

On a class of Einstein space-time manifolds

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Abstract. We deal with a general space-time (M, g) with usual differentiability conditions and hyperbolic metric g of index 1, which carries 3 skew-symmetric Killing vector fields X, Y, Z having as generative the unit time-like vector field e of the hyperbolic metric g . It is shown that such a space-time (M, g) is an Einstein manifold of curvature -1 , which is foliated by space-like hypersurfaces M_s normal to e and the immersion $x : M_s \rightarrow M$ is pseudo-umbilical. In addition, it is proved that the vector fields X, Y, Z and e are exterior concurrent vector fields and X, Y, Z define a commutative Killing triple, M admits a Lorentzian transformation which is in an orthochronous Lorentz group and the distinguished spatial 3-form of M is a relatively integral invariant of the vector fields X, Y and Z .

0. Introduction

Let (M, g) be a general space-time with usual differentiability conditions and hyperbolic metric g of index 1.

We assume in this paper that (M, g) carries 3 skew-symmetric Killing vector fields (abbr. SSK) X, Y, Z having as generative the unit time-like vector field e of the hyperbolic metric g (see [R1], [MRV]). Therefore, if ∇ is the Levi-Civita connection and $\wedge$ means the wedge product of vector

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fields, one has

$$\nabla X = X \wedge e, \quad \nabla Y = Y \wedge e, \quad \nabla Z = Z \wedge e. \quad (0.1)$$

Setting $\alpha = X^\flat$, $\beta = Y^\flat$, $\gamma = Z^\flat$ ($\flat : TM \rightarrow T^*M$, $X \mapsto X^\flat$, denotes the musical isomorphism defined by g), one derives by (0.1)

$$d\alpha = 2\omega \wedge \alpha, \quad \omega = e^\flat, \quad (0.2)$$

and clearly similar equations for β and γ .

Since one finds that ω is an exact form, equations (0.2) may be written in terms of d^ω -cohomology as $d^{-2\omega}\alpha = 0$ and say that α , β , γ are $d^{-2\omega}$ -exact forms. From (0.2) it also follows that the existence of such a space-time is determined by an exterior differential system in involution [C2].

The following theorem is proved:

Any space-time (M, g) satisfying (0.1) is an Einstein manifold of curvature -1 , which is foliated by space-like hypersurfaces M_s normal to e and the immersion $x : M_s \rightarrow M$ is pseudo-umbilical [Ch].

The following additional properties are also obtained.

- i) the vector fields X , Y , Z and e are exterior concurrent vector fields (abbr. EC) and X , Y , Z define a commutative Killing triple;
- ii) M admits a Lorentzian transformation which is in an orthochronous Lorentz group [CWD];
- iii) the 3-form $\varphi = \alpha \wedge \beta \wedge \gamma$ is a relatively integral invariant of the vector fields X , Y and Z .

1. Preliminaries

Let (M, g) be a pseudo-Riemannian C^∞ -manifold and let ∇ be the covariant differential operator defined by the metric tensor g . We assume that M is oriented and that ∇ is the Levi-Civita connection. Let ΓTM be the set of sections of the tangent bundle TM and $\flat : TM \rightarrow T^*M$ and $\sharp : T^*M \rightarrow TM$ the classical musical isomorphisms defined by g .

Following [P], we set $A^q(M, TM) = \Gamma \text{Hom}(\Lambda^q TM, TM)$ and notice that elements of $A^q(M, TM)$ are vector valued q -forms, $q \leq \dim M$.

Denote by $d^\nabla : A^q(M, TM) \rightarrow A^{q+1}(M, TM)$ the *exterior covariant derivative operator* with respect to ∇ (it should be noticed that generally $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$).

If $p \in M$, then the vector valued 1-form $dp \in A^1(M, TM)$ is called the *soldering form* of M (dp is the canonical vector valued 1-form of M and one has $d^\nabla(dp) = 0$). The operator $d^\omega = d + e(\omega)$ acting on ΛM is called the *cohomology operator*, where $e(\omega)$ means the exterior product by the closed 1-form $\omega \in \Lambda^1 M$, i.e. $d^\omega u = du + \omega \wedge u$, for any $u \in \Lambda M$. We have $d^\omega \circ d^\omega = 0$ and if $d^\omega u = 0$, u is said to be *d^ω -closed* [GL].

Any vector field $X \in \Gamma TM$ such that

$$d^\nabla(\nabla X) = \nabla^2 X = \pi \wedge dp \in A^2(M, TM)$$

is said to be an *exterior concurrent* vector field.

The 1-form π , which is called the *concurrence form*, is given by

$$\pi = fX^\flat, \quad f \in C^\infty M.$$

If $\mathcal{R}$ denotes the *Ricci tensor* of ∇ , we have

$$\mathcal{R}(X, Y) = -(n-1)fg(X, Z), \quad Z \in \Gamma TM, \quad n = \dim M,$$

and consequently

$$f = -\frac{1}{n-1} \text{Ric}(X),$$

where $\text{Ric}(X)$ means the *Ricci curvature* of M with respect to X .

Let $\mathcal{O} = \{e_A \mid A \in \{1, \dots, n\}\}$ be an adapted local field of orthonormal frames on M and let $\mathcal{O}^* = \{\omega^A\}$ be its associated coframe.

With respect to $\mathcal{O}$ and $\mathcal{O}^*$, the soldering form dp and E. Cartan's structure equations in indexless form are

$$dp = \omega^A \otimes e_A \in A^1(M, TM), \quad (1.1)$$

$$\nabla e = \theta \otimes e \in A^1(M, TM), \quad (1.2)$$

$$d\omega = -\theta \wedge \omega, \quad (1.3)$$

$$d\theta = -\theta \wedge \theta + \Theta. \quad (1.4)$$

In the above equations, θ , respectively Θ , are the *local connection forms* in the bundle $\mathcal{O}(M)$, respectively the *curvature forms* on M .

2. Main result

Let (M, g) be a general space-time with usual differentiability conditions and normal *hyperbolic* metric g , i.e. of index 1. Let $\mathcal{O} = \{e_A \mid A \in \{1, \dots, n\}\}$ be an adapted local field of orthonormal frames on M and let $\mathcal{O}^* = \{\omega^A\}$ be its associated coframe. We agree to denote by e_a , $a, b \in \{1, 2, 3\}$ and by e_4 the space-like vector basis and the time-like vector basis, respectively, w.r.t. g . Then, by reference to [C1] (see also [MRV]), one has

$$dp = -\omega^a \otimes e_a + \omega^4 \otimes e_4 \implies \langle dp, dp \rangle = (\omega^4)^2 - \sum_{a=1}^3 (\omega^a)^2, \quad (2.1)$$

and Cartan's structure equations are expressed by

$$\begin{cases} \nabla e_a = -\theta_a^b e_b + \theta_a^4 \otimes e_4, \\ \nabla e_4 = -\theta_4^a \otimes e_a, \end{cases} \quad (2.2)$$

$$\begin{cases} d\omega^a = -\omega^b \wedge \theta_b^a + \omega^4 \otimes \theta_4^a, \\ d\omega^4 = -\omega^a \wedge \theta_a^4, \end{cases} \quad (2.3)$$

$$\begin{cases} d\theta_a^b = \Theta_a^b - \theta_a^c \wedge \theta_c^b + \theta_a^4 \wedge \theta_4^b, \\ d\theta_4^a = \Theta_4^a - \theta_4^c \wedge \theta_c^a. \end{cases} \quad (2.4)$$

In the following, in order to simplify, we set $\omega^4 = \omega$ and $e_4 = e$.

In this paper we assume that the manifold M under consideration carries 3 space-like vector fields X, Y, Z which are skew-symmetric Killing vector fields (abbr. SSK) [R1], [MRV] having as generative the unit time-like vector field e . In order to simplify, we also set

$$X^\flat = \alpha, \quad Y^\flat = \beta, \quad Z^\flat = \gamma. \quad (2.5)$$

Under these conditions, by reference to [R1], [MRV], one has

$$\begin{cases} \nabla X = X \wedge e = \omega \otimes X - \alpha \otimes e, \\ \nabla Y = Y \wedge e = \omega \otimes Y - \beta \otimes e, \\ \nabla Z = Z \wedge e = \omega \otimes Z - \gamma \otimes e, \end{cases} \quad (2.6)$$

and since X, Y, Z are space-like, we set

$$X = X^a e_a, \quad Y = Y^a e_a, \quad Z = Z^a e_a, \quad (2.7)$$

i.e.

$$\|X\|^2 = -\sum_{a=1}^3 (X^a)^2, \quad \|Y\|^2 = -\sum_{a=1}^3 (Y^a)^2, \quad \|Z\|^2 = -\sum_{a=1}^3 (Z^a)^2. \quad (2.8)$$

From (2.2) and (2.6), one derives

$$dX^a - X^b \theta_b^a = X^a \omega \quad (2.9)$$

and

$$\alpha = -X^a \theta_a^4. \quad (2.10)$$

Hence, it follows from (2.10)

$$\theta_a^4 = -\omega^a \quad (2.11)$$

Setting $2\varphi_x = \sum_{a=1}^3 (X^a)^2$, $2\varphi_y = \sum_{a=1}^3 (Y^a)^2$, $2\varphi_z = \sum_{a=1}^3 (Z^a)^2$, one gets at once from (2.9)

$$\frac{d\varphi_x}{2\varphi_x} = \omega, \quad \frac{d\varphi_y}{2\varphi_y} = \omega, \quad \frac{d\varphi_z}{2\varphi_z} = \omega, \quad (2.12)$$

which shows that the generative ω (i.e. the time-like covector) is exact. Further, taking the exterior differential of α , one finds by (2.9) and by the structure equations (2.3)

$$d\alpha = 2\omega \wedge \alpha, \quad (2.13)$$

which is the general equation of a SSK vector field [R1].

Now, by (2.11) and the second equation (2.4), one derives on behalf of the structure equations (2.3) regarding the space-like covectors ω^a

$$\Theta_4^a = -\omega^a \wedge \omega. \quad (2.14)$$

Further, taking the exterior differentials of (2.4)

$$\begin{cases} X^2(\Theta_1^2 - \omega^1 \wedge \omega^2) + X^3(\Theta_1^3 - \omega^1 \wedge \omega^3) = 0, \\ X^1(\Theta_2^1 - \omega^2 \wedge \omega^1) + X^3(\Theta_2^3 - \omega^2 \wedge \omega^3) = 0. \end{cases}$$

Since, clearly, similar equations hold for Y and Z , one infers

$$\Theta_b^a = \omega^b \wedge \omega^a. \quad (2.15)$$

So, by reference to a known formula regarding space forms, we conclude by (2.14) and (2.15) that the space-time manifold (M, g) under consideration is an *Einstein manifold* of curvature -1 .

Since $d\omega = 0$, it is seen that (M, g) is foliated by space-like hypersurfaces M_s and by (2.11) and the equation (2.2), one has

$$\nabla e = \omega^a \otimes e_a = -dp_s, \quad (2.16)$$

where dp_s is the *spatial component* of the soldering form dp . Then the second fundamental form $II = -\langle dp_s, dp_s \rangle$ associated with the immersion $x : M_s \rightarrow M$ being conformal to the metric tensor g_s of M_s , it follows [Ch] that the immersion x is *pseudo-umbilical*.

On the other hand, clearly, Y and Z enjoy the same properties as X and one may write

$$\begin{cases} d\beta = 2\omega \wedge \beta \Leftrightarrow d^{-2\omega}\beta = 0, \\ d\gamma = 2\omega \wedge \gamma \Leftrightarrow d^{-2\omega}\gamma = 0 \end{cases} \quad (2.17)$$

(since ω is an exact form one may say that, cohomologically, the dual forms α, β, γ of the SSK vector fields X, Y, Z are $d^{-2\omega}$ -exact).

Finally, by (2.12), (2.13) and (2.17), it is seen that the existence of the considered space-time is defined by an exterior differential system Σ , whose characteristic numbers [C2] are $r = 4$, $s_0 = 1$, $s_1 = 3$. Consequently, we conclude that Σ is in *involution*, in the sense of CARTAN [C2], and depends of 3 arbitrary functions of one argument.

Since the vector fields X, Y, Z are orthogonal to e , one easily finds by (2.6)

$$[X, Y] = 0, \quad [X, Z] = 0, \quad [Y, Z] = 0,$$

which proves that the SSK vector fields X, Y, Z define a *commutative triple*.

Summing up, we state the following.

Theorem. *Let (M, g) be a space-time manifold with normal hyperbolic metric g . Assume that M carries 3 skew-symmetric Killing vector*

fields X, Y, Z having as generative the unit time-like vector field e of the hyperbolic metric g . Then M is an Einstein manifold of curvature -1 . The existence of the triple $\{X, Y, Z\}$ is assured by an exterior differential system Σ in involution. Such a manifold (M, g) has also the following properties:

- i) M is foliated by space-like hypersurfaces M_s tangent to $\{X, Y, Z\}$, normal to e and the immersion $x : M_s \rightarrow M$ is pseudo-umbilical;
- ii) the vector fields $\{X, Y, Z\}$ define a commutative triple of Killing vector fields.

3. Additional properties

In this section we shall make some additional considerations regarding the Einstein manifold defined in Section 2.

By using (2.16) and operating by d^∇ , one may write

$$d^\nabla(dp_s) = \omega \wedge dp_s. \quad (3.1)$$

Because $dp = \omega \otimes e - dp_s$, one derives

$$\nabla^2 e = \omega \wedge dp. \quad (3.2)$$

By reference to [R1], [PRV], the above equation proves that e is an EC vector field. Hence, following the general theory [PRV], if W is any vector field on M , one may write $\mathcal{R}(e, W) = -3g(e, W)$ (we notice that for any space-like vector field Z_s one has $\mathcal{R}(e, Z_s) = 0$).

Recall that the sectional curvature $K_{U \wedge V}$ of any vector fields U, V is expressed by

$$K_{U \wedge V} = \frac{g(R(U, V)V, U)}{\|U\|^2 \|V\|^2 - g(U, V)^2}. \quad (3.3)$$

Then by (2.6) and recalling that X, Y, Z are space-like vector fields, one finds $K_{X \wedge Y} = K_{Y \wedge Z} = K_{Z \wedge X} = -1$, which means that M is of curvature -1 .

Next, operating by d^∇ on the vector fields X, Y, Z , one derives

$$\nabla^2 X = \alpha \wedge dp, \quad \nabla^2 Y = \beta \wedge dp, \quad \nabla^2 Z = \gamma \wedge dp. \quad (3.4)$$

From (3.4) it follows that the SSK vector fields X , Y , Z are also exterior concurrent, as the time-like vector field e .

Next, setting $\alpha = \lambda U^1$, $\beta = \lambda U^2$, $\gamma = \lambda U^3$ as a space-like covector basis, one may define a subgroup of Lorentz by the group of space-like rotations $O(3)$ (*orthocronous transformations* [CWD]) preserving the time-like vector field e .

Also, one has

$$U^4 = \omega^4 = \omega, \quad \alpha^2 + \beta^2 + \gamma^2 = \lambda^2 \sum_{a=1}^3 (\omega^a)^2, \quad (3.5)$$

where λ is a scalar field.

Equations (3.5) imply

$$\sum_{a=1}^3 [(X^a)^2 + (Y^a)^2 + (Z^a)^2] = \lambda^2 \quad (3.6)$$

and

$$\begin{cases} \sum_{a \neq b} [X^a X^b + Y^a Y^b + Z^a Z^b] = 0, \\ \sum_{a \neq b} [X^a X^c + Y^a Y^c + Z^a Z^c] = 0, \\ \sum_{a \neq b} [X^b X^c + Y^b Y^c + Z^b Z^c] = 0. \end{cases} \quad (3.7)$$

Making use of equations (2.9), one finds that the differentiation of (3.7) holds good and the differentiation of (3.6) gives

$$\frac{d\lambda}{\lambda} = \omega \quad (3.8)$$

(recall that ω is an exact form).

Finally, we shall outline a certain property of the Lie algebra induced by the space-like vector fields X , Y , Z . We agree to call the 3-form

$$\varphi = \alpha \wedge \beta \wedge \gamma \quad (3.9)$$

the *distinguished spatial form* of M .

By (2.13) one gets at once

$$d\varphi = 6\omega \wedge \varphi \iff d^{-6\omega}\varphi = 0 \iff \mathcal{L}_e\varphi = 6\varphi, \quad (3.10)$$

which shows that φ is an $d^{-6\omega}$ -exact form and e defines an *infinitesimal conformal transformation* of φ .

By (2.12) one has

$$\frac{dg(X, X)}{4g(X, X)} = \frac{dg(Y, Y)}{4g(Y, Y)} = \frac{dg(Z, Z)}{4g(Z, Z)} = \omega, \quad (3.11)$$

and one derives

$$i_X\varphi = g(X, X)\beta \wedge \gamma + g(X, Y)\gamma \wedge \alpha + g(X, Z)\alpha \wedge \beta$$

and similar relations for Y and Z .

So, by (2.13), one infers $d(i_X\varphi) = 8\omega \wedge i_X\varphi$, which gives $\mathcal{L}_X\varphi = 2\omega \wedge i_X\varphi$.

Finally, by exterior differentiation, one may write $d(\mathcal{L}_X\varphi) = 0$ and clearly similar relations for Y and Z hold.

Hence, following a known definition [A], one may state that the distinguished spatial 3-form φ is a *relatively integral invariant* of the SSK vector fields X, Y, Z .

Consequently, the following results were obtained.

Theorem. *Let (M, g) be the space-time manifold defined in the Section 2 and let X, Y, Z be the 3 skew-symmetric Killing vector fields which determine M and e the unit time-like vector field of the hyperbolic metric g . One has the following properties:*

- i) *the vector fields X, Y, Z and e are exterior concurrent vector fields;*
- ii) *M admits an orthogonal transformation of a space-like Lorentz subgroup $O(3)$;*
- iii) *the 3-form $\varphi = \alpha \wedge \beta \wedge \gamma$ is a relatively integral invariant of the vector fields X, Y and Z .*

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