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Harmonic maps and the topology of complete submanifolds

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Abstract. In this paper, we obtain several Liouville theorems for harmonic maps and use them to study the topology of minimal submanifolds in a Euclidaen space and open submanifolds in manifolds with non-negative bi-Ricci curvature.

1. Introduction

Harmonic maps are critical points of the energy functional defined on the space of maps between two Riemannian manifolds. The Liouville type properties for harmonic maps have been studied extensively in the past years (cf. [Ch], [C], [EL1], [EL2], [ES], [H], [HJW], [J], [SY], [S], [Y1], etc.). It has been shown by Schoen and Yau that a harmonic map of finite energy from a stable minimal hypersurface in complete Riemannian manifolds with non-negative sectional curvature to a complete manifold with non-positive sectional curvature is constant [SY]. This Liouville theorem of Schoen–Yau was used [SY] to show the important result which states that any smooth map of finite energy from a stable minimal hypersurface of complete manifolds with non-negative sectional curvature to a compact manifold with non-positive sectional curvature is homotopic to constant on each compact set. In this paper, we use the same idea of Schoen–Yau to study the minimal submanifolds in Euclidean space and open submanifolds

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in manifolds with non-negative bi-Ricci curvature. In order to state our results, let us firstly fix some notation.

Definition 1.1 ([W]). Let M be an n-dimensional complete minimal immersed submanifold in R^m, the m-dimensional Euclidean space. Denote by $|A|^2$ the squared norm of the second fundamental form A of M in R^m. M is said to be super stable if for all compactly supported $\psi \in H_1^2(M)$, it holds

$$\int_M |\nabla\psi|^2 \geq \int_M |A|^2\psi^2. \tag{1.1}$$

From [Sp], we know that a *super stable* minimal immersed submanifold in R^m is always stable. In the case that $n = m - 1$, the inequality (1.1) is just the usual stability inequality for minimal hypersurface in Euclidean space.

Definition 1.2 ([ShY], [T]). Let $\overline{M}$ be an m-dimensional complete Riemannian manifold, and u, v be orthonormal tangent vectors. We set

$$\text{b-}\operatorname{Ric}(u,v) = \overline{\operatorname{Ric}}(u,u) + \overline{\operatorname{Ric}}(v,v) - \overline{K}(u,v),$$

and call it the bi-Ricci curvature in the directions u, v. Here $\overline{\operatorname{Ric}}$ and $\overline{K}$ denote the Ricci and sectional curvatures of $\overline{M}$, respectively.

Some interesting results about stable minimal hypersurfaces in manifolds with non-negative bi-Ricci curvature have been obtained, e.g. in [ShY], [T].

Definition 1.3. Let M be a complete oriented hypersurface immersed in an $(n+1)$-dimensional oriented Riemannian manifold $\overline{M}$. Let ∇ be the gradient operator of M and denote by $|A|^2$ the squared norm of the second fundamental form of M in $\overline{M}$. We say that M is stable if

$$\int_M \{|\nabla\psi|^2 - (|A|^2 + \overline{\operatorname{Ric}}(\mu,\mu))\psi^2\} \geq 0 \tag{1.2}$$

for all all compactly supported $\psi \in H_1^2(M)$, where $\overline{\operatorname{Ric}}(\mu,\mu)$ is the Ricci curvature of $\overline{M}$ in the unit normal direction μ of M.

In the Definition 1.3 above, we do not assume that M is minimal or have constant mean curvature. In the case that M is minimal in $\overline{M}$, the inequality (1.2) is the usual stability inequality for minimal hypersurfaces.

Now we can state our results in this paper as follows.

Theorem 1.1. *Let M be an n-dimensional complete super stable minimal submanifold immersed in R^{n+p} and let N be a complete Riemannian manifold with non-positive sectional curvature. Then any harmonic map from M to N with finite energy must be a constant.*

It has been shown by Schoen–Yau [SY] that any smooth map of finite energy from a complete Riemannian manifold to a compact manifold with non-positive sectional curvature is homotopic to a constant on each compact set. Thus Theorem 1.1 implies immediately the following

Corollary 1.1. *Let M be an n-dimensional complete super stable minimal submanifold immersed in R^{n+p} and let N be a complete Riemannian manifold with non-positive sectional curvature. If $f : M \to N$ is a smooth map with finite energy, then f is homotopic to constant on each compact set.*

As an application of this corollary, one has the following result the proof of which is similiar to that of the corollary to Theorem 1 in [SY].

Corollary 1.2. *Let M be as in Theorem 1.1 and let D be a compact domain in M with smooth simply connected boundary. Then there exists no non-trivial homomorphism from $\pi_1(D)$ into the fundamental group of a compact manifold with non-positive sectional curvature.*

Our next result is a Liouville theorem for harmonic maps from stable hypersurfaces in manifolds with non-negative bi-Ricci curvature.

Theorem 1.2. *Let M be an $n(2 \leq n \leq 5)$-dimensional complete noncompact oriented stable hypersurface in a complete Riemannian manifold $\overline{M}$ with non-negative bi-Ricci curvature. Assume that N is a complete Riemannian manifold with non-positive sectional curvature. Then any harmonic map from M to N with finite energy is constant.*

Combining Theorem 1.2 and the Schoen–Yau's work mentioned before, one gets

Corollary 1.3. *Let M be an $n(2 \leq n \leq 5)$-dimensional complete noncompact oriented stable hypersurface in a complete Riemannian manifold $\overline{M}$ with non-negative bi-Ricci curvature. Then any smooth map with* *finite energy from M to a compact manifold with non-positive sectional curvature is homotopic to constant on each compact set.*

Corollary 1.4. *Let M be as in Theorem 1.2 and let D be a compact domain in M with smooth simply connected boundary. Then there exists no non-trivial homomorphism from $\pi_1(D)$ into the fundamental group of a compact manifold with non-positive sectional curvature.*

2. Preliminaries

Let M and N be complete Riemannian manifolds of dimensions n and s, respectively. Let $f : M \to N$ be a harmonic map. Let $\{e_i\}_{i=1}^n$ and $\{\overline{e}_\alpha\}_{\alpha=1}^s$ be local orthonormal frames of M and N, respectively. Suppose $\{\omega_i\}_{i=1}^n$ and $\{\theta_\alpha\}_{\alpha=1}^s$ are the dual coframes of $\{e_i\}_{i=1}^n$ and $\{\overline{e}_\alpha\}_{\alpha=1}^s$, respectively, and $\{\omega_{ij}\}_{i,j=1}^n$ and $\{\theta_{\alpha\beta}\}_{\alpha,\beta=1}^s$ are the corresponding connection forms. Denote by R_{ijkl} and $K_{\alpha\beta\gamma\delta}$ the curvature tensors of M^n and N^s, respectively. Then we have the structure equations:

$$\begin{cases} d\omega_i = \sum_j \omega_{ij} \wedge \omega_j \\ \omega_{ij} + \omega_{ji} = 0 \\ d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \dfrac{1}{2} \sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \end{cases}$$

$$\begin{cases} d\theta_\alpha = \sum_\beta \theta_{\alpha\beta} \wedge \theta_\beta \\ \theta_{\alpha\beta} + \theta_{\beta\alpha} = 0 \\ d\theta_{\alpha\beta} = \sum_\gamma \theta_{\alpha\gamma} \wedge \theta_{\gamma\beta} - \dfrac{1}{2} \sum_{\gamma,\delta} K_{\alpha\beta\gamma\delta}\theta_\gamma\theta_\delta. \end{cases}$$

Define $f_{\alpha i}$, $1 \le \alpha \le s$, $1 \le i \le n$ by

$$f^*(\theta_\alpha) = \sum_i f_{\alpha i}\omega_i. \tag{2.1}$$

The energy density $e(f)$ of f is given by

$$e(f) = \sum_{\alpha,i} f_{\alpha i}^2.$$

Taking the exterior differentiation of (2.1), we get

$$f^*(d\theta_\alpha) = \sum_i \left(df_{\alpha i} \wedge \omega_i + f_{\alpha i} d\omega_i\right),$$

which gives

$$\sum_i \left(df_{\alpha i} - \sum_j f_{\alpha j}\omega_{ij} - f^*(\theta_{\alpha\beta}) f_{\beta i} \right) \wedge \omega_i = 0. \tag{2.2}$$

Define $f_{\alpha ij}$ by

$$df_{\alpha i} + \sum_\beta f_{\beta i} f^*(\theta_{\beta\alpha}) + \sum_j f_{\alpha j}\omega_{ji} = \sum_j f_{\alpha ij}\omega_j. \tag{2.3}$$

Then (2.2) and (2.3) imply that $f_{\alpha ij} = f_{\alpha ji}$ and f is harmonic means

$$\sum_i f_{\alpha ii} = 0, \quad \forall\ \alpha = 1, \ldots, s.$$

Exterior differentiating (2.3), we have

$$\begin{aligned} &\sum_l \left(df_{\alpha il} + \sum_j (f_{\alpha ij}\omega_{jl} + f_{\alpha jl}\omega_{ji}) + \sum_\beta f_{\beta il} f^*(\theta_{\beta\alpha}) \right) \wedge \omega_l \\ &\qquad = \frac{1}{2}\sum_{j,k,l} R_{ijkl} f_{\alpha j}\omega_k \wedge \omega_l + \frac{1}{2}\sum_{\beta,\delta,\gamma,k,l} K_{\alpha\beta\gamma\delta} f_{\beta i} f_{\gamma k} f_{\delta l}\omega_k \wedge \omega_l. \end{aligned} \tag{2.4}$$

Define

$$\sum_k f_{\alpha ijk}\omega_k = df_{\alpha ij} + \sum_k (f_{\alpha ik}\omega_{kj} + f_{\alpha kj}\omega_{ki}) + \sum_\beta f_{\beta ij} f^*(\theta_{\alpha\beta});$$

then (2.4) implies that

$$f_{\alpha ikl} - f_{\alpha ilk} = \sum_j R_{ijlk} f_{\alpha j} + \sum_{\beta,\gamma,\delta} K_{\alpha\beta\gamma\delta} f_{\beta i} f_{\gamma l} f_{\delta k}.$$

Set $e = e(f)$ and let Δ be the Laplacian operator acting on functions on M^n. From the above formula, one can easily get the following Bochner type formula for harmonic maps which was first derived by Eells–Sampson [ES].

$$\frac{1}{2}\Delta e = \sum_{\alpha,i,j} f_{\alpha ij}^2 + \sum_{\alpha,i,j} R_{ij} f_{\alpha i} f_{\alpha j} - \sum_{\alpha,\beta,\gamma,\delta,i,j} K_{\alpha\beta\gamma\delta} f_{\alpha i} f_{\beta j} f_{\gamma i} f_{\delta j}, \tag{2.5}$$

where (R_{ij}) is the Ricci tensor of M. It is known that (cf. [SY])

$$\sum_{\alpha,i,j} f_{\alpha ij}^2 \geq \left(1 + \frac{1}{2ns}\right) |\nabla\sqrt{e}|^2. \tag{2.6}$$

3. Proofs of the theorems

Proof of Theorem 1.1. We use the methods in [SY]. Let A be the second fundamental form of M in R^{n+p} which is defined by

$$A(X,Y)=\overline{\nabla}_XY-\nabla_XY,\quad \forall\ \ X,Y\in TM,$$

where $\overline{\nabla}$ and ∇ are the Riemannian connections on R^{n+p} and M, respectively. Let $s=\dim N$ and let $e=e(f)$ be the energy density of f. Replacing ψ in the stability inequality (1.1) by $\sqrt{e}\phi$ with $\phi\in C_0^\infty(M)$, we get

$$\begin{aligned}\int_M e\phi^2|A|^2 &\le \int_M e|\nabla\phi|^2+2\int_M\sqrt{e}\phi\langle\nabla\sqrt{e},\nabla\phi\rangle\phi^2+\int_M\phi^2|\nabla\sqrt{e}|^2\\ &=\int_M e|\nabla\phi|^2-\frac{1}{2}\int_M\phi^2\Delta e+\int_M\phi^2|\nabla\sqrt{e}|^2. \qquad (3.1)\end{aligned}$$

Let $e_1,\ldots,e_n,e_{n+1},\ldots,e_{n+p}$ be a local orthonormal frame on R^{n+p} such that when restricted to M, $e_1,\ldots,e_n$ are tangent to M. We will make the following index conventions:

$$1\le i,j,k,l\le n,\quad n+1\le a,b,c,d\le n+p,\quad 1\le\alpha,\beta,\gamma\le m.$$

Set

$$h_{ij}^a=\langle A(e_i,e_j),e_a\rangle,\quad\forall\ \ i,j,a;$$

then

$$|A|^2=\sum_{a,i,j}(h_{ij}^a)^2.$$

Since M is minimal, the Gauss equation implies that the Ricci curvature tensor Ric of M satisfies

$$R_{ij}\equiv\mathrm{Ric}(e_i,e_j)=-\sum_k\langle A(e_i,e_k),A(e_k,e_j)\rangle=-\sum_{k,a}h_{ik}^ah_{kj}^a. \qquad (3.2)$$

It follows from (2.5), (3.2), the non-positivity of the sectional curvature of N and the Schwarz inequality that

$$\frac{1}{2}\Delta e\ge\sum_{\alpha,i,j}f_{\alpha ij}^2+\sum_{\alpha,i,j}R_{ij}f_{\alpha i}f_{\alpha j}$$

$$
\begin{aligned}
&= \sum_{\alpha,i,j} f_{\alpha ij}^2 - \sum_{a,\alpha,i,j,k} h_{ik}^a h_{kj}^a f_{\alpha i} f_{\alpha j} \\
&= \sum_{\alpha,i,j} f_{\alpha ij}^2 - \sum_{a,\alpha,k} \left(\sum_i h_{ik}^a f_{\alpha i} \right)^2 \\
&\geq \sum_{\alpha,i,j} f_{\alpha ij}^2 - \sum_{k,a,\alpha} \left(\sum_i (h_{ik}^a)^2 \right) \left(\sum_i f_{\alpha i}^2 \right) = \sum_{\alpha,i,j} f_{\alpha ij}^2 - |A|^2 e.
\end{aligned}
$$

Substituting the above inequality into (3.1), we obtain

$$
\int_M e|\nabla\phi|^2 \geq \int_M \phi^2 \Big(\sum_{\alpha,i,j} f_{\alpha ij}^2 - |\nabla\sqrt{e}|^2 \Big).
$$

Thus we have from (2.6) that

$$
\frac{1}{2ns} \int_M \phi^2 |\nabla\sqrt{e}|^2 \leq \int_M e|\nabla\phi|^2. \tag{3.3}
$$

Fix a $p \in M$ and choose ϕ to be a cut-off function with the properties

$$
\phi = \begin{cases} 1 & \text{on } B(p,r) \\ 0 & \text{on } M \setminus B(p,3r) \end{cases}
$$

and

$$
|\nabla\phi| \leq \frac{1}{r},
$$

where $B(p,r)$ denotes the geodesic ball of M of radius r with center p. It then follows from (3.3) that

$$
\int_{B(p,r)} |\nabla\sqrt{e}|^2 \leq \frac{2ns}{r^2} E(f).
$$

Letting $r \to \infty$, it follows that e is a constant.

On the other hand, we obtain by choosing the above function ϕ into the inequality (1.1) that

$$
\int_{B(p,r)} |A|^2 \leq \frac{\mathrm{vol}(M)}{r^2}. \tag{3.4}
$$

If $\mathrm{vol}(M) < \infty$, then we conclude by letting $\to \infty$ that M is totally geodesic and so M is an affine n-plane. This is a contradiction. Therefore $\mathrm{vol}(M) = \infty$, which implies from $E(f) < \infty$ that $e = 0$. Consequently, f is a constant. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Set $s = \dim N$ and assume that $f : M \to N$ is a harmonic map. Let $e_1, \ldots, e_n, \mu$ be a local orthonormal frame on $\overline{M}$ such that $e_1, \ldots, e_n$ when restricted to M form a local orthonormal frame. Set

$$h_{ij} = \langle A(e_i, e_j), \mu \rangle, \quad i, j = 1, \ldots, n.$$

The squared norm of the second fundamental form of M is then given by

$$|A|^2 = \sum_{i,j} h_{ij}^2.$$

Replacing ψ in the stability inequality (1.2) by $\sqrt{e}\phi$ with $\phi \in C_0^\infty(M)$, we get

$$\begin{aligned}\int_M & e\phi^2(|A|^2 + \overline{\mathrm{Ric}}(\mu,\mu)) \\ &\le \int_M e|\nabla\phi|^2 + 2\int_M \sqrt{e}\phi\langle\nabla\sqrt{e}, \nabla\phi\rangle\phi^2 + \int_M \phi^2|\nabla\sqrt{e}|^2 \\ &= \int_M e|\nabla\phi|^2 - \frac{1}{2}\int_M \phi^2\Delta e + \int_M \phi^2|\nabla\sqrt{e}|^2. \end{aligned} \tag{3.5}$$

From Gauss equation, we know that the Ricci curvature tensor of M is given by

$$R_{ij} = \sum_k \left(\overline{R}(e_i, e_k, e_j, e_k) + h_{kk}h_{ij} - h_{ik}h_{jk}\right), \tag{3.6}$$

where $\overline{R}$ is the curvature tensor of $\overline{M}$. Set $u_\alpha = \sum_{k=1}^n f_{\alpha k}e_k$ and let u'_α be the unit vector in the direction u_α; it then follows from (2.5),(3.6) and the non-positivity of the sectional curvature of N that

$$\begin{aligned}\frac{1}{2}\Delta e &\ge \sum_{\alpha,i,j} f_{\alpha ij}^2 + \sum_{\alpha,i,j,k}\left(\overline{R}(e_i, e_k, e_j, e_k) + h_{kk}h_{ij} - h_{ik}h_{jk}\right) f_{\alpha i}f_{\alpha j} \\ &= \sum_{\alpha,i,j} f_{\alpha ij}^2 + \sum_{\alpha,k}|u_\alpha|^2\overline{R}(u_{\alpha'}, e_k, u_{\alpha'}, e_k) + \sum_{\alpha,i,j,k}(h_{kk}h_{ij} - h_{ik}h_{jk}) f_{\alpha i}f_{\alpha j} \\ &= \sum_{\alpha,i,j} f_{\alpha ij}^2 + \sum_{\alpha,k}|u_\alpha|^2\left(\overline{\mathrm{Ric}}(u_{\alpha'}, u_{\alpha'}) - \overline{K}(u_{\alpha'}, \mu)\right) \\ &\quad + \sum_{\alpha,i,j,k}(h_{kk}h_{ij} - h_{ik}h_{jk}) f_{\alpha i}f_{\alpha j}. \end{aligned} \tag{3.7}$$

We *claim* that

$$\sum_{\alpha,i,j,k} (h_{kk}h_{ij} - h_{ik}h_{jk})\, f_{\alpha i} f_{\alpha j} \geq -|A|^2 e.$$

In order to see this, let us take $e_1, \ldots, e_n$ to be the orthonormal principal directions corresponding to the principal curvatures $\lambda_1, \ldots, \lambda_n$. That is,

$$h_{ij} = \lambda_i \delta_{ij}, \quad \forall \;\; i, j.$$

Then we have

$$\begin{aligned}
\sum_{\alpha,i,j,k} (h_{kk}h_{ij} - h_{ik}h_{jk})\, f_{\alpha i} f_{\alpha j} &= \sum_{\alpha,i,j} \left(\Big(\sum_k \lambda_k \Big) \delta_{ij}\lambda_i - \delta_{ij}\lambda_i\lambda_j \right) f_{\alpha i} f_{\alpha j} \\
&= \Big(\sum_k \lambda_k \Big) \Big(\sum_{\alpha,i} \lambda_i f_{\alpha i}^2 \Big) - \sum_{\alpha,i} \lambda_i^2 f_{\alpha i}^2 \\
&= \sum_{\alpha,i} \Big(\sum_{k \neq i} \lambda_k \Big) \lambda_i f_{\alpha i}^2.
\end{aligned}$$

For any fixed i, set

$$G_i = \Big(\sum_{k \neq i} \lambda_k \Big) \lambda_i.$$

When $n = 2$, $G_i \geq -(\lambda_1^2 + \lambda_2^2) = -|A|^2$.
When $n = 3$,

$$G_1 = (\lambda_2 + \lambda_3)\lambda_1 \geq -\frac{\lambda_1^2 + \lambda_2^2}{2} - \frac{\lambda_1^2 + \lambda_3^2}{2} \geq -|A|^2,$$

and similarly, $G_i \geq -|A|^2$, $i = 2, 3$.
When $n = 4$,

$$\begin{aligned}
G_1 &= (\lambda_2\lambda_1 + \lambda_3\lambda_1 + \lambda_4\lambda_1) \\
&\geq -\frac{\lambda_1^2}{3} - \frac{3\lambda_2^2}{4} - \frac{\lambda_1^2}{3} - \frac{3\lambda_3^2}{4} - \frac{\lambda_1^2}{3} - \frac{3\lambda_4^2}{4} \geq -|A|^2,
\end{aligned}$$

and similarly, $G_i \geq -|A|^2$, $i = 2, 3, 4$.

When $n = 5$,

$$\begin{aligned} G_1 &= (\lambda_2\lambda_1 + \lambda_3\lambda_1 + \lambda_4\lambda_1 + \lambda_5\lambda_1) \\ &\geq -\frac{\lambda_1^2}{4} - \lambda_2^2 - \frac{\lambda_1^2}{4} - \lambda_3^2 - \frac{\lambda_1^2}{4} - \lambda_4^2 - \frac{\lambda_1^2}{4} - \lambda_2^2 \\ &= -|A|^2, \end{aligned}$$

and similarly, $G_i \geq -|A|^2$, $i = 2, 3, 4, 5$.

Hence, we obtain

$$\begin{aligned} \sum_{\alpha,i,j,k} (h_{kk}h_{ij} - h_{ik}h_{jk})\, f_{\alpha i} f_{\alpha j} &= \sum_{\alpha,i} G_i f_{\alpha i}^2 \\ &\geq -|A|^2 \sum_{\alpha,i} f_{\alpha i}^2 = -|A|^2 e. \end{aligned}$$

Thus, our *claim* is true and so we have

$$\frac{1}{2}\Delta e \geq \sum_{\alpha,i,j} f_{\alpha ij}^2 + \sum_{\alpha} |u_\alpha|^2 \left(\overline{\mathrm{Ric}}(u'_\alpha, u'_\alpha) - \overline{K}(u'_\alpha, \mu)\right) - |A|^2 e. \tag{3.8}$$

Substituting (3.8) into (3.5), we obtain

$$\begin{aligned} \int_M e|\nabla\phi|^2 \geq \int_M \phi^2 \bigg(e\overline{\mathrm{Ric}}(\mu,\mu) + \sum_\alpha |u_\alpha|^2 \left(\overline{\mathrm{Ric}}(u'_\alpha, u'_\alpha) - \overline{K}(u'_\alpha, \mu)\right) \bigg) \\ + \int_M \phi^2 \bigg(\sum_{\alpha,i,j} f_{\alpha ij}^2 - |\nabla\sqrt{e}|^2 \bigg). \end{aligned} \tag{3.9}$$

It follows from the non-negativity of the bi-Ricci curvature of $\overline{M}$ that

$$\begin{aligned} e\overline{\mathrm{Ric}}(\mu,\mu) + \sum_\alpha |u_\alpha|^2 \left(\overline{\mathrm{Ric}}(u'_\alpha, u'_\alpha) - \overline{K}(u'_\alpha, \mu)\right) \\ = \sum_\alpha \left\{ |u_\alpha|^2 \left(\overline{\mathrm{Ric}}(\mu,\mu) + \overline{\mathrm{Ric}}(u'_\alpha, u'_\alpha) - \overline{K}(u'_\alpha, \mu)\right)\right\} \geq 0. \end{aligned} \tag{3.10}$$

Therefore

$$\int_M e|\nabla\phi|^2 \geq \int_M \phi^2 \bigg(\sum_{\alpha,i,j} f_{\alpha ij}^2 - |\nabla\sqrt{e}|^2 \bigg). \tag{3.11}$$

Thus we know from (2.6) that

$$\frac{1}{2ns}\int_M \phi^2|\nabla\sqrt{e}|^2 \le \int_M e|\nabla\phi|^2. \tag{3.12}$$

Fix a $p \in M$ and choose ϕ to be a cut-off function with the properties

$$\phi = \begin{cases} 1 & \text{on } B(p,r) \\ 0 & \text{on } M^n \setminus B(p,3r) \end{cases}$$

and

$$|\nabla\phi| \le \frac{1}{r}.$$

Substituting the above ϕ into (3.12), we get

$$\int_{B(p,r)} |\nabla\sqrt{e}|^2 \le \frac{2ns}{r^2}E(f).$$

Letting $r \to \infty$, one finds that e is constant.

Introducing the above ϕ into (3.11), taking $r \to \infty$ and noticing that e is constant, we conclude that

$$f_{\alpha ij} \equiv 0, \quad \forall\ \alpha, i, j,$$

which in turn implies from (3.8) that

$$\sum_\alpha |u_\alpha|^2\left(\overline{\mathrm{Ric}}(u'_\alpha, u'_\alpha) - \overline{K}(u'_\alpha,\mu)\right) - |A|^2 e \le 0.$$

Also, it follows by introducing the above ϕ into (3.9) and using (3.10) that

$$e\overline{\mathrm{Ric}}(\mu,\mu) + \sum_\alpha |u_\alpha|^2\left(\overline{\mathrm{Ric}}(u'_\alpha, u'_\alpha) - \overline{K}(u'_\alpha,\mu)\right) = 0.$$

Therefore

$$e\left(\overline{\mathrm{Ric}}(\mu,\mu) + |A|^2\right) \ge 0.$$

On the other hand, substituting the above ϕ into (3.5) and taking $r \to \infty$, one has

$$\int_M e\left(|A|^2 + \overline{\mathrm{Ric}}(\mu,\mu)\right) \le 0.$$

Hence

$$e\left(|A|^2 + \overline{\mathrm{Ric}}(\mu,\mu)\right) = 0 \tag{3.13}$$

holds on M.

Let v be an arbitrary unit vector tangent to M. From the non-negativity of the bi-Ricci curvature of $\overline{M}$ for the orthonormal pair $\{v, \mu\}$, it follows that

$$\overline{\mathrm{Ric}}(v,v) + \overline{\mathrm{Ric}}(\mu,\mu) - \overline{K}(v,\mu) \geq 0. \tag{3.14}$$

Set

$$v = \sum_i a_i e_i, \quad \sum_i a_i^2 = 1;$$

then (3.6) implies that the Ricci curvature of M in the direction v satisfies

$$\begin{aligned}
\mathrm{Ric}(v,v) &= \sum_{i,j} a_i a_j R_{ij} \\
&= \sum_{i,j,k} a_i a_j \overline{R}(e_i, e_k, e_j, e_k) + \sum_{i,j,k} (h_{kk} h_{ij} - h_{ik} h_{jk}) a_i a_j \\
&= \overline{\mathrm{Ric}}(v,v) - \overline{K}(v,\mu) + \sum_{i,j,k} (h_{kk} h_{ij} - h_{ik} h_{jk}) a_i a_j.
\end{aligned}$$

Using the same arguments as in the proof of the *claim* above, one deduces that

$$\sum_{i,j,k} (h_{kk} h_{ij} - h_{ik} h_{jk}) a_i a_j \geq -|A|^2 \sum_i a_i^2 = -|A|^2.$$

Therefore,

$$\mathrm{Ric}(v,v) \geq \overline{\mathrm{Ric}}(v,v) - \overline{K}(v,\mu) - |A|^2. \tag{3.15}$$

If $e \neq 0$, we get from (3.13)–(3.15) that

$$\mathrm{Ric}(v,v) \geq -\overline{\mathrm{Ric}}(\mu,\mu) - |A|^2 = 0.$$

That is, M has non-negative Ricci curvature and so it has infinite volume since it is non-compact [Y2]. But this contradicts to the fact that $E(f)$ is finite. Consequently, we conclude that $e \equiv 0$ and f is constant. This completes the proof of Theorem 1.2. □

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