

On the torsion-free connections on higher order frame bundles

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Abstract. Using the r -jets of flows of vector fields, we show that every torsion-free linear r -th order connection Γ on the tangent bundle of a manifold M determines a reduction of the $(r + 1)$ -st order frame bundle of M to the general linear group. We deduce that this reduction coincides with the well known reduction determined by the principal connection induced by Γ on the r -th order frame bundle of M .

Introduction

Our starting point is the fact that the principal connections on the r -th order frame bundle $P^r M$ of a manifold M are in bijection with the linear r -th order connections on the tangent bundle TM , i.e. with the linear splittings

$$\Gamma : TM \rightarrow J^r TM \tag{1}$$

of the jet projection $J^r TM \rightarrow TM$. We shall write

$$\tilde{\Gamma} : P^r M \rightarrow J^1 P^r M \tag{2}$$

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for the principal connection corresponding to (1). Its lifting map, denoted by the same symbol,

$$\tilde{\Gamma} : P^r M \times_M TM \rightarrow TP^r M$$

is determined by

$$\tilde{\Gamma}(u, A) = \mathcal{P}^r X(u), \quad u \in P^r M, \quad A \in T_x M, \quad x \in M, \quad (3)$$

where $\mathcal{P}^r X$ is the flow prolongation of a vector field $X : M \rightarrow TM$ satisfying $j_x^r X = \Gamma(A)$, [3].

The structure group of $P^r M$ is G_m^r , $m = \dim M$. There is a canonical $\mathbb{R}^m \times \mathfrak{g}_m^{r-1}$ -valued one-form Θ_r on $P^r M$. P. C. YUEN, [5], introduced the torsion of $\tilde{\Gamma}$ as the exterior covariant differential

$$D_{\tilde{\Gamma}} \Theta_r, \quad (4)$$

see also [1]. On the other hand, the $(r-1)$ -jet at x of the bracket $[X, Y]$ of two vector fields X, Y on M depends on $j_x^r X$ and $j_x^r Y$ only. This defines a map

$$[\cdot, \cdot]_{r-1} : J^r TM \times_M J^r TM \rightarrow J^{r-1} TM. \quad (5)$$

The torsion of Γ can be introduced, [6], as a map

$$\tau_\Gamma : TM \times_M TM \rightarrow J^{r-1} TM$$

defined by

$$\tau_\Gamma(A, B) = [\Gamma(A), \Gamma(B)]_{r-1}, \quad A, B \in T_x M. \quad (6)$$

In [3] we deduced that the torsions $D_{\tilde{\Gamma}} \Theta_r$ and τ_Γ coincide in a natural way.

There is a canonical injection $i_{r+1} : P^{r+1} M \rightarrow J^1 P^r M$, see formula (13) below. If $\tilde{\Gamma}$ is torsion-free, there is a well-known map $\varrho(\tilde{\Gamma}) : P^1 M \rightarrow P^{r+1} M$ defined by the induction

$$i_{r+1} \circ \varrho(\tilde{\Gamma}) = \tilde{\Gamma} \circ \varrho(\tilde{\Gamma}_{r-1}), \quad (7)$$

where $\tilde{\Gamma}_{r-1}$ is the underlying connection on $P^{r-1} M$, so that $\varrho(\tilde{\Gamma}_{r-1}) : P^1 M \rightarrow P^r M$ by the induction hypothesis. If we consider the canonical injection $GL(m, \mathbb{R}) \hookrightarrow G_m^{r+1}$, [4], p. 130, then $\varrho(\tilde{\Gamma})(P^1 M)$ is a reduction of

$P^{r+1}M$ to $GL(m, \mathbb{R})$. This establishes the well-known bijection between the torsion-free connections on P^rM and the reductions of $P^{r+1}M$ to $GL(m, \mathbb{R})$, [2], [3].

In Section 1 of the present paper we use the r -jets of flows of vector fields to construct a map $\sigma(\Gamma) : P^1M \rightarrow P^{r+1}M$ for every torsion-free linear r -th order connection Γ on TM . Our proof of the fact that $\sigma(\Gamma)(P^1M)$ is a reduction of $P^{r+1}M$ to $GL(m, \mathbb{R})$ is based on two interesting lemmas concerning the r -jets of the bracket of vector fields, the proofs of which we postpone to Section 3. In Section 2 we deduce $\sigma(\Gamma) = \varrho(\tilde{\Gamma})$, i.e. both constructions of a reduction of $P^{r+1}M$ to $GL(m, \mathbb{R})$ coincide.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from the book [4].

1. The reduction $\sigma(\Gamma)$

Consider $\Gamma : TM \rightarrow J^rTM$. For a linear frame $u \in P_x^1M$, $u = (A_1, \dots, A_m)$, $A_i \in T_xM$, we take vector fields X_i satisfying $j_x^r X_i = \Gamma(A_i)$, $i = 1, \dots, m$. Then

$$(Fl_{t^1}^{X_1} \circ \dots \circ Fl_{t^m}^{X_m})(x)$$

is a local map $\mathbb{R}^m \rightarrow M$ and we define

$$\sigma(\Gamma)(u) = j_0^{r+1}(Fl_{t^1}^{X_1} \circ \dots \circ Fl_{t^m}^{X_m})(x) \in P_x^{r+1}M, \quad (8)$$

where $0 \in \mathbb{R}^m$. One verifies easily that $\sigma(\Gamma)(u)$ depends on u and Γ only.

Proposition 1. *If Γ is torsion-free, then $\sigma(\Gamma)(P^1M)$ is a reduction of $P^{r+1}M$ to $GL(m, \mathbb{R})$.*

The proof will be based on the following two lemmas, the proofs of which we postpone to the last section.

Consider two vector fields X and Y on M . Then $(Fl_t^X \circ Fl_\tau^Y)(x)$ is a local map $\mathbb{R}^2 \rightarrow M$, so that $j_{0,0}^{r+1}(Fl_t^X \circ Fl_\tau^Y)(x) \in (T_2^{r+1}M)_x$ is a $(2, r+1)$ -velocity on M .

Lemma 1. *If $j_x^{r-1}[X, Y] = 0$, then*

$$j_{0,0}^{r+1}(Fl_t^X \circ Fl_\tau^Y)(x) = j_{0,0}^{r+1}(Fl_\tau^Y \circ Fl_t^X)(x). \quad (9)$$

Further, $(Fl_t^X \circ Fl_t^Y)(x)$ is a local map $\mathbb{R} \rightarrow M$, so that $j_0^{r+1}(Fl_t^X \circ Fl_t^Y)(x) \in (T_1^{r+1}M)_x$ is a $(1, r+1)$ -velocity on M .

Lemma 2. *If $j_x^{r-1}[X, Y] = 0$, then*

$$j_0^{r+1}(Fl_t^X \circ Fl_t^Y)(x) = j_0^{r+1}(Fl_t^{X+Y})(x). \quad (10)$$

Now we prove Proposition 1 by using Lemmas 1 and 2. We shall also use the well known formula

$$Fl_{at}^X = Fl_t^{aX}, \quad a \in \mathbb{R}. \quad (11)$$

Take $g = (a_j^i) \in GL(m, \mathbb{R})$, $u = (A_i) \in P_x^1 M$ and consider $ug = (a_i^j A_j)$. Write $\Gamma(A_i) = j_x^r X_i$. Since Γ is torsion-free, by (10), (11) and (9) we obtain gradually

$$\begin{aligned} \sigma(\Gamma)(ug) &= j_0^{r+1}(Fl_{t^1}^{a_1^1 X_1 + \dots + a_1^m X_m} \circ \dots \circ Fl_{t^m}^{a_m^1 X_1 + \dots + a_m^m X_m}) \\ &= j_0^{r+1}(Fl_{t^1}^{a_1^1 X_1} \circ \dots \circ Fl_{t^1}^{a_1^m X_m} \circ \dots \circ Fl_{t^m}^{a_m^1 X_1} \circ \dots \circ Fl_{t^m}^{a_m^m X_m}) \\ &= j_0^{r+1}(Fl_{a_1^1 t^1}^{X_1} \circ \dots \circ Fl_{a_1^m t^1}^{X_m} \circ \dots \circ Fl_{a_m^1 t^m}^{X_1} \circ \dots \circ Fl_{a_m^m t^m}^{X_m}) \\ &= j_0^{r+1}(Fl_{a_1^1 t^1 + \dots + a_1^m t^m}^{X_1} \circ \dots \circ Fl_{a_m^1 t^1 + \dots + a_m^m t^m}^{X_m}). \end{aligned}$$

This proves Proposition 1.

2. The main result

We shall use the following form of the canonical injection $i_{r+1} : P^{r+1}M \rightarrow J^1 P^r M$. We have $P^r M \subset T_m^r M$, where T_m^r is the functor of (m, r) -velocities. Clearly, $j_0^r f \in T_m^r M$, $f : \mathbb{R}^m \rightarrow M$, can be expressed in the form

$$j_0^r f = (T_m^r f)(e), \quad e = j_0^r \text{id}_{\mathbb{R}^m}. \quad (12)$$

Write $E_i = \frac{\partial}{\partial t}|_0 j_0^r \tau_t^i \in T_e T_m^r \mathbb{R}^m$, where $\tau_t^i : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the translation $\bar{t}^1 = t^1, \dots, \bar{t}^i = t^i + t, \dots, \bar{t}^m = t^m$. If we consider $j_0^{r+1} \psi \in P^{r+1}M$, then

$$(TT_m^r \psi)(E_i) \quad (13)$$

is an m -tuple of tangent vectors at $j_0^r \psi \in P^r M$. The linear span of these vectors defines $i_{r+1}(j_0^{r+1} \psi) \in J^1 P^r M$.

Proposition 2. *If Γ is a torsion-free linear r -th order connection on TM and $\tilde{\Gamma}$ is the corresponding principal connection on $P^r M$, then $\sigma(\Gamma) = \varrho(\tilde{\Gamma})$.*

PROOF. We proceed by induction. If Γ_{r-1} and $\tilde{\Gamma}_{r-1}$ are the underlying connections in the order $r-1$, then

$$\sigma(\Gamma_{r-1}) = \varrho(\tilde{\Gamma}_{r-1}) \quad (14)$$

by the induction hypothesis. Consider $u = (A_1, \dots, A_m) \in P_x^1 M$ and write

$$v = \sigma(\Gamma_{r-1})(u) = \varrho(\tilde{\Gamma}_{r-1})(u).$$

By (13), $i_{r+1}(j_0^{r+1}(Fl_{t^1}^{X_1} \circ \dots \circ Fl_{t^m}^{X_m})(x))$ is the linear span of the vectors

$$TT_m^r(Fl_{t^1}^{X_1} \circ \dots \circ Fl_{t^m}^{X_m})(E_i). \quad (15)$$

Using the basic properties of flows, Lemma 1 and (12), we deduce that (15) is equal to

$$\begin{aligned} & \left. \frac{\partial}{\partial t} \right|_0 T_m^r(Fl_{t^1}^{X_1} \circ \dots \circ Fl_{t+t^i}^{X_i} \circ \dots \circ Fl_{t^m}^{X_m})(e) \\ &= \left. \frac{\partial}{\partial t} \right|_0 (Fl_t^{\mathcal{T}_m^r X_i} \circ Fl_{t^1}^{\mathcal{T}_m^r X_1} \circ \dots \circ Fl_{t^m}^{\mathcal{T}_m^r X_m})(e) \\ &= \mathcal{T}_m^r X_i(T_m^r(Fl_{t^1}^{X_1} \circ \dots \circ Fl_{t^m}^{X_m})(e)) = \mathcal{T}_m^r X_i(v), \end{aligned}$$

where $\mathcal{T}_m^r X_i$ denotes the flow prolongation of X_i . By (3) and by the induction hypothesis, this m -tuple spans $\varrho(\tilde{\Gamma})(v)$. $\square$

3. The proofs of Lemmas 1 and 2

In general, if we have two maps $f, g : \mathbb{R}^m \rightarrow \mathbb{R}^m$, it suffices to verify the condition $j_0^r f = j_0^r g$ on all curves of the form $x^i = a^i t$, $i = 1, \dots, m$, [4]. By the flow property (11), Lemma 1 follows from the fact that $j_x^{r-1}[X, Y] = 0$ implies

$$j_0^{r+1}(Fl_t^X \circ Fl_t^Y)(x) = j_0^{r+1}(Fl_t^Y \circ Fl_t^X)(x) \in T_1^{r+1} M. \quad (16)$$

But this is a direct consequence of Lemma 2. So it suffices to prove Lemma 2. We have the following 3 cases.

I. If $X(x) = Y(x) = 0$, then the $(r+1)$ -jets of the flows of X and Y are in the group of all invertible $(r+1)$ -jets of M into M with source x and target x and we have a well known result concerning Lie groups.

II. If $X(x) \neq 0$, we can consider such local coordinates on M that $X = \frac{\partial}{\partial x^1}$. Then $j_x^{r-1}[\frac{\partial}{\partial x^1}, Y] = 0$ means

$$D_\alpha \frac{\partial Y^i(x)}{\partial x^1} = 0, \quad 0 \leq \|\alpha\| \leq r-1, \quad (17)$$

where Y^i are the coordinate components of Y and D_α denotes the partial derivative with respect to a multiindex α of the range m .

The flow $\psi^i(t, x)$ of the vector field $\frac{\partial}{\partial x^1} + Y$ satisfies

$$\frac{\partial \psi^i(t, x)}{\partial t} = \delta_1^i + Y^i(\psi(t, x)). \quad (18)$$

If $\eta^i(t, x)$ denotes the flow of Y , then the coordinate expression of $Fl_t^X \circ Fl_t^Y$ is

$$\mu^i(t, x) = \delta_1^i t + \eta^i(t, x). \quad (19)$$

Hence

$$\frac{\partial \mu^i(t, x)}{\partial t} = \delta_1^i + \frac{\partial \eta^i(t, x)}{\partial t} = \delta_1^i + Y^i(\eta(t, x)). \quad (20)$$

From (19) we obtain

$$\frac{\partial^k \mu^i(t, x)}{\partial t^k} = \frac{\partial^k \eta^i(t, x)}{\partial t^k}, \quad k \geq 2. \quad (21)$$

For $t = 0$, (18) and (20) yield directly $\frac{\partial \psi^i(0, x)}{\partial t} = \frac{\partial \mu^i(0, x)}{\partial t}$. Then we find by direct evaluation

$$\frac{\partial^2 \psi^i(t, x)}{\partial t^2} = \frac{\partial Y^i(\psi(t, x))}{\partial x^j} \frac{\partial \psi^j(t, x)}{\partial t}, \quad (22)$$

$$\frac{\partial^2 \mu^i(t, x)}{\partial t^2} = \frac{\partial Y^i(\eta(t, x))}{\partial x^j} \frac{\partial \eta^j(t, x)}{\partial t}. \quad (23)$$

Hence (17) implies

$$\frac{\partial^2 \psi^i(0, x)}{\partial t^2} = \frac{\partial^2 \mu^i(0, x)}{\partial t^2}. \quad (24)$$

By iteration we deduce (10) for every r .

III. The case $Y(x) \neq 0$ can be reduced to II by using $Fl_t^X \circ Fl_t^Y = (Fl_{-t}^Y \circ Fl_{-t}^X)^{-1}$.

This proves Lemmas 1 and 2.

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