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The set of pseudo solutions of the differential equation $x^{(m)} = f(t, x)$ in Banach spaces

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Abstract. In this paper we prove the existence theorem for the equation $x^{(m)} = f(t, x(t))$ in Banach spaces where f is weakly-weakly sequentially continuous. Moreover, we prove that the set of pseudo-solutions of our equation is compact and connected.

1. Introduction

In this paper we will deal with the Cauchy problem

$$\begin{cases} x^{(m)} = f(t, x(t)) \\ x(0) = 0, \\ x'(0) = \eta_1, \dots, x^{(m-1)}(0) = \eta_{m-1}, \end{cases} \qquad t \in I = \langle 0, a\rangle,\ a \in \mathbb{R}_+ \tag{1.1}$$

where $\eta_1, \dots, \eta_{m-1} \in E$, $m \in \mathbb{N}$.

Throughout this paper $(E, \|\cdot\|)$ will be denote a real Banach space, E^* the dual space, $(R)\int_0^t f(s)ds$ the weak Riemann integral, $(P)\int_0^t f(s)ds$ the Pettis integral ([8], [9], [12], [16]).

By $(C(I, E), \omega)$ we will denote the space of all continuous functions from I to E endowed with the topology $\sigma(C(I, E), C(I, E)^*)$.

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This paper is divided into two main sections. In Section 1 we prove an existence theorem for the problem (1.1). In Section 2 we prove that, the set of pseudo-solutions of the equation (1.1) is compact and connected.

The result presented in this paper extends the results for Cichoń [5], Cichoń, Kubiaczyk [6], Cramer, Laksmikantham and Mitchell [7], O'Regan [15], Szufla [17], Szufla and Szukała [18].

Assume that $B = \{x \in E : \|x\| < b,\ b > 0\}$ and $f : I \times B \to E$. Moreover, let $M = \sup\{\|f(t,x)\| : t \in I,\ x \in B\}$. Choose a positive number d such that $d \le a$, $\sum_{j=1}^{m-1} \|\eta_j\| \frac{d^j}{j!} + M\frac{d^m}{m!} < b$, $d^m < 1$, $(m > 1)$.

Let $J = \langle 0, d\rangle$. We set $\widetilde{B} = \{x \in C(J,E) : x(t) \in B,\ t \in J\}$.

We will consider the problem

$$x(t) = p(t) + (R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_0^{t_{m-1}} f(t_m, x(t_m))dt_m \dots dt_2 dt_1, \quad (1.2)$$

where $p(t) = \begin{cases} 0, & m = 1 \\ \sum_{j=1}^{m-1} \eta_j \cdot \frac{t^j}{j!}, & m > 1 \end{cases}$ is a continuous function.

Now we recall the notion of the pseudo-solution. For such solutions, the problem (1.1) is equivalent to the integral problem (1.2).

Fix $x^* \in E^*$. Let us introduce the following definition:

Definition 1.1. A function $x : I \to E$ is said to be a *pseudo-solution* of the equation (1.1) if it satisfies the following conditions:

(i) x is a strongly absolutely continuous, $(m-1)$-times weakly differentiable,

(ii) $\forall_{x^* \in E^*} \exists_{\substack{\operatorname{mes} A(x^*)=0 \\ A(x^*) \subset I}} A(x^*)$ $x^* x : I \to E$ is m-times differentiable,

(iii) $(x^* x^{(m-1)})'(t) = x^* f(t, x(t))$ for each $t \notin A(x^*)$ and $x(0) = 0$, $x'(0) = \eta_1, \dots, x^{(m-1)}(0) = \eta_{m-1}$.

In this paper we will use the measure of weak noncompactness developed by DeBlasi [3]. The proofs of properties of the measure of week noncompactness see in [2].

Let A be a bounded nonvoid subset of E.

The de Blasi measure of weak noncompactness $\beta(A)$ is defined by

$$\beta(A) = \inf\{t > 0 : \text{there exist } C \in K^\omega \text{ such that } A \subset C + tB_0\},$$

where K^ω is the set of weakly compact subsets of E and B_0 is the norm unit ball.

The properties of measure of weak noncompactness $\beta(A)$ are:

(i) if $A \subset B$ then $\beta(A) \le \beta(B)$;

(ii) $\beta(A) = \beta(\overline{A})$, where $\overline{A}$ denotes the closure of A;

(iii) $\beta(A) = 0$ if and only if A is a weakly relatively compact;

(iv) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$;

(v) $\beta(\lambda A) = |\lambda|\beta(A)$, $(\lambda \in \mathbb{R})$;

(vi) $\beta(A + B) \le \beta(A) + \beta(B)$;

(vii) $\beta(\operatorname{conv} A) = \beta(A)$.

We can construct many other measures of noncompactness with the above properties, by using a scheme from [1], [4].

We recall that a function $f : I \times \widetilde{B} \to E$ is called *a Carathéodory function* if for each $x \in \widetilde{B}$, $f(t, x)$ is measurable in t and for almost all $t \in I$, $f(t, x)$ is continuous. A function $f : I \to E$ is said to be *weakly continuous* if it is continuous from I to E endowed with its weak topology.

A function $g : E \to E_1$, where E and E_1 are Banach spaces, is said to be *weakly – weakly sequentially continuous* if for each weakly convergent sequence $(x_n) \subset E$, a sequence $(g(x_n)) \subset E_1$.

2. Existence of solution

We will use the following lemmas:

Lemma 2.1 ([14])**.** *Let $H \subset C(I, E)$ be a family of strongly equicontinuous functions. Then $\beta_C(H) = \sup_{t \in I} \beta(H(t)) = \beta(H(I))$, where $\beta_C(H)$ denotes the measure of weak noncompactness in $C(I, E)$ and the function $t \to \beta(H(t))$ is continuous.*

Lemma 2.2 ([6])**.** *Let (X, d) be a metric space and let $g : X \to (E, \omega)$ be sequentially continuous. If $A \subset X$ is a connected subset in X, then $g(A)$ is a connected subset in (E, ω).*

Similar as in [10] we can prove the following lemma.

Lemma 2.3. *For each bounded, equicontinuous set $X \subset C(I,E)$ and for each $c,d \in I$ we have*

$$\beta\bigg(\int_c^d X(s)ds\bigg) \leq \int_c^d \beta(X(s))ds,$$

where $\int_c^d X(s)ds = \big\{\int_c^d x(s)ds : x \in X\big\}$.

In the proof of the main theorem we will apply the following fixed point theorem.

Theorem 2.1 ([13])**.** *Let D be a closed convex subset of E, and let F be a weakly sequentially continuous map from D into itself. If for some $x \in D$ the implication*

$$\overline{V} = \overline{\mathrm{conv}}\,(\{x\} \cup F(V)) \Rightarrow V \text{ is relatively wekly compact,} \tag{2.1}$$

Now we prove an existence theorem for the problem (1.1).

Theorem 2.2. *Assume, that for each strongly absolutely continuous function $x : J \to E$, $f(\cdot, x(\cdot))$ is Pettis integrable, $f(t,\cdot)$ is weakly-weakly sequentially continuous and*

$$\beta(f(J \times X)) \leq h(\beta(X)) \quad \text{for each } X \subset B, \tag{2.2}$$

where h is a function such that $h(u) < u$ for $u \in \mathbb{R}_+$. Then there exists a pseudo-solution of the problem (1.1) *on J.*

Proof. By F_x we define a mapping

$$F_x(t) = p(t) + (R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_0^{t_{m-1}} f(t_m, x(t_m))dt_m \dots dt_2 dt_1,$$

where $p(t) = \begin{cases} 0, & m = 1, \\ \sum_{j=1}^{m-1} \eta_j \cdot \frac{t^j}{j!}, & m > 1. \end{cases}$

We require that $F_x : \widetilde{B} \to \widetilde{B}$ is weakly sequentially continuous.

(i) For any $x^* \in E^*$ such that $\|x^*\| \leq 1$ and for any $x \in B$ as $|x^* f(t,x(t))| \leq M$ we have

$$\begin{aligned}
&|x^* F_x(t)| \\
&\quad = \bigg|x^*\bigg[p(t) + (R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_0^{t_{m-1}} f(t_m, x(t_m))dt_m \dots dt_2 dt_1\bigg]\bigg|
\end{aligned}$$

$$\leq \|x^*\| \cdot \sum_{j=1}^{m-1} \|\eta_j\| \frac{\|t^j\|}{j!}$$

$$+ (R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_0^{t_{m-1}} |x^*(f(t_m, x(t_m)))| dt_m \dots dt_2 dt_1$$

$$\leq \sum_{j=1}^{m-1} \|\eta_j\| \frac{d^j}{j!} + (R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_0^{t_{m-1}} M dt_m \dots dt_2 dt_1$$

$$\leq \sum_{j=1}^{m-1} \|\eta_j\| \frac{d^j}{j!} + \frac{M \cdot d^m}{m!} < b.$$

Hence

$$\sup\{|x^* F_x(t)| : x^* \in E^*,\ \|x^*\| \leq 1\} \text{ and } \|F_x(t)\| \leq b \text{ so } F_x \in \widetilde{B}.$$

(ii) Now we will prove that the set $F_x(\widetilde{B})$ is equicontinuous. Because

$$\|F_x(t) - F_x(s)\| \leq \|p(t) - p(s)\|$$

$$+ \left\| (R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_0^{t_{m-1}} f(t_m, x(t_m)) dt_m \dots dt_2 dt_1 \right\|$$

$$\leq \|p(t) - p(s)\| + \frac{M d^{m-1}}{(m-1)!}|t - s|, \quad \text{for each } x \in C(J, E),$$

so $F_x(\widetilde{B})$ is strongly equicontinuous.

(iii) Now we will show weakly sequentially continuity of F_x. Let $x_n \to x$ in $(C(I, E), \omega)$.

$$|x^*[F_{x_n}(t) - F_x(t)]|$$

$$= \left| x^* \left[p(t) + (R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_0^{t_{m-1}} f(t_m, x_n(t_m)) dt_m \dots dt_2 dt_1 \right.\right.$$

$$\left.\left. - p(t) - (R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_0^{t_{m-1}} f(t_m, x(t_m)) dt_m \dots dt_2 dt_1 \right] \right|$$

$$= \left| x^*\left[(R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_0^{t_{m-1}} [f(t_m, x_n(t_m)) - f(t_m, x(t_m))]dt_m \dots dt_2 dt_1\right]\right|$$

$$\leq (R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_0^{t_{m-1}} |x^*[f(t_m, x_n(t_m)) - f(t_m, x(t_m))]| dt_m \dots dt_2 dt_1.$$

Because $x_n \to x$ in $(C(I,E),\omega)$ and f is weakly sequentially continuous so F_x is weakly sequentially continuous.

Suppose that $\overline{V} = \overline{\mathrm{conv}}\,(F_x(V) \cup \{0\})$ for some $V \subset \widetilde{B}$.

We will prove that V is relatively weakly compact, thus (2.1) is satisfied. As $F_x(V)$ is equicontinuous, the function $v(t) \to \beta(V(t))$ is continuous (by Lemma 2.1).

By the definition of V, the mean valued theorem for the Pettis integral, Lemma 2.3, the strongly equicontinuity of the family of Riemann integrals, by the properties of β and (2.2) we obtain:

$$\beta(V(t)) = \beta(\overline{\mathrm{conv}}\,(F_x(V) \cup \{0\})) \leq \beta(F_x(V))$$

$$= \beta\left(p(t) + (R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_0^{t_{m-1}} f(t_m, x(t_m))dt_m \dots dt_2 dt_1\right)$$

$$\leq \beta\left((R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_0^{t_{m-1}} f(t_m, x(t_m))dt_m \dots dt_2 dt_1\right)$$

$$\leq (R)\int_0^t (R)\int_0^{t_1} \dots (R)\int_0^{t_{m-2}} \beta\left[(P)\int_0^{t_{m-1}} f(t_m, x(t_m))dt_m\right] dt_{m-1} \dots dt_2 dt_1$$

$$\leq (R)\int_0^t (R)\int_0^{t_1} \dots (R)\int_0^{t_{m-2}} \beta[t_{m-1} \cdot \overline{\mathrm{conv}}\, f(J \times V(J))] dt_{m-1} \dots dt_1$$

$$\leq (R)\int_0^t (R)\int_0^{t_1} \dots (R)\int_0^{t_{m-2}} t_{m-1} \cdot h(\beta(V(J))) dt_{m-1} \dots dt_1$$

$$\leq \frac{d^m}{m!} \cdot h(\beta(V(J))).$$

By our assumptions about the function h we have

$$\beta(V(t)) \leq \frac{d^m}{m!}\beta(V(J)).$$

So

$$\beta(V(J)) \leq \frac{d^m}{m!}\beta(V(J)).$$

Because $d^m < 1$, we get $v(t) = \beta(V(t)) = 0$ for $t \in J$.

By Arzelá–Ascoli's theorem, V is relatively weakly compact. So, by Theorem 2.1 F_x has a fixed point in $\widetilde{B}$ which is actually a pseudo-solution of the problem (1.1). □

3. Compactness and connectedness

In this part we show that the set of pseudo-solutions of our equation (1.1) is compact and connected.

Theorem 3.1. *Under the assumptions of Theorem 2.2 the set S of all pseudo-solutions of the Cauchy problem* (1.1) *on J is compact and connected in $(C(J,E),\omega)$.*

Proof. As $S = F_x(S)$, by repeating the above argument, with $V = S$ we can show that S is relatively compact in $(C(J,E),\omega)$. Since F is weakly continuous on $\overline{S(J)}^{\omega}$, S is weakly closed and consequently weakly compact.

For any $\eta > 0$ denotes by S_η the set of all functions $u : J \to E$ satisfying the following conditions:

(i) $u(0) = 0,\ u'(0) = \eta_1, \dots, u^{(m-1)}(0) = \eta_{m-1}$,

$$\|u(t)-u(s)\| \leq K|t-s|, \text{ for } t,s \in J, \text{ where } K = \sum_{j=1}^{m-1}\|\eta_j\|\frac{d^{j-1}}{j!}+\frac{Md^{m-1}}{(m-1)!},$$

(ii) $$\sup_{t\in J}\left\|u(t)-p(t)-(R)\int_0^t(R)\int_0^{t_1}\dots(P)\int_0^{t_{m-1}} f(t_m,x(t_m))dt_m\dots dt_2dt_1\right\| < \eta.$$

The set S_η is nonempty as $S \subset S_\eta$.

Let $\rho = \min(a, \eta/K)$. For any $\varepsilon \in (0, \rho)$ let $v(\cdot, \varepsilon) : J \to E$ be defined by the formula:

$$v(t,\varepsilon) = \begin{cases} p(t), & \text{for } 0 \le t \le \varepsilon \\ p(t) + (R)\int_0^t (R)\int_0^{t_1} \dots \\ (P)\int_0^{t_{m-1}-\varepsilon} f(t_m, x(t_m))dt_m \dots dt_2 dt_1, & \text{for } \varepsilon < t \le d \end{cases}$$

Clearly $v(\cdot, \varepsilon)$ satisfies (i).
Furthermore we have:

$$\left\| v(t,\varepsilon) - p(t) - (R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_0^{t_{m-1}} f(t_m, x(t_m))dt_m \dots dt_2 dt_1 \right\|$$

$$= \begin{cases} \left\| (R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_0^{t_{m-1}} f(t_m, x(t_m))dt_m \dots dt_2 dt_1 \right\|, & \text{for } 0 \le t \le \varepsilon \\ \left\| (R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_{t_{m-1}-\varepsilon}^{t_{m-1}} f(t_m, x(t_m))dt_m \dots dt_2 dt_1 \right\|, & \text{for } \varepsilon < t \le d \end{cases}$$

$$\le \frac{M \cdot \varepsilon \cdot d^{m-1}}{(m-1)!} < \eta$$

thus $v(\cdot, \varepsilon)$ satisfies (ii).

Now, we will prove that S_η is connected. Define

$$v_\varepsilon(t) = \begin{cases} p(t), & \text{for } 0 \le t \le \varepsilon \\ F_x(v_\varepsilon)(t-\varepsilon), & \text{for } \varepsilon < t \le d \end{cases}$$

where $v_\varepsilon = v(\cdot, \varepsilon)$. We will show that the mapping $\varepsilon \to v_\varepsilon(\cdot)$ is sequentially continuous from $(0, \rho)$ into $(C(J,E), \omega)$.

Let $0 < \varepsilon < \delta \le d$ (when $\delta \le \varepsilon$ the argument is similar).

For $t \in \langle 0, \varepsilon\rangle$

$$|x^*(v_\varepsilon(t) - v_\delta(t))| = 0. \tag{3.1}$$

For $t \in (\varepsilon, \delta\rangle$

$$\begin{aligned}
&|x^*(v_\varepsilon(t) - v_\delta(t))| \\
&= \Bigg| x^* \Bigg[(R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_0^{t_{m-1}-\varepsilon} f(t_m, v_\varepsilon(t_m))dt_m \dots dt_2 dt_1 \\
&\quad - (R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_0^{t_{m-1}-\delta} f(t_m, v_\varepsilon(t_m))dt_m \dots dt_2 dt_1 \Bigg]\Bigg| \\
&\le \|x^*\| \Bigg\| (R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_{t_{m-1}-\delta}^{t_{m-1}-\varepsilon} f(t_m, v_\varepsilon(t_m))dt_m \dots dt_2 dt_1 \Bigg\| \\
&\le \|x^*\| \cdot |\delta - \varepsilon| \cdot M \frac{d^{m-1}}{(m-1)!}.
\end{aligned} \tag{3.2}$$

For $t \in (\delta, 2\delta\rangle$

$$\begin{aligned}
|x^*(v_\varepsilon(t) - v_\delta(t))| &= |x^*(F_x(v_\varepsilon)(t-\varepsilon) - F_x(v_\delta)(t-\delta))| \\
&\le |x^*[F_x(v_\varepsilon)(t-\varepsilon) - F_x(v_\epsilon)(t-\delta)]| \\
&\quad + |x^*[F_x(v_\varepsilon)(t-\delta) - F_x(v_\delta)(t-\delta)]| \\
&\le |x^*[F_x(v_\varepsilon)(t-\delta) - F_x(v_\delta)(t-\delta)]| \\
&\quad + \|x^*\| \cdot M \cdot \frac{d^{m-1}}{(m-1)!} |t - \varepsilon - t\delta| \\
&= |x^*(F_x(v_\varepsilon)(t-\delta) - F_x(v_\delta)(t-\delta)]| \\
&\quad + \|x^*\| \cdot M \cdot \frac{d^{m-1}}{(m-1)!} |\delta - \varepsilon|.
\end{aligned} \tag{3.3}$$

Let (δ_n) be a sequence such that $\delta_n \to \varepsilon$ $(\delta_n \ge \varepsilon)$.

By (3.1) and (3.2), it follows that $v_{\delta_n}(t)$ converges weakly to $v_\varepsilon(t)$, uniformly for $t \in \langle 0, \delta\rangle$. So $F_x(v_{\delta_n})(t) \to F_x(v_\varepsilon)(t)$ weakly on $\langle 0, \delta\rangle$. Now, by (3.3) $v_{\delta_n}(t)$ tends to $v_\varepsilon(t)$ weakly for each $t \in \langle 0, 2\delta\rangle$.

By repeating the above argument and using induction, we obtain that the map $\varepsilon \to v_\varepsilon(t)$ from $(0,d)$ into $(C(J,E),\omega)$ is sequentially continuous. Therefore, by Lemma 2.2, the set $\{v_\varepsilon(\cdot) : 0 < \varepsilon < d\}$ is connected in $(C(J,E),\omega)$.

Let $x \in S_\eta$. Choose $\varepsilon > 0$ such that $0 < \varepsilon < d$ and

$$\sup_{t\in J}\left\| x(t) - p(t) - (R)\int_0^t (R)\int_0^{t_1} \dots (P)\int_0^{t_{m-1}} f(t_m, x(t_m))dt_m \dots dt_2 dt_1 \right\| + M\varepsilon \cdot \frac{d^{m-1}}{(m-1)!} < \eta.$$

For any q, $0 \le q \le d$ let $y(\cdot,q) : J \to E$ be defined by the formula:

$$y(t,q) = \begin{cases} x(t), & \text{for } 0 \le t \le q \\ x(q) + \dfrac{p(t) - x(q)}{\varepsilon}(t-q), & \text{for } q < t \le \min(d, q+\varepsilon) \\ p(t) + (R)\displaystyle\int_0^t (R)\int_0^{t_1} \dots & \\ (P)\displaystyle\int_q^{t_{m-1}-\varepsilon} f(t_m, y(t_m,q))dt_m \dots dt_2 dt_1, & \text{for } \min(d,q+\varepsilon) < t < d \end{cases}$$

By repeating the above consideration, with $y(\cdot,q)$ in the place of $v(\cdot,\varepsilon)$, one can show that $y(\cdot,q) \in S_\eta$ for each $q \in \langle 0,d\rangle$ and the mapping $q \to y(\cdot,q)$ from J into $(C(J,E),\omega)$ is sequentially continuous. Consequently, by Lemma 2.2, the set $T_x = \{y(\cdot,q) : 0 \le q \le d\}$ is connected in $(C(J,E),\omega)$.

As $y(\cdot,0) = v(\cdot,\varepsilon) \in V \cap T_x$, the set $V \cup T_x$ is connected, and therefore the set $W = \bigcup_{x\in S_\eta} T_x \cup V$ is connected in $(C(J,E),\omega)$.

Moreover $S_\eta \subset W$, because $x = y(\cdot,d) \in T_x$ for each $x \in S_\eta$. On the other hand $W \subset S_\eta$, since $T_x \subset S_\eta$ and $V \subset S_\eta$. Finally $S_\eta \subset W$ is a connected subset of $(C(J,E),\omega)$.

Suppose that the set S is not connected. As S weakly compact, there exist nonempty weakly compact sets W_1 and W_2 such that $S = W_1 \cup W_2$

and $W_1 \cap W_2 = \emptyset$. Consequently there exists two disjoint weakly open sets U_1, U_2 such that $W_1 \subset U_1$, $W_2 \subset U_2$. Suppose that for every $n \in N$, there exists a $u_n \in V_n \setminus U$, where $V_n = \overline{S^{\omega}_{1/n}}$ and $U = U_1 \cup U_2$.

Put $H = \overline{\{u_n : n \in N\}^{\omega}}$. Since $u_n - F_x(u_n) \to 0$ in $C(J, E)$ as $n \to \infty$ and $H(t) \subset \{u_n(t) - F_x(u_n)(t) : u_n \in H\} + F_x(H)(t)$ repeating the argument from Theorem 2.2, one can show that there exists $u_0 \in H$ such that $u_0 = F_x(u_0)$, i.e. $u_0 \in S \setminus U$. Furthermore, $S \subset (C(J, E), \omega) \setminus U$, since U is weakly open and hence $u_0 \in S$, a contradiction.

Therefore, there is $m \in N$ such that $V_m \subset U$. Since $U_1 \cap V_m \neq \emptyset \neq U_2 \cap V_m$, V_m is not connected, a contradiction with the connectedness of each V_n. Consequently, S is connected in $(C(J, E), \omega)$. $\square$

References

[1] J. Banaś and K. Goebel, Measures of Noncompactness in Banach spaces, Vol. 60, Lecture Notes in Pure and Appl. Math., *Marcel Dekker*, *New York and Basel*, 1980.

[2] J. Banaś and J. Rivero, On measures of weak noncompactness, *Ann. Math. Pura Appl.* **125** (1987), 213–224.

[3] F. S. DeBlasi, On a property of the unit sphere in a Banach space, *Bull. Math. Soc. Sci. Math. R.S. Roumanie* **21** (1977), 259–262.

[4] M. Cichoń, On measures of weak noncompactness, *Publicationes Math. Debrecen* **45** (1994), 93–102.

[5] M. Cichoń, Weak solutions of differential equations in Banach spaces, *Disc. Math. Diff. Inclusions* **15** (1995), 5–14.

[6] M. Cichoń and I. Kubiaczyk, On the set of solutions of the Cauchy problem in Banach spaces, *Arch. Math.* **63** (1994), 251–257.

[7] F. Cramer, V. Lakshmikantham and A. R. Mitchell, On the existence of weak solution of differential equations in nonreflexive Banach spaces, *Nonlin. Anal. TMA* **2** (1978), 169–177.

[8] J. Diestel and J. J. Jr. Uhl, Vector Measures Math. Surveys, Vol. 15, *Providence, Rhode Island*, 1977.

[9] R. F. Geitz, Pettis integration, *Proc. Amer. Math. Soc.* **82** (1991), 81–86.

[10] K. Goebel and W. Rzymowski, An existence theorem for the equation $x' = f(t, x)$ in Banach space, *Bull. Acad. Pol. Sci. Math.* **18** (1970), 367–370.

[11] P. Hartman, Ordinary Differential Equations, *New York*, 1964.

[12] W. J. Knight, Solutions of differential equations in B-spaces, *Duke Math. J.* **41** (1974), 437–442.

[13] I. Kubiaczyk, On a fixed point theorem for weakly sequentially continuous mappings, *Disc. Math. Diff. Inclusions* **15** (1995), 15–20.

[14] A. R. Mitchell and Ch. Smith, An existence theorem for weak solutions of differential equations in Banach spaces, In: Nonlinear Equations in Abstract Spaces, (V. Lakshmikantham, ed.), *Orlando*, 1978, 387–404.

[15] D. O'Regan, Weak solutions of ordinary differential equations in Banach spaces, *Applied Mathematics Letters* **12** (1999), 101–105.

[16] B. J. Pettis, On integration in vector spaces, *Trans. Amer. Math. Soc.* **44** (1938), 277–304.

[17] S. Szufla, On the differential equation $x^{(m)} = f(t, x)$ in Banach spaces, *Func. Ekv.* **41** (1998), 101–105.

[18] S. Szufla and A. Szukała, Existence theorems for weak solutions of n-th order differential equations in Banach spaces, *Functiones at Approximatio* **26** (1998), 313–319.

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