- [1] J. ACZÉL, Lectures on functional equations and their applications, *Academic Press, New York, London*, 1966.
- [2] W. BENZ, A common characterization of euclidean and hyperbolic geometry by functional equations, *Publ. Math. Debrecen* **63** (2003), 495–510.
- [3] W. BENZ, Mappings preserving two hyperbolic distances, *J. Geom.* **70** (2001), 8–16.
- [4] Z. DARÓCZY, Elementare Lösung einer mehrere unbekannte Funktionen enthaltenden Funktionalgleichung, *Publ. Math. Debrecen* **8** (1961), 160–168.
- [5] Z. MOSZNER and J. TABOR, L'équation de translation sur une structure avec zéro, *Ann. Polon. Math.* **31** (1975), 255–264.

WALTER BENZ
 MATHEMATISCHES SEMINAR
 DER UNIVERSITÄT
 BUNDESSTR. 55, 20146 HAMBURG
 GERMANY

(Received December 30, 2003)