

On Stetkær type functional equations and Hyers–Ulam stability

By BOUIKHALENE BELAID (Kenitra) and
ELQORACHI ELHOUCIEN (Agadir)

Abstract. Let G be a locally compact group, K a compact subgroup of morphisms of G , $\chi : K \longrightarrow \{z \in \mathbb{C} \mid |z| = 1\}$ a continuous homomorphism and μ a K -invariant bounded measure on G . In this paper we study functional equations of the form

$$\int_G \int_K f(xtk \cdot y) \overline{\chi}(k) dk d\mu(t) = g(x)h(y), \quad x, y \in G,$$

in which $f, g, h \in C_b(G)$ are unknown functions. These equations may be viewed as a generalization of the functional equations considered by Stetkær in many of his works. We show how the solutions g and h are closely related to the solutions of Badora's functional equation solved in [4] and [13]. We treat examples and we give some applications. The case where G is a Lie group is considered. Furthermore, we investigate the Hyers–Ulam stability problem of these functional equations.

1. Introduction

Let G be a locally compact group endowed with a left Haar measure dx , and K a compact subgroup of morphisms of G i.e. of mappings k of G onto itself that are either automorphisms and homeomorphisms ($k \in K^+$), or antiautomorphisms and homeomorphisms ($k \in K^-$). The action of $k \in K$

Mathematics Subject Classification: 39B32, 39B42, 22D10, 22D12, 22D15.

Key words and phrases: functional equation, Gelfand measure, spherical function, Hyers–Ulam stability.

on $x \in G$ will be denoted by $k \cdot x$. The mapping $\chi : K \longrightarrow \{z \in \mathbb{C} \mid |z| = 1\}$ is a continuous homomorphism. For μ a complex bounded measure on G , $\check{\mu}$ (resp. $\overline{\mu}$) will denote the measure defined by $\langle \check{\mu}, f \rangle = \langle \mu, \check{f} \rangle$ (resp. $\langle \overline{\mu}, f \rangle = \langle \mu, \overline{f} \rangle$), where $\check{f}(x) = f(x^{-1})$, $\overline{f}(x) = \overline{f(x)}$ for all continuous and bounded functions f on G . $C(G)$ (resp. $C_b(G)$) designates the space of continuous (resp. continuous and bounded) complex valued functions. We assume that K has a topology making it a compact Hausdorff group with the property that the canonical map $K \times G \longrightarrow G$ sending each pair (k, x) onto $k \cdot x$ is continuous. For any $k \in K$, and for any function f on G , we put $(k \cdot f)(x) = f(k^{-1} \cdot x)$, and we say that f is K -invariant if $k \cdot f = f$ for all $k \in K$. The algebra of all regular and complex bounded measures on G will be denoted by $M(G)$. We recall that the convolution of $M(G)$ is given by

$$\langle \mu * \nu, f \rangle = \int_G \int_G f(ts) d\mu(t) d\nu(s), \quad \text{for all } f \in C_b(G).$$

For any $\mu \in M(G)$ and any $k \in K$, we put $\langle k \cdot \mu, f \rangle = \langle \mu, k \cdot f \rangle$ for all $f \in C_b(G)$, and we say that μ is K -invariant if $k \cdot \mu = \mu$ for all $k \in K$. A function $f \in C_b(G)$ is bi- μ -invariant if $f_\mu = f$, where f_μ is the continuous and bounded function defined by

$$f_\mu(x) = \int_G \int_G f(sxt) d\mu(s) d\mu(t), \quad \text{for all } x \in G.$$

We notice that if $\mu * \mu = \mu$, then f is bi- μ -invariant if and only if it is both left and right μ invariant, i.e. $\int_G f(tx) d\mu(t) = \int_G f(xt) d\mu(t) = f(x)$ for all $x \in G$.

Finally, $L_1(G, dx)$ designates the Banach algebra of all integrable functions on G .

Definition 1.1 ([2]). Let $\mu \in M(G)$; μ is said to be a Gelfand measure if $\check{\check{\mu}} = \mu * \mu = \mu$ and the Banach algebra $L_1^\mu(G) = \mu * L_1(G) * \mu$ is commutative under the convolution.

A non-zero function $\phi \in C_b(G)$ is a μ -spherical function if it satisfies the functional equation

$$\int_G \phi(xty) d\mu(t) = \phi(x)\phi(y), \quad x, y \in G. \quad (1.1)$$

We will say that a function $f \in C_b(G)$ satisfying

$$\int_G f(xty) d\mu(t) = f(x)\phi(y), \quad x, y \in G \quad (1.2)$$

is associated to the μ -spherical function ϕ .

The μ -spherical functions and related notions have been introduced by M. AKKOUCI and A. BAKALI [2]. When H is a compact subgroup of G and dh is the normalized Haar measure of H , then dh is a Gelfand measure on G if and only if (G, H) is a GELFAND pair [11]. A function $f \in C_b(G)$ satisfies a KANNAPPAN type condition $K(\mu)$ [25], [12] if

$$\int_G \int_G f(zsxy) d\mu(s) d\mu(t) = \int_G \int_G f(zsytx) d\mu(s) d\mu(t), \quad x, y, z \in G.$$

In the series of papers [28]–[32], a number of results has been obtained by STETKÆR for functional equations of the form

$$\int_K f(xk \cdot y) \overline{\chi(k)} dk = \sum_{i=1}^n g_i(x) h_i(y), \quad x, y \in G, \quad (1.3)$$

where the functions $f, g_1, \dots, g_n, h_1, \dots, h_n$ to be determined are continuous complex-valued functions on a locally compact group G and K is a compact subgroup of automorphisms of G .

In the present paper we study a generalization of the equation (1.3)

$$\int_G \int_K f(xtk \cdot y) \overline{\chi(k)} dk d\mu(t) = g(x)h(y), \quad x, y \in G, \quad (1.4)$$

where μ is a complex bounded measure on G and K is a compact subgroup of morphisms of G , not just a compact subgroup of automorphisms of G .

Our approach is to consider $\mu = \delta_e$: The Dirac measure concentrated at the identity element e as a complex bounded measure on G , and then the functional equation (1.3) can be written in the form

$$\int_G \int_K f(xtk \cdot y) \overline{\chi(k)} dk d\delta_e(t) = \sum_{i=1}^n g_i(x) h_i(y), \quad x, y \in G. \quad (1.5)$$

It is the same point of view as in [12] and [16] except that the compact subgroup K of morphisms of G is new; it was $\{I, \sigma\}$ in [12], [16] (σ is a continuous involution of G), where ELQORACHI and AKKOUCI have introduced and studied the functional equation

$$\int_G f(xty)d\mu(t) \mp \int_G f(xt\sigma(y))d\mu(t) = 2 \sum_{i=1}^n g_i(x)h_i(y). \quad (1.6)$$

The class of equations (1.4) contains also the functional equation of spherical functions

$$\int_K f(xk \cdot y)dk = f(x)f(y), \quad x, y \in G, \quad (1.7)$$

which has attracted the attention of many mathematicians. The first significant results were obtained in [9], [4], [31] and [32] for bounded and continuous solutions. For continuous solutions of (1.7), recently SHIN'YA [27] described the non-zero solutions in the following form:

$$f(x) = \int_K \varphi(k \cdot x)dk \quad \text{for all } x \in G,$$

where $\varphi : G \longrightarrow \mathbb{C} \setminus \{0\}$ is a continuous homomorphism of the abelian group G , (cf. [27] Corollary 3.12). BADORA's functional equation is considered in [4]:

$$\int_G \int_K f(x + t + k \cdot y)dkd\mu(t) = f(x)f(y), \quad x, y \in G. \quad (1.8)$$

The non-zero essentially bounded solutions of equation (1.8) are of the form

$$f(x) = \int_K (\varphi * \mu_{k \cdot x})(e)dk, \quad x \in G, \quad (1.9)$$

where φ is a character of G and e is the identity element of the abelian group G (cf. [4]). For G non necessarily abelian and μ a K -invariant generalized Gelfand measure, the non-zero continuous and bounded solutions of (1.8) are given by

$$f(x) = \int_K \varphi(k \cdot x)dk, \quad x \in G,$$

where φ is a μ -spherical function on G (cf. [13]).

We shall notice here that the additional assumption that every closed ideal of the commutative Banach algebra $\mu * L_1(G, dx) * \mu$ is contained in some maximal ideal of $\mu * L_1(G, dx) * \mu$, used in Section 3 of [13], is superfluous, because the commutative Banach algebra $\mu * L_1(G, dx) * \mu$ approximates the identity.

Equation (1.4) contains also the functional equation of μ -spherical functions

$$\int_G f(xty) d\mu(t) = f(x)f(y), \quad x, y \in G, \quad (1.10)$$

which was studied in [2] and [3]. It should be motioned here that if $\mu \in M(G)$, then the continuous solutions of (1.10) are only given when G is compact (cf. [3]). They are of the form

$$f(x) = \langle \pi(x)\xi, \eta \rangle, \quad (1.11)$$

where $(\pi, \mathcal{H})$ is an irreducible, continuous and unitary representation of G such that $\pi(\mu)$ is of rank one, $\eta \in \mathcal{H} \setminus \{0\}$ and ξ is a unit vector in $\mathfrak{R}(\pi(\mu))$, the range of the operator $\pi(\mu)$.

The classical examples of equation (1.4) with $K = \{I, -I\}$ and $\chi = 1$ are: D’ALEMBERT’s equation [25], [7], [10]

$$f(x+y) + f(x-y) = 2f(x)f(y), \quad x, y \in G, \quad (1.12)$$

and WILSON’s equation [37], [38]

$$f(x+y) + f(x-y) = 2f(x)g(y), \quad x, y \in G. \quad (1.13)$$

Other references and informations on detailed discussions of classical equations can be found in the monographs by ACZÉL and DHOMBRES (cf. [1]). An example of transformation groups K other than those two, and an example of homomorphisms other than $\chi = 1$ in connection with functional equation (1.4) is $K = \mathbb{Z}_n$ acting on $G = \mathbb{C}$

$$\frac{1}{n} \sum_{j=1}^{n-1} f(x + \omega^j y) = f(x)f(y), \quad x, y \in \mathbb{C}, \quad (1.14)$$

where $\omega = \exp(2\pi i/n)$. This functional equation occurs in FÖRG-ROB and SCHWAIGER [18] and STETKÆR [30].

Our discussion in the present paper is organized as follows. In Section 2 we establish some general properties of the solutions of (1.4). We show how they are closely related to the solutions of Badora's equation. This is an extension of STETKÆR's results ([30], III, Theorem 1). The conclusion is the same if we replace the functional equation of spherical functions in [30] by BADORA's functional equation, but the assumptions are weaker. K is not assumed to act by homomorphisms only, but by homomorphisms and also antihomomorphisms, and f satisfies $K(\mu)$. In the case when K is a compact subgroup of the group $\text{Aut}(G)$ of all mappings of G onto G that are simultaneously automorphisms and homeomorphisms and μ is a Gelfand K -invariant measure, we prove that the solutions g and h of (1.4) are associated to $\mu \otimes dk$ -spherical functions on the semi-direct product group $K \ltimes G$. In Section 3 we treat examples. In Section 4 we study the functional equation

$$\int_G \int_K f(xtk \cdot y) \overline{\chi(k)} dk d\mu(t) = g(x)f(y), \quad x, y \in G, \quad (1.15)$$

as a particular case of (1.4). In Section 5, G is a connected Lie group and μ is a K -invariant idempotent measure with compact support. We show that the solutions of (1.4) are the eigenfunctions of a system of operators associated to left invariant differential operators on G . This extends the previous results obtained by STETKÆR for equation (1.4) ([30], II, Theorem 2) and those of the authors in [13] to BADORA's equation. In the last section we deal with the stability of Badora's functional equation and of the equation (1.15).

The results obtained in this paper may be viewed as a continuation and a generalization of BADORA's work [4], [5], FÖRG-ROB's and SCHWAIGER's work [18], [19] and STETKÆR's work [30].

2. On the second generalization of functional equations of Stetkær type

In this section we study the properties of the functional equation

$$\int_G \int_K f(xtk \cdot y) \overline{\chi}(k) dk d\mu(t) = g(x)h(y), \quad x, y \in G. \quad (2.1)$$

The ideas are inspired by the STETKÆR's work [30] just mentioned. By easy computation, we get the following

Proposition 2.1. *Let μ be a K -invariant measure. Let $f, g, h \in C_b(G)$ be a solution of (2.1) such that f satisfies the Kannappan type condition $K(\mu)$, $g \neq 0$ and $h \neq 0$. Then, for all $x, y \in G$, we have*

$$\int_G h(xty) d\mu(t) = \int_G h(ytx) d\mu(t)$$

and g satisfies $K(\mu)$.

Theorem 2.2. *Let μ be a K -invariant measure. Let $f, g, h \in C_b(G)$ be a solution of (2.1) such that f satisfies the Kannappan type condition $K(\mu)$, $g \neq 0$ and $h \neq 0$. Then*

- i) $h(k \cdot x) = \chi(k)h(x)$ for all $k \in K$, $x \in G$,
- ii) there exists a function ϕ , solution of Badora's functional equation, such that

$$\int_G \int_K g(xtk \cdot y) dk d\mu(t) = g(x)\phi(y), \quad x, y \in G \quad (2.2)$$

and

$$\int_G \int_K \check{h}(xtk \cdot y) dk d\check{\mu}(t) = \check{h}(x)\check{\phi}(y), \quad x, y \in G. \quad (2.3)$$

- iii) If G is a unimodular group, K a compact subgroup of automorphisms of G and μ a Gelfand K -invariant measure, then ϕ is a $\mu \otimes \omega_K$ -spherical function and g (resp. $\check{h}$) is associated to ϕ (resp. $\check{\phi}$).

PROOF. Let $x, y \in G$ and let $k_0 \in K$, then we have

$$g(x)h(k_0 \cdot y) = \int_G \int_K f(xtkk_0 \cdot y) \overline{\chi}(k) dk d\mu(t)$$

$$= \int_K f(xtk \cdot y) \overline{\chi}(kk_0^{-1}) dk d\mu(t) = \chi(k_0) g(x) h(y),$$

from which we deduce (i).

Let $x_0, y_0 \in G$ such that $g(x_0) \neq 0$ and $h(y_0) \neq 0$, then by using $K(\mu)$, the K -invariance of μ and equation (2.1), we get

$$\begin{aligned} & h(y_0) \int_G \int_K g(x_0 tk \cdot x) dk d\mu(t) \\ &= \int_G \int_K \int_G \int_K f(x_0 tk \cdot xsk_1 \cdot y_0) \overline{\chi}(k_1) dk_1 d\mu(s) dk d\mu(t) \\ &= \int_G \int_K \int_G \int_{K^+} f(x_0 tk \cdot xsk_1 \cdot y_0) \overline{\chi}(k_1) dk_1 dk d\mu(s) d\mu(t) \\ &\quad + \int_G \int_K \int_G \int_{K^-} f(x_0 tk_1 \cdot y_0 sk \cdot x) \overline{\chi}(k_1) dk_1 dk d\mu(s) d\mu(t) \\ &= \int_G \int_K \int_G \int_{K^+} f(x_0 tk_1 \cdot [k_1^{-1} k \cdot xsy_0]) \overline{\chi}(k_1) dk_1 d\mu(s) dk d\mu(t) \\ &\quad + \int_G \int_K \int_G \int_{K^-} f(x_0 tk_1 \cdot [k_1^{-1} k \cdot xsy_0]) \overline{\chi}(k_1) dk_1 dk d\mu(s) d\mu(t) \\ &= \int_G \int_K \int_G \int_K f(x_0 tk_1 \cdot (k \cdot xsy_0)) \overline{\chi}(k_1) dk_1 d\mu(t) dk d\mu(s) \\ &= g(x_0) \int_G \int_K h(k \cdot xsy_0) dk d\mu(s). \end{aligned}$$

Now, in view of Proposition 2.1 and the K -invariance of μ , we obtain

$$\begin{aligned} &= g(x_0) \int_G \int_K h(k \cdot xsy_0) dk d\mu(s) \\ &= g(x_0) \int_G \int_{K^+} h(k \cdot (xtk^{-1} \cdot y_0)) dk d\mu(t) \\ &\quad + g(x_0) \int_G \int_{K^-} h(k \cdot (k^{-1} \cdot y_0 tx)) dk d\mu(t) \\ &= g(x_0) \int_G \int_{K^+} \chi(k) h(xtk^{-1} \cdot y_0) dk d\mu(t) \\ &\quad + g(x_0) \int_G \int_{K^-} \chi(k) h(k^{-1} \cdot y_0 tx) dk d\mu(t) \end{aligned}$$

$$\begin{aligned}
 &= g(x_0) \int_G \int_K \chi(k) h(xtk^{-1} \cdot y_0) dk d\mu(t) \\
 &= g(x_0) \int_G \int_K \overline{\chi(k)} h(xtk \cdot y_0) dk d\mu(t),
 \end{aligned}$$

from which we get

$$\int_G \int_K g(xtk \cdot y) dk d\mu(t) = g(x) \phi(y), \quad x, y \in G,$$

where ϕ is given by

$$\begin{aligned}
 \phi(x) &= \frac{1}{g(x_0)} \int_G \int_K g(x_0 tk \cdot x) dk d\mu(t) \\
 &= \frac{1}{h(y_0)} \int_G \int_K h(xtk \cdot y_0) \overline{\chi(k)} dk d\mu(t).
 \end{aligned}$$

Now, using Proposition 2.1 and the definition of ϕ , we show that ϕ is a solution of Badora's functional equation.

$$\int_G \int_K \phi(xtk \cdot y) dk d\mu(t) = \phi(x) \phi(y), \quad x, y \in G,$$

and h, ϕ satisfy the equation

$$\int_K \int_G \check{h}(xtk \cdot y) dk d\check{\mu}(t) = \check{h}(x) \check{\phi}(y).$$

This proves (ii). For iii), let K be a compact subgroup of $\text{Aut}(G)$, and let $K \rtimes G$ be the semi-direct product group with the group law

$$(k_1, x)(k_2, y) = (k_1 k_2, x k_1 \cdot y), \quad k_1, k_2 \in K, \quad x, y \in G.$$

A function $F : K \rtimes G \longrightarrow \mathbb{C}$ that is bi- $\mu \otimes dk$ -invariant can be regarded as a function $F(k, x) = f(x)$ on G such that f is both bi- μ -invariant and K -invariant. Accordingly, we obtain the bijection

$$\begin{aligned}
 L_1^{\mu \otimes dk}(K \rtimes G) &\longrightarrow L_1^\mu(G) \cap L_1^K(G) \\
 F &\longrightarrow f,
 \end{aligned}$$

where $L_1^K(G) = \{f \in L_1(G) : k \cdot f = f, k \in K\}$, so that $L_1^{\mu \otimes dk}(K \rtimes G) \cong L_1^\mu(G) \cap L_1^K(G) = \mu * L_1^K(G) * \mu = M_K(\mu * L_1(G) * \mu)$, where $M_K(f)(x) = \int_K f(k \cdot x) dk$, $x \in G$, and $f \in L_1(G)$. Then $\mu \otimes dk$ is a Gelfand measure on $K \rtimes G$. Furthermore, by using ([13], Theorem 2.2), we get that the $\mu \otimes dk$ -spherical functions are solutions of Badora's functional equation. $\square$

Remark 2.3. In Theorem 2.2 it is not necessary to assume that f satisfies the condition $K(\mu)$ if K is a compact subgroup of homomorphisms of G .

Corollary 2.4. *Let G be a locally compact group and let H be a compact subgroup of G such that H is K -invariant (i.e. $K \cdot H \subset H$). Let $(f, g, l) \in C_b(G)$ be a solution of*

$$\int_H \int_K f(xhk \cdot y) \overline{\chi}(k) dk dh = g(x)l(y), \quad x, y \in G, \quad (2.4)$$

such that $g \neq 0$, $l \neq 0$ and f satisfies a Kannappan type condition

$$\int_H \int_H f(zh_1 x h_2 y) dh_1 dh_2 = \int_K \int_K f(zh_1 y h_2 x) dh_1 dh_2, \quad x, y, z \in G.$$

Then

- i) $l(k \cdot x) = \chi(k)l(x)$ for all $k \in K$, $x \in G$,
- ii) there exists a function ϕ solution of the functional equation

$$\int_H \int_K \phi(xhk \cdot y) dk dh = \phi(x)\phi(y), \quad x, y \in G, \quad (2.5)$$

such that

$$\int_H \int_K g(xhk \cdot y) dk dh = g(x)\phi(y), \quad x, y \in G, \quad (2.6)$$

and

$$\int_H \int_K \check{l}(xhk \cdot y) dk dh = \check{l}(x)\check{\phi}(y), \quad x, y \in G. \quad (2.7)$$

- iii) If G is a unimodular group and K a compact subgroup of $\text{Aut}(G)$, then ϕ is a $K \rtimes H$ -spherical function.

Corollary 2.5. *Let G be a locally compact group and H a compact subgroup of G such that $K.H \subset H$. Let τ be a continuous, unitary and irreducible representation of H and let χ_τ be a normalized character of τ such that $\chi_\tau * \chi_\tau = \chi_\tau$ and $\mu_\tau = \chi_\tau dh$. Moreover, let $(f, g, l) \in C_b(G)$ be a solution of*

$$\int_H \int_K f(xhk \cdot y) \overline{\chi_\tau}(k) \chi_\tau(h) dk dh = g(x)l(y) \quad (2.8)$$

such that $g \neq 0$, $h \neq 0$ and f satisfies a Kannappan type condition

$$\begin{aligned} \int_H \int_H f(zh_1 x h_2 y) \chi_\tau(h_1) \chi_\tau(h_2) dh_1 dh_2 \\ = \int_K \int_K f(zh_1 y h_2 x) \chi_\tau(h_1) \chi_\tau(h_2) dh_1 dh_2. \end{aligned}$$

Then

- i) $l(k \cdot x) = \chi(k)l(x)$ for all $k \in K$, $x \in G$,
- ii) there exists a function ϕ , solution of the functional equation

$$\int_H \int_K \phi(xhk \cdot y) \chi_\tau(h) dk dh = \phi(x)\phi(y), \quad x, y \in G, \quad (2.9)$$

such that

$$\int_H \int_K g(xhk \cdot y) \chi_\tau(h) dk dh = g(x)\phi(y), \quad x, y \in G, \quad (2.10)$$

and

$$\int_H \int_K \check{l}(xhk \cdot y) \overline{\chi_\tau}(h) dk dh = \check{l}(x)\check{\phi}(y), \quad x, y \in G. \quad (2.11)$$

- iii) If G is unimodular and K a compact subgroup of $\text{Aut}(G)$, then ϕ is a $K \propto H$ -spherical function of type τ .

Corollary 2.6. *Let G be an unimodular group and μ a K -invariant Gelfand measure on G . Then the corresponding $\mu \otimes dk$ -spherical functions have the form*

$$\phi(x) = \int_K \omega(k \cdot x) dk, \quad x \in G,$$

for some μ -spherical function ω . Furthermore, if ϕ is integrable or G is a compact group, then ϕ has the form

$$\phi(x) = \int_K \langle \pi(\mu)\pi(k \cdot x)\xi, \eta \rangle dk, \quad x \in G,$$

where $(\pi, \mathcal{H}_\pi)$ is an irreducible, continuous and unitary representation of G , such that $\pi(\mu)$ is a rank one operator and $\xi, \eta \in \mathcal{H}_\pi$.

PROOF. By using [3] and [13], we derive the proof. $\square$

3. Examples

The next examples extend those obtained by STETKÆR in [30].

3.1. Let K be a compact subgroup of morphisms of G . Let μ be a K -invariant measure, $\omega \in C_b(G)$ a solution of (1.1), and $a \in C(G)$. Put

$$f(x) := \int_K a(k)\omega(k \cdot x)\overline{\chi}(k)dk, \quad x \in G,$$

$$g(x) := \int_K a(k)\omega(k \cdot x)dk, \quad x \in G,$$

$$h(x) := \int_K \omega(k \cdot x)\overline{\chi}(k)dk, \quad x \in G.$$

Then (f, g, h) is a solution of (2.1) and the corresponding function ϕ given by Theorem 2.2 has the form $\phi(x) = \int_K \omega(k \cdot x)dk$, $x \in G$.

3.2. Let $\chi = 1$ and let $f \neq 0$ be a right μ -invariant function which satisfies the condition $K(\mu)$, and $(f, g, h) \in C_b(G)$ a solution of (2.1). By putting $y = e$ in (2.1) we get $f(x) = g(x)h(e)$. So $h(e) \neq 0$, and (2.1) becomes

$$\int_G \int_K f(xtk \cdot y)dkd\mu(t) = 2\frac{f(x)}{h(e)}h(y) = 2f(x)\phi(y), \quad x, y \in G. \quad (3.1)$$

By Theorem 2.2, $\phi = \frac{h}{h(e)}$ is a solution of Badora's functional equation.

An example of (2.1) with $K = \{I, \sigma\}$, where σ is a continuous involution of G , is

$$\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2g(x)h(y), \quad x, y \in G, \quad (3.2)$$

which reduces to the generalized form of Wilson’s functional equation

$$\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2f(x)\phi(y), \quad x, y \in G, \quad (3.3)$$

where $\phi(y) = \frac{h(y)}{h(e)}$, for all $y \in G$. The solutions of (3.3) and (3.2) are described in [12] and [16].

3.3. By taking G an abelian locally compact group and $\mu = \delta_e$ we may derive other examples (see [1]).

4. On the first generalization of a functional equation of Stetkær type

In this section we will study a functional equation of the form

$$\int_G \int_K f(xtk \cdot y) \overline{\chi}(k) dk d\mu(t) = g(x)f(y), \quad x, y \in G. \quad (4.1)$$

This equation is a special case of the equation (2.1) in which $f = h$. Using Theorem 2.2, we deduce the following

Theorem 4.1. *Let μ be a K -invariant measure. Let $(f, g) \in C_b(G)$ such that $g \neq 0$ and f satisfies $K(\mu)$. Then*

- (1) *If (f, g) is a solution of (4.1) and $f \neq 0$ then g is a solution of (1.3).*
- (2) *(f, g) is a solution of (4.1) if and only if*

i) $f(k \cdot x) = \chi(k)f(x)$ for all $k \in K, x \in G$, and

$$\text{ii) } \int_G \int_K \check{f}(xtk \cdot y) dk d\check{\mu}(t) = \check{f}(x)\check{g}(y), \quad x, y \in G. \quad (4.2)$$

Corollary 4.2 ([12]). *Let σ be a continuous involution of G . Let μ be a σ -invariant measure. Let $(f, g) \in C_b(G) \setminus \{0\}$ such that f satisfies $K(\mu)$. The solutions of the functional equation*

$$\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2g(x)f(y), \quad x, y \in G \quad (4.3)$$

are given as follows:

- i) there exists a $\check{\mu}$ -spherical function φ such that $g = \frac{\varphi + \varphi \circ \sigma}{2}$,
- ii) if $\varphi \circ \sigma \neq \varphi$, then there exist $\alpha, \beta \in \mathbb{C}$ such that $f = \alpha \frac{\varphi + \varphi \circ \sigma}{2} + \beta \frac{\varphi - \varphi \circ \sigma}{2}$,
- iii) if $\varphi \circ \sigma = \varphi$, then there exists $\gamma \in \mathbb{C}$ such that $f = \gamma\varphi + l$, where $l \circ \sigma = -l$ and l is a solution of the functional equation

$$\int_G l(xty) d\check{\mu}(t) = l(x)\varphi(y) + l(y)\varphi(x), \quad x, y \in G. \quad (4.4)$$

PROOF. By taking $\chi = 1$ in (4.1) and by using Theorem 4.1 and ([12], Theorem 3.1), we get the proof. $\square$

5. On generalized functional equations of Stetkær type on Lie groups

In this section we characterize the solutions $f, g \in C^\infty(G)$ of the functional equation

$$\int_G \int_K f(xtk \cdot y) \overline{\chi}(k) dk d\mu(t) = g(x)h(y), \quad x, y \in G \quad (5.1)$$

on a connected Lie group G as joint eigenfunctions of certain operators associated to the left invariant differential operators, where in this case K is a compact subgroup of the group $\text{Aut}(G)$ of all mappings of G onto G that are simultaneously automorphisms and homeomorphisms. This extends the previous results obtained by STETKÆR in [30] to equation (1.7) and those of the authors in [13] to Badora's functional equation.

To formulate our results, we need the following notations:

Let G be a connected Lie group and K a compact subgroup of the group $\text{Aut}(G)$ of all mappings of G onto G that are simultaneously automorphisms and homeomorphisms. $\mathbb{D}(G)$ denotes the algebra of the left invariant differential operators on G , i.e. for all $D \in \mathbb{D}(G)$, $a \in G$, and for all $f \in C^\infty(G)$ we have $(L_a D)f = D(L_a f)$, where $(L_a f)(x) = f(a^{-1}x)$ for all $x \in G$. We recall (see [30], Proposition II.3) that K has a Lie group structure, the canonical map $K \times G \longrightarrow G$ sending (k, x) onto $k.x$ is C^∞ , and if $f \in C^\infty$ then so does $k \cdot f$ for any $k \in K$, because continuous homomorphisms between Lie groups automatically are C^∞ . Throughout the rest of the present section, we assume that μ satisfies the following conditions:

- i) μ is a K -invariant measure with compact support on G and
- ii) $\mu * \mu = \mu$.

The symbol $C_\mu^\infty(G) = \check{\mu} * C^\infty(G) * \Delta\check{\mu}$ will stand for all functions $f \in C^\infty(G)$ which are μ -invariant on G . The subspace of $C_\mu^\infty(G)$ consisting of the functions which are K -invariant will be denoted $C_{\mu,K}^\infty(G)$. For any $D \in C^\infty(G)$, we define the new operator $D_\mu^K f$ by

$$(D_\mu^K f)(x) = D\{M_K(L_{x^{-1}}f)_\mu\}(e)$$

for all $f \in C^\infty(G)$ and $x \in G$ [13]. In view of ([13] Proposition 4.1 and Proposition 4.2), D_μ^K has the following properties:

- Theorem 5.1.**
- i) D_μ^K is left invariant.
 - ii) $k \cdot D_\mu^K f = D_\mu^K k \cdot f$, for all $k \in K$ and $f \in C^\infty(G)$,
 - iii) $(D_\mu^K f)(e) = D(M_K f_\mu)(e)$. In particular if f is a bi- μ -invariant and K -invariant function on G , then we have $(D_\mu^K f)(e) = (Df)(e)$.
 - iv) g and $h \in C^\infty(G)$.
 - v) If (f, g, h) is a solution of (5.1), such that $g \neq 0$, $h \neq 0$ and satisfying $\int_G \check{h}(xt) d\check{\mu}(t) = \check{h}(x)$ and $\int_G g(xt) d\mu(t) = g(x)$, then $D_\mu^K g = (D\phi)(e)g$ and $D_\mu^K \check{g} = (D\check{\phi})(e)\check{g}$, where ϕ is a solution of the functional equation (3.1). Consequently g and h are analytic.
 - vi) If $D \in \mathbb{D}(G)$, then for all $f \in C_{\mu,K}^\infty(G)$ we have

$$D_\mu^K f = M_K(Df * \Delta\check{\mu}).$$

In particular, the restriction of D_μ^K to $C_{\mu,K}^\infty(G)$ is an endomorphism.

The next theorem extends the result obtained by the authors to Badora's functional equation ([13]).

Theorem 5.2. Let $\mu \in M(G)$ be a K -invariant, idempotent measure on G with compact support. If $(f, g) \in C(G) \setminus \{0\}$, then the following statements are equivalent:

- (1) (f, g) is a solution of

$$\int_G \int_K f(xtk \cdot y) \overline{\chi}(k) dk d\mu(t) = g(x)f(y), \quad x, y \in G, \quad (5.2)$$

- (2) a) $f(k \cdot y) = \chi(k)f(y)$,
 b) f and $g \in C^\infty(G)$,
 c) f and g are analytic,
 d) $\int_G \check{f}(xt)d\check{\mu}(t) = \check{f}(x)$ for all $x \in G$,
 e) $D_\mu^K \check{f} = (D\check{g})(e)\check{f}$ for all $D \in \mathbb{D}(G)$.

PROOF. (1) $\implies$ (2) follows directly from Theorem 5.1. Conversely, suppose that (a), (b), (c), (d) and (e) hold. For a fixed $x \in G$, we define the function

$$F(y) = \int_G \int_K \check{f}(xtk \cdot y)dkd\check{\mu}(t), \quad y \in G.$$

It is easy to verify that F is K -invariant. Furthermore, since $\mu * \mu = \mu$, μ is K -invariant and $\check{f}$ is right μ -invariant hence F is bi- μ -invariant. Now $F(y)$ can be written

$$F(y) = \int_G \int_K (L_{(k^{-1}.xt)}^{-1}\check{f})(y)\overline{\chi}(k)dkd\check{\mu}(t).$$

Consequently, for all $D \in \mathbb{D}(G)$ we have

$$(D_\mu^K F)(y) = D(\check{g})(e)F(y).$$

In particular, for $y = e$ we have

$$(D_\mu^K F)(e) = D(\check{g})(e)F(e).$$

Hence, by Theorem 5.1, it follows that

$$(DF)(e) = D(\check{g})(e)F(e)$$

i.e

$$D(F - F(e)\check{g})(e) = 0$$

for all $D \in \mathbb{D}(G)$. Since $F - F(e)\check{g}$ is an analytic function on the connected Lie group G , by [21] we obtain

$$F - F(e)\check{g} \equiv 0$$

on G . We conclude that

$$\int_G \int_K \check{f}(xk \cdot y) dk d\check{\mu}(t) = \check{f}(x)\check{g}(y), \quad x, y \in G.$$

Finally, by using (a) we obtain

$$\int_G \int_K f(xk \cdot y) \overline{\chi}(k) dk d\mu(t) = g(x)f(y), \quad x, y \in G.$$

This ends the proof of the theorem. $\square$

6. Hyers–Ulam stability of generalized equations of Stetkær type

In 1940 S. M. Ulam posed the following problem on the stability of homomorphisms:

Given a group G_1 , a metric group (G_2, d) , and a positive number ε , does there exist a $\lambda > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \varepsilon$$

for all $x, y \in G$, then a homomorphism $a : G_1 \longrightarrow G_2$ exists with

$$d(f(x), a(x)) < \lambda \quad \text{for all } x \in G?$$

See S. M. ULAM [35] or [36] for a discussion of such problems, as well as D. H. HYERS [22], D. H. HYERS and S. M. ULAM [24], TH. M. RASSIAS [26], D. H. HYERS, G. I. ISAC and T. M. RASSIAS [23]. Later, the above question became a source of stability theory in the Hyers–Ulam sense. The first affirmative answer to Ulam’s question was given by D. H. HYERS in [22], under the assumption that G_1 and G_2 are Banach spaces. The Hyers–Ulam–Rassias stability was taken up by a number of mathematicians and the study of this area has grown to be one of the central subjects in mathematical analysis. There is a strong stability phenomenon which is known as superstability. An equation is called superstable if for any approximate homomorphism, (i.e. $d(f(xy), f(x)f(y)) \leq \delta$), either f is bounded or f is a true homomorphism. This property was first observed when the following theorem was proved by J. BAKER, J. LAWRENCE, and F. ZORZITTO [8]:

Theorem. *Let V be a vector space. If a function $f : V \longrightarrow \mathbb{R}$ satisfies the inequality*

$$|f(x+y) - f(x)f(y)| \leq \varepsilon$$

for some $\varepsilon > 0$ and for all $x, y \in V$, then either f is a bounded function or $f(x+y) = f(x)f(y)$, for all $x, y \in V$.

Later this result was generalized by J. BAKER [7] and L. SZÉKELYHIDI [33], [34].

The aim of the present section is to investigate the stability of the following family of functional equations:

$$\int_K \int_G f(xtk \cdot y) dk d\mu(t) = f(x)g(y), \quad x, y \in G, \quad (6.1)$$

$$\int_K \int_G f(xtk \cdot y) \overline{\chi(k)} dk d\mu(t) = f(y)g(x), \quad x, y \in G, \quad (6.2)$$

where $\mu \in M(G)$ is a K -invariant measure with compact support on G .

Particular cases of (6.1) and (6.2) are

$$\int_K f(x+k \cdot y) dk = f(x)g(y), \quad x, y \in G \quad (6.3)$$

and

$$\int_K f(x+k \cdot y) \overline{\chi(k)} dk = f(y)g(x), \quad x, y \in G, \quad (6.4)$$

where G is a commutative group and $\mu = \delta_e$, the Dirac measure concentrated at the identity element of G . The stability properties of the equations (6.3), (6.4) have been obtained by BADORA [5]. For K -spherical functions (i.e. (6.3) with $f = g$) with K finite this problem was solved by W. FÖRG-ROB and J. SCHWAIGER in [19] and by R. BADORA in [6], and for $K = \{Id, -Id\}$, i.e. d'Alembert's functional equation, by J. BAKER [7].

For the noncommutative case, some results for some particular equations of type (6.1) were obtained by ELQORACHI and AKKOUCHI [14], [15], [17]. The stability of the classical examples

$$f(x+y) = f(x)f(y), \quad (6.5)$$

$$f(x+y) + f(x-y) = 2f(x)f(y) \quad (6.6)$$

of equations (6.1) and (6.2) has attracted the attention of many mathematicians. The interested reader should refer to [23] for a thorough account on the subject of stability of functional equations.

Throughout this section μ is assumed to be a compactly supported measure on G which is K -invariant, and f satisfies the Kannappan condition $K(\mu)$.

Theorem 6.1. *Let $f, g : G \rightarrow \mathbb{C}$ be continuous functions. Assume that there exists $\delta \geq 0$ such that*

$$\left| \int_K \int_G f(xtk \cdot y) dk d\mu(t) - f(x)g(y) \right| \leq \delta, \quad x, y \in G, \quad (6.7)$$

and f fulfills $K(\mu)$. Then either

- i) f, g are bounded or
- ii) f is unbounded and g satisfies Badura's equation

$$\int_K \int_G g(xtk \cdot y) dk d\mu(t) = g(x)g(y), \quad x, y \in G, \quad (6.8)$$

or

- iii) g is unbounded and f satisfies the equation (6.1) (if $f \neq 0$, then g satisfies (6.8)).

PROOF. The proof of the theorem is related to the one in [15], (see Theorem 3.1), where $K = \{Id, \sigma\}$ and σ is a continuous involution of G . If f is unbounded, then by using the inequality (6.7), we get

$$\begin{aligned} & |f(z)| \left| \int_K \int_G g(xtk \cdot y) dk d\mu(t) - g(x)g(y) \right| \\ & \leq \left| f(z) \int_K \int_G g(xtk \cdot y) dk d\mu(t) - g(y) \int_K \int_G f(ztk \cdot x) dk d\mu(t) \right| \\ & \quad + |g(y)| \left| \int_K \int_G f(ztk \cdot x) dk d\mu(t) - f(z)g(x) \right| \\ & \leq \left| f(z) \int_K \int_G g(xtk \cdot y) dk d\mu(t) - g(y) \int_K \int_G f(ztk \cdot x) dk d\mu(t) \right| + |g(y)|\delta, \end{aligned}$$

for all $x, y, z \in G$. Since

$$\begin{aligned}
& \left| \int_K \int_K \int_G \int_G f(ztk \cdot xsk' \cdot y) dk dk' d\mu(t) d\mu(s) \right. \\
& \quad \left. - \int_K \int_G f(ztk \cdot x) dk d\mu(t) g(y) \right| \\
& \leq \int_K \int_G \left| \int_K \int_G f(ztk \cdot xsk' \cdot y) dk' d\mu(s) - f(ztk \cdot x) g(y) \right| dk d|\mu|(t) \\
& \leq \delta \|\mu\| dk(K) = \delta \|\mu\|, \\
& \left| \int_K \int_K \int_G \int_G f(ztk \cdot (xsk' \cdot y)) dk dk' d\mu(t) d\mu(s) \right. \\
& \quad \left. - f(z) \int_K \int_G g(xsk' \cdot y) dk' d\mu(t) d\mu(s) \right| \\
& \leq \int_K \int_G \left| \int_K \int_G f(ztk \cdot (xsk' \cdot y)) dk d\mu(t) - f(z) g(xsk' \cdot y) \right| dk' d|\mu|(s) \\
& \leq \delta \|\mu\|,
\end{aligned}$$

and from the relation

$$\begin{aligned}
& \int_K \int_K \int_G \int_G f(ztk \cdot (xsk' \cdot y)) dk dk' d\mu(t) d\mu(s) \\
& = \int_K \int_{K^+} \int_G \int_G f(ztk \cdot xk \cdot s(kk') \cdot y) dk dk' d\mu(t) d\mu(s) \\
& \quad + \int_K \int_{K^-} \int_G \int_G f(zt(kk') \cdot yk \cdot sk \cdot x) dk dk' d\mu(t) d\mu(s) \\
& = \int_K \int_{K^+} \int_G \int_G f(ztk \cdot xs(kk') \cdot y) dk dk' d\mu(t) d\mu(s) \\
& \quad + \int_K \int_{K^-} \int_G \int_G f(ztk \cdot xs(kk') \cdot y) dk dk' d\mu(t) d\mu(s) \\
& = \int_K \int_K \int_G \int_G f(ztk \cdot xs(kk') \cdot y) dk dk' d\mu(t) d\mu(s) \\
& = \int_K \int_G \int_G f(ztk \cdot xsk' \cdot y) dk dk' d\mu(t) d\mu(s)
\end{aligned}$$

we obtain

$$\left| f(z) \int_K \int_G g(xtk \cdot y) dk d\mu(t) - g(y) \int_K \int_G f(ztk \cdot x) dk d\mu(t) \right| \leq 2\delta \|\mu\|,$$

and finally

$$|f(z)| \left| \int_K \int_G g(xtk \cdot y) dk d\mu(t) - g(x)g(y) \right| \leq 2\delta \|\mu\| + |g(y)|\delta.$$

Since f is unbounded, it follows that

$$\int_K \int_G g(xtk \cdot y) dk d\mu(t) = g(x)g(y), \quad \text{for all } x, y \in G,$$

which ends the proof in this case.

If g is unbounded, equation (6.1) holds if $f = 0$. Let us assume now that $f \neq 0$. Then there exists $z \in G$ such that $f(z) \neq 0$. From inequality (6.7), we obtain

$$\left| \frac{\int_K \int_G f(ztk \cdot x) dk d\mu(t)}{f(z)} - g(x) \right| \leq \frac{\delta}{|f(z)|}, \quad \text{for all } x \in G.$$

Since g is unbounded, the function defined by

$$h(x) = \frac{\int_K \int_G f(ztk \cdot x) dk d\mu(t)}{f(z)}$$

is also unbounded.

On the other hand h satisfies the following inequality:

$$\left| \int_K \int_G h(xtk \cdot y) dk d\mu(t) - h(x)g(y) \right| \leq \frac{\delta \|\mu\|}{|f(z)|}, \quad \text{for all } x, y \in G. \quad (6.9)$$

Now, by the preceding discussion, we conclude that g satisfies the equation (6.8). To see that f, g satisfy (6.1), let $x, y, z \in G$. Using inequality (6.7) and the fact that g satisfies the equation (6.8), we get

$$\begin{aligned} & |g(z)| \left| \int_K \int_G f(xtk \cdot y) dk d\mu(t) - f(x)g(y) \right| \\ & \leq \left| \int_K \int_K \int_G \int_G f(xtk \cdot ysk' \cdot z) dk dk' d\mu(t) d\mu(s) \right| \end{aligned}$$

$$\begin{aligned}
& -g(z) \int_K \int_G f(xtk \cdot y) dk d\mu(t) \Big| \\
& + \Big| \int_K \int_K \int_G \int_G f(xtk \cdot ysk' \cdot z) dk dk' d\mu(s) d\mu(t) \\
& - f(x) \int_K \int_G g(ysk' \cdot z) dk' d\mu(t) \Big| \leq 2\delta \|\mu\|.
\end{aligned}$$

Hence f, g satisfy the equation (6.1) and the proof of the theorem is complete. $\square$

As a consequence, we have the superstability of the equation (6.8).

Corollary 6.2 ([17] Theorem 2.1). *Let $f : G \longrightarrow \mathbb{C}$ be a continuous function. Assume that there exists $\delta \geq 0$ such that*

$$\left| \int_K \int_G f(xtk \cdot y) dk d\mu(t) - f(x)f(y) \right| \leq \delta, \quad x, y \in G. \quad (6.10)$$

Then either

$$|f(x)| \leq \frac{\|\mu\| + \sqrt{\|\mu\|^2 + 4\delta}}{2}, \quad x \in G, \quad (6.11)$$

or f is a solution of the equation (6.8).

Remark 6.3. In Theorem 6.1 it is not necessary to assume that f satisfies the condition $K(\mu)$ if K is a compact subgroup of homomorphisms of G .

In the following theorem we shall investigate the stability of the functional equation (6.2), under the additional condition that f satisfies the Kannappan type condition $K_1(\mu)$:

$$\begin{cases} \int_G \int_G f(ztxsy) d\mu(t) d\mu(s) = \int_G \int_G f(ztyxs) d\mu(t) d\mu(s), \\ \int_G f(xsy) d\mu(s) = \int_G f(ysx) d\mu(s), \quad \text{for all } x, y, z \in G. \end{cases}$$

Theorem 6.4. *Let $f, g : G \longrightarrow \mathbb{C}$ be continuous functions. Assume that there exists $\delta \geq 0$ such that*

$$\left| \int_K \int_G f(xtk \cdot y) \overline{\chi(k)} dk d\mu(t) - f(y)g(x) \right| \leq \delta, \quad x, y \in G, \quad (6.12)$$

and f fulfills $K_1(\mu)$. Then either

- i) f, g are bounded or
- ii) f is unbounded and g satisfies

$$\int_K \int_G \check{g}(x t k \cdot y) d k d \check{\mu}(t) = \check{g}(x) \check{g}(y), \quad x, y \in G, \quad (6.13)$$

or

- iii) g is unbounded and f, g satisfy the equation (6.2).

PROOF. In the proof, we use ideas and methods that are analogous to those used in [5].

In order to apply Theorem 6.1, we recall the following formula proved for $\mu = \delta_e$ by BADORA (see [5]).

$$\begin{aligned} & \chi(k) \int_K \int_G f(x s k' \cdot y) \overline{\chi(k')} d k' d \mu(s) - \chi(k) f(y) g(x) \\ & - \int_K \int_G f(x s k' \cdot (k \cdot y)) \overline{\chi(k')} d k' d \mu(s) + g(x) f(k \cdot y) \\ & = g(x) (f(k \cdot y) - \chi(k) f(y)), \end{aligned} \quad (6.14)$$

for all $x, y \in G$.

On the other hand, by using the condition $K_1(\mu)$, the K -invariance of μ and some computations used in [5], we prove that

$$\begin{aligned} & g(z) \left[\int_K \int_G f(y t k \cdot x) \overline{\chi(k)} d k d \mu(t) - g(y) f(x) \right] \\ & + \int_K \left[\int_K \int_G \int_G f(z s k' \cdot (y t k \cdot x)) \overline{\chi(k')} d k' d \mu(t) d \mu(s) \right. \\ & \left. - f(z) \int_G g(y t k \cdot x) d \mu(t) \right] \overline{\chi(k)} d k \\ & - \int_K \left[\int_K \int_G \int_G f(z s k' \cdot (k^{-1} \cdot y t x)) \overline{\chi(k')} d k' d \mu(t) d \mu(s) \right. \\ & \left. - f(z) \int_G g(k^{-1} \cdot y t x) d \mu(t) \right] d k \\ & = g(z) \left[\int_K \int_G f(k^{-1} \cdot y t x) d k d \mu(t) - g(y) f(x) \right]. \end{aligned} \quad (6.15)$$

Now we are ready to prove the theorem. If f is unbounded, $f = 0$ satisfies the equation (6.13). If $g \neq 0$, then in view of (6.15), there exists some constant $\delta' \geq 0$ such that

$$\left| \int_K \int_G f(k^{-1} \cdot ytx) dk d\mu(t) - g(y)f(x) \right| \leq \delta', \quad x, y \in G, \quad (6.16)$$

which can be written

$$\left| \int_K \int_G \check{f}(xtk \cdot y) dk d\check{\mu}(t) - \check{f}(x)\check{g}(y) \right| \leq \delta', \quad x, y \in G. \quad (6.17)$$

It follows from Theorem 6.1, that g satisfies the equation (6.13). If g is unbounded, then by (6.14)

$$f(k \cdot x) = \chi(k)f(x), \quad \text{for all } x \in G. \quad (6.18)$$

By using the equation (6.15), we obtain that f, g satisfy some inequality like (6.17) and hence by Theorem 6.1 we deduce that f, g are solutions of the equation

$$\int_K \int_G \check{f}(xtk \cdot y) dk d\check{\mu}(t) = \check{f}(x)\check{g}(y), \quad x, y \in G. \quad (6.19)$$

Now from Theorem 4.1 of the section 4, we deduce that f, g are solutions of (6.2) and the proof is completes. $\square$

ACKNOWLEDGMENT. The authors thank very much the referees for their helpful comments.

References

- [1] J. ACZÉL and J. DHOMBRES, Functional equations in several variables, *Cambridge University Press Cambridge, New York – New Rochelle – Melbourne – Sydney*, 1989.
- [2] M. AKKOUCI et A. BAKALI, une généralisation des paires de Gelfand, *Boll. Un. Mat. Ital.* (7) **6-B** (1992), 795–822.
- [3] M. AKKOUCI, A. BAKALI and I. KHALIL, A class of functional equations on a locally compact group, *J. London Math. Soc.* (2) **57** (1998), 694–705.
- [4] R. BADORA, On a joint generalization of Cauchy's and d'Alembert's functional equations, *Aequationes Math.* **43** (1992), 92–89.

- [5] R. BADORA, On Hyers–Ulam stability of Wilson’s functional equation, *Aequationes Math.* **60**, no. 3 (2000), 211–218.
- [6] R. BADORA, On the stability of some functional equation for generalized trigonometric functions, (*preprint*).
- [7] J. A. BAKER, The stability of the cosine equation, *Proc. Amer. Math. Soc.* **80** (1980), 411–416.
- [8] J. A. BAKER, J. LAWRENCE and F. ZORZITO, The stability of the equation $f(x+y) = f(x)f(y)$, *Proc. Amer. Math. Soc.* **74** (1979), 242–246.
- [9] W. CHOJNACKI, On some functional equations generalizing Cauchy’s and d’Alembert’s functional equations, *Colloq. Math.* **55** (1988), 169–178.
- [10] W. CHOJNACKI, Functions cosinus hilbertiennes bornées dans les groupes commutatifs localement compacts, *Compositio Math.* **57** (1986), 15–60.
- [11] J. DIEUDONNÉ, *Eléments d’Analyse Tome, Vol. 6*, Gautier-Villars, 1968.
- [12] E. ELQORACHI and M. AKKOUCHI, On generalized d’Alembert and Wilson functional equations, *Aequationes Math.* **66** (2003), 241–256.
- [13] E. ELQORACHI, M. AKKOUCHI, A. BAKALI, and B. BOUIKHALENE, Badora’s equation on non-abelian locally compact groups, *Georgian Math. J.* **11**, no. 3 (2004), 449–466.
- [14] E. ELQORACHI and M. AKKOUCHI, The superstability of the generalized d’Alembert Functional Equation, *Georgian Math. J.*, **10**, no. 3 (2003), 503–509.
- [15] E. ELQORACHI and M. AKKOUCHI, On Hyers–Ulam stability of Cauchy and Wilson equations, *Georgian Math. J.* **11**, no. 1 (2004), 69–82.
- [16] E. ELQORACHI and M. AKKOUCHI, Functional equations and Gelfand measures, *Publicationes Math. Debrecen* **63** (2003), 643–666.
- [17] E. ELQORACHI, The stability of the generalized form for the Cauchy and d’Alembert functional equations, (*submitted*).
- [18] W. FÖRG-ROB and J. SCHWAIGER, A generalization of the cosine equation to n summands, *Grazer Math. Ber.* **316** (1992), 219–226.
- [19] W. FÖRG-ROB and J. SCHWAIGER, The stability of some functional equations for generalized hyperbolic functions and the generalized cosine equation, *Results in Math.* **26** (1994), 274–280.
- [20] S. HELGASON, *Differential Geometry and symmetric spaces*, Academic Press, New York and London, 1962.
- [21] S. HELGASON, *Groups and Geometric Analysis*, Academic Press, New York – London, 1984.
- [22] D. H. HYERS, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. USA* **27**, no. 2 (1941), 222–224.
- [23] D. H. HYERS, G. I. ISAC and T. M. RASSIAS, *Stability of functional equations in several variables*, Birkhäuser, Basel, 1998.
- [24] D. H. HYERS and S. M. ULAM, On approximate isometry, *Bull. Amer. Math. Soc.* **51** (1945), 228–292.
- [25] PL. KANNAPPAN, The functional equation $f(xy) + f(xy^{-1}) = 2f(x)f(y)$, for groups, *Proc. Amer. Math. Soc.* **19** (1968), 69–74.

- [26] TH. M. RASSIAS, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72** (1978), 297–300.
- [27] H. SHIN'YA, Spherical matrix functions and Banach representability for locally compact motion groups, *Japan. J. Math.* **28** (2002), 163–201.
- [28] H. STETKÆR, D'Alembert's equation and spherical functions, *Aequationes Math.* **48** (1994), 220–227.
- [29] H. STETKÆR, Harmonic Analysis and Functional Equations, Harmonic Analysis and Hypergroups, (Trends in Mathematics), *Birkhäuser, Boston*, 1995, 131–143.
- [30] H. STETKÆR, Wilson's functional equation on groups, *Aequationes Math.* **49** (1995), 252–275.
- [31] H. STETKÆR, Functional equations and matrix-valued spherical functions (*to appear*).
- [32] H. STETKÆR, Functional equations and Spherical functions, *Aarhus Univ. Preprint Series* **18** (1994).
- [33] L. SZÉKELYHIDI, On a Theorem of Baker, Lawrence and Zorzitto, *Proc. Amer. Math. Soc.* **84** (1982), 95–96.
- [34] L. SZÉKELYHIDI, On a stability theorem, *C. R. Math. Rep. Acad. Sci. Canada* **3** **1981**, 253–255.
- [35] S. M. ULAM, A collection of mathematical problems, *Interscience Publ. New York*, 1960.
- [36] S. M. ULAM, Sets, numbers and universes, *Mass. Inst. of Tech Press, Cambridge, MA*, 1974.
- [37] W. H. WILSON, On certain related functional equations, *Bull. Amer. Math. Soc.* **26** (1919), 300–312.
- [38] W. H. WILSON, Two general functional equations., *Bull. Amer. Math. Soc.* **31** (1925), 330–334.

BOUIKHALENE BELAID
 DEPARTMENT OF MATHEMATICS
 UNIVERSITY OF IBN TOFAIL
 FACULTY OF SCIENCES
 BP 133, KENITRA 14000
 MOROCCO

E-mail: bbouikhalene@yahoo.fr

ELQORACHI ELHOUCIEN
 DEPARTMENT OF MATHEMATICS
 UNIVERSITY OF IBN ZOHR
 FACULTY OF SCIENCES
 AGADIR
 MOROCCO

E-mail: elqorachi@hotmail.com

(Received June 14, 2004; revised February 10, 2005)