

An estimate for the length of an arithmetic progression the product of whose terms is almost square

By SHANTA LAISHRAM (Mumbai)

Abstract. Erdős conjectured that

$$n(n+d)\dots(n+(k-1)d) = y^2 \tag{1}$$

in positive integers n , $k \geq 3$, $d > 1$, y with $\gcd(n, d) = 1$, implies that k is bounded by an absolute constant. SHOREY and TIJDEMAN [16] showed that (1) implies that k is bounded by an effectively computable number depending only on $\omega(d)$, the number of distinct prime divisors of d . In this paper, an explicit bound for k in terms of $\omega(d)$ is presented.

1. Introduction

For an integer $x > 1$, we denote by $P(x)$ and $\omega(x)$ the greatest prime factor of x and the number of distinct prime divisors of x , respectively. Further we put $P(1) = 1$ and $\omega(1) = 0$. Let n, d, k, b, y be positive integers such that b is square free, $d \geq 1$, $k \geq 3$, $P(b) \leq k$ and $\gcd(n, d) = 1$. We consider the equation

$$n(n+d)\dots(n+(k-1)d) = by^2 \quad \text{in } n, d, k, b, y \quad \text{with } P(b) \leq k. \tag{2}$$

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For a survey of results on (2), see [16], [4], [14] and [15]. Equation (2) with $d = 1$ has been solved completely in [3] with $P(b) < k$ and in [11] with $P(b) = k$. Therefore we assume from now onwards that $d > 1$. MARSZALEK [7] proved that (2) implies k is bounded by an effectively computable number k_0 depending only on d . In fact the above assertion holds with k_0 depending only on $\omega(d)$. This is due to SHOREY and TIJDEMAN [16], who proved that $2^{\omega(d)} > c \frac{k}{\log k}$ where c is an effectively computable absolute constant. However the bound k_0 is very large. Further (2) with $\omega(d) = 1$ and $k \notin \{3, 5\}$ has been solved completely in [12] and [8]. Therefore we shall always assume that $\omega(d) \geq 2$. In this paper, we give an explicit bound for k in terms of $\omega(d)$ whenever (2) holds.

For $2 \leq \omega(d) \leq 11$, we define $\kappa_0 = \kappa_0(\omega(d))$ as in the table below.

$\omega(d)$	$\kappa_0(d \text{ even})$	$\kappa_0(d \text{ odd})$	$\omega(d)$	$\kappa_0(d \text{ even})$	$\kappa_0(d \text{ odd})$
2	500	800	7	2.643×10^5	1.376×10^6
3	700	3400	8	1.172×10^6	6.061×10^6
4	2900	15300	9	5.151×10^6	2.649×10^7
5	13100	69000	10	2.247×10^7	1.149×10^8
6	59000	3.096×10^5	11	9.73×10^7	4.95×10^8

For $\omega(d) \geq 12$, we define $\kappa_0 = \kappa_0(\omega(d))$ as

$$\kappa_0(\omega(d)) = \begin{cases} 2.25\omega(d)4^{\omega(d)} & \text{if } d \text{ is even,} \\ 11\omega(d)4^{\omega(d)} & \text{if } d \text{ is odd.} \end{cases}$$

We prove

Theorem 1. *Equation (2) implies that*

$$k < \kappa_0. \quad (3)$$

Theorem 1 is a direct consequence of the following two propositions.

Proposition 2. *Let $k \geq \kappa_0$. Then (2) implies that*

$$d < 4c_1(k-1)^2, \quad (4)$$

$$n < c_1(k-1)^3 \quad (5)$$

and hence

$$n + (k - 1)d < 5c_1(k - 1)^3 \quad (6)$$

where

$$c_1 = \begin{cases} \frac{1}{16} & \text{if } d \text{ is odd,} \\ \frac{1}{8} & \text{if } \text{ord}_2(d) = 1, \\ \frac{1}{4} & \text{if } \text{ord}_2(d) \geq 2. \end{cases}$$

Proposition 3. *Let $k \geq \kappa_0$. Then (2) implies that*

$$n + (k - 1)d \geq 2^\delta \frac{5}{16} k^3 \quad (7)$$

where

$$\delta = \min\{\text{ord}_2(d), 3\}.$$

2. Notation and preliminaries

From (2), we have

$$n + \text{id} = A_i X_i^2 \quad (8)$$

for $0 \leq i < k$ with $P(A_i) \leq k$ and $(X_i, \prod_{p \leq k} p) = 1$. Also we have

$$n + \text{id} = a_i x_i^2 \quad (9)$$

for $0 \leq i < k$ with a_i squarefree. Since $\gcd(n, d) = 1$, we see that

$$(A_i, d) = (a_i, d) = (X_i, d) = (x_i, d) = 1 \quad \text{for } 0 \leq i < k. \quad (10)$$

Let

$$T = \{i \mid 0 \leq i < k, X_i = 1\}, \quad T_1 = \{i \mid 0 \leq i < k, X_i \neq 1\}.$$

Note that $X_i > k$ for $i \in T_1$. For $0 \leq i < k$, let

$$\nu(A_i) = |\{j \in T_1, A_j = A_i\}|. \quad (11)$$

We always suppose that there exist $i_0 > i_1 > \cdots > i_{\nu(A_i)-1}$ such that $A_{i_0} = A_{i_1} = \cdots = A_{i_{\nu(A_i)-1}}$. Similarly we define

$$R = \{a_i \mid 0 \leq i < k\}$$

and

$$\nu(a_i) = |\{j \mid 0 \leq j < k, a_i = a_j\}|. \quad (12)$$

Define

$$\rho := \rho(d) = \begin{cases} 1 & \text{if } 3 \nmid d, \\ 3 & \text{if } 3 \mid d. \end{cases} \quad (13)$$

The letter p always denotes a prime number and p_i the i -th prime number. Let $P_1 < P_2 < \cdots$ be odd prime divisors of d . Let $r := r(d) \geq 0$ be the unique integer such that

$$P_1 P_2 \cdots P_r < (4c_1)^{\frac{1}{3}}(k-1)^{\frac{2}{3}} \text{ but } P_1 P_2 \cdots P_{r+1} \geq (4c_1)^{\frac{1}{3}}(k-1)^{\frac{2}{3}}. \quad (14)$$

If $r = 0$, we understand that the product $P_1 \cdots P_r = 1$.

Let $d' \mid d$ and $d'' = \frac{d}{d'}$ be such that $\gcd(d', d'') = 1$. We write

$$d'' = d_1 d_2, \quad \gcd(d_1, d_2) = \begin{cases} 1 & \text{if } \text{ord}_2(d'') \leq 1, \\ 2 & \text{if } \text{ord}_2(d'') \geq 2 \end{cases}$$

and we always suppose that d_1 is odd if $\text{ord}_2(d'') = 1$. We call such pairs (d_1, d_2) as partitions of d'' .

We observe that the number of partitions of d'' is $2^{\omega(d'')-\theta_1}$ where

$$\theta_1 := \theta_1(d'') = \begin{cases} 1 & \text{if } \text{ord}_2(d'') = 1, 2, \\ 0 & \text{otherwise} \end{cases}$$

and we write θ for $\theta_1(d)$. In particular, by taking $d' = 1$ and $d'' = d$, the number of partitions of d is $2^{\omega(d)-\theta}$.

Suppose that $A_i = A_j$, $i > j$. Then from (8) and (10), we have

$$(i-j)d = A_i(X_i^2 - X_j^2) = A_i(X_i - X_j)(X_i + X_j) \quad (15)$$

such that $\gcd(d, X_i - X_j, X_i + X_j) = 1$ if d is odd and 2 if d is even. Hence for any divisor d'' of d , we have a partition (d_1, d_2) of d'' corresponding to $A_i = A_j$ such that $d_1 \mid (X_i - X_j)$ and $d_2 \mid (X_i + X_j)$ and it is the unique partition of d'' corresponding to the pair (i, j) . Similarly, we have unique partition of d'' corresponding to every pair (i, j) whenever $a_i = a_j$.

As in SHOREY and TIJDEMAN [16], the proof depends on comparing an upper bound and a lower bound for $n + (k - 1)d$. The upper bound of $n + (k - 1)d$ given by Proposition 2 is a consequence of Lemmas 5, 8, 11, 12, 13 which are refinements of results in [16], [1] and [12]. It is proved by counting the number of distinct a_i 's and looking at the number of partitions of d . The proof of Proposition 3 is by counting the number of X_i 's greater than k and calculating the maximal value of A_i . Proposition 3 is a consequence of Lemmas 4, 6, 7, 9, 10, 14. The new features of the paper are the refinement of the upper bound of the multiplicities of A_i with respect to partitions of d , counting the number of A_i 's with multiplicity greater than 1 and the use of r to improve the lower bounds of the maximum of A_i 's.

We shall follow the notation of this section throughout the paper. We use MATHEMATICA for the computations in the paper. This is a part of my Master's thesis [6].

3. Lemmas

We begin with some estimates from Prime number theory.

Lemma 1. *We have*

- (i) $\pi(\nu) \leq \frac{\nu}{\log \nu} \left(1 + \frac{1.5}{\log \nu}\right)$ for $\nu > 1$,
- (ii) $\pi(\nu) \geq \frac{\nu}{\log \nu} \left(1 + \frac{0.5}{\log \nu}\right)$ for $\nu \geq 59$,
- (iii) $p_i \geq i \log i$ for $i \geq 2$,
- (iv) $\sum_{p \leq \nu} \log p < 1.000081\nu$ for $\nu > 0$,
- (v) $\text{ord}_p(k!) \geq \frac{k-p}{p-1} - \frac{\log(k-1)}{\log p}$ for $p < k$.

PROOF. The estimates (i), (ii) and (iii) are due to ROSSER and SCHOENFELD [10]. For estimate (iv), see [13, p. 360] and [2, Prop. 1.7]. For a proof of (v), see [5, Lemma 2(i)]. $\square$

The next result is Stirling's formula, see [9].

Lemma 2. *For a positive integer ν , we have*

$$\sqrt{2\pi\nu} e^{-\nu} \nu^\nu e^{\frac{1}{2\nu+1}} < \nu! < \sqrt{2\pi\nu} e^{-\nu} \nu^\nu e^{\frac{1}{12\nu}}.$$

Lemma 3. *Let $\pi_d(k) \leq \pi(k) - 1$. Then*

$$|T_1| > k - \frac{(k-1) \log(k-1)}{\log(n + (k-1)d) - \log 2} - \pi(k). \quad (16)$$

PROOF. We use [12, Lemma 3] with $t = k$, $-\log \prod_{p|d} p^{-\text{ord}_p((k-1)!)} \geq 0$ and $\pi_d(k) \leq \pi(k) - \omega(d) + 2$. Let $n \geq (k-1)d$. Then $\log n \geq \log(n + (k-1)d) - \log 2$. This with [12, (4.2)] and Lemma 1 (i) gives (16). For $n < (k-1)d$, we have $\log(k-1) + \log d > \log(n + (k-1)d) - \log 2$. This with [12, (4.1)] and Lemma 1 (i) gives (16). $\square$

Lemma 4. *Let $d = d'd''$ with $\gcd(d', d'') = 1$. Let $i_0 \in T_1$ be such that $A_{i_0} \geq d'$. Then*

$$\nu(A_{i_0}) \leq 2^{\omega(d'') - \theta_1(d'')}. \quad (17)$$

PROOF. For simplicity, we write $\theta_1 = \theta_1(d'')$. Assume that $\nu(A_{i_0}) > 2^{\omega(d'') - \theta_1}$. Then there exists a sequence of indices $i_0 > i_1 > \dots > i_{2^{\omega(d'') - \theta_1}}$ such that $A_{i_0} = A_{i_1} = \dots = A_{i_{2^{\omega(d'') - \theta_1}}}$. For each pair (i_0, i_r) , $r = 1, 2, \dots, 2^{\omega(d'') - \theta_1}$, we have a unique partition corresponding to the pair. But there are at most $2^{\omega(d'') - \theta_1}$ partitions of d'' . Since $(i_0 - i_r)d = A_{i_0}(X_{i_0} - X_{i_r})(X_{i_0} + X_{i_r})$ and $A_{i_0} \geq d'$, we have

$$\begin{aligned} k > i_0 - i_r &= \frac{A_{i_0}}{d'} \left(\frac{X_{i_0} - X_{i_r}}{d_1} \right) \left(\frac{X_{i_0} + X_{i_r}}{d_2} \right) \\ &\geq \left(\frac{X_{i_0} - X_{i_r}}{d_1} \right) \left(\frac{X_{i_0} + X_{i_r}}{d_2} \right), \end{aligned}$$

where (d_1, d_2) is the partition of d'' corresponding to pair (i_0, i_r) . This shows that we cannot have the partition $(\frac{d''}{2^{\theta_1}}, 2^{\theta_1})$ corresponding to any

pair. Hence there can be at most $2^{\omega(d'')-\theta_1}-1$ partitions of d'' with respect to $2^{\omega(d'')-\theta_1}$ pairs of (i_0, i_r) , $r = 1, \dots, 2^{\omega(d'')-\theta_1}$. Hence by Box Principle, there exist pairs (i_0, i_r) , (i_0, i_s) with $1 \leq r < s \leq 2^{\omega(d'')-\theta_1}$ and a partition (d_1, d_2) of d'' corresponding to these pairs. Thus

$$d_1 \mid (X_{i_0} - X_{i_r}), d_2 \mid (X_{i_0} + X_{i_r}) \text{ and } d_1 \mid (X_{i_0} - X_{i_s}), d_2 \mid (X_{i_0} + X_{i_s})$$

so that $\text{lcm}(d_1, d_2) \mid (X_{i_r} - X_{i_s})$. Since $A_{i_r} = A_{i_s} = A_{i_0}$ and $\gcd(d_1, d_2) \leq 2$, we have

$$k > (i_r - i_s) \frac{d'}{A_{i_0}} = \frac{(X_{i_r} - X_{i_s})(X_{i_r} + X_{i_s})}{\text{lcm}(d_1, d_2) \gcd(d_1, d_2)} > \frac{(X_{i_r} + X_{i_s})}{2} > \frac{2k}{2} = k,$$

a contradiction. $\square$

By taking $d' = 1$ and $d'' = d$, the following result is immediate from Lemma 4 since $\theta_1(d) = \theta$.

Corollary 1. *For $i_0 \in T_1$, we have $\nu(A_{i_0}) \leq 2^{\omega(d)-\theta}$.*

Lemma 5. *Let $k \geq 17$. Suppose $n \geq c_1(k-1)^3$ or $d \geq 4c_1(k-1)^2$. Then for $0 \leq i_0 < k$, we have*

$$\nu(a_{i_0}) \leq 2^{\omega(d)-\theta}. \quad (18)$$

PROOF. Suppose that $\nu(a_{i_0}) > 2^{\omega(d)-\theta}$. We note that both $x_i + x_j$ and $x_i - x_j$ are even when d is even. Continuing as in the proof of (17) with $d'' = d$, we see that there exists i, j with $i > j$ and

$$k > \frac{a_{i_0}(x_i + x_j)}{2}$$

where $\frac{d}{2} \mid (x_i - x_0)$ if d is even and $d \mid (x_i - x_0)$ if d is odd. We have $x_i \geq x_j + \frac{d}{2}$ so that $k > \frac{1}{2}a_{i_0}(x_i + x_j) \geq (a_j x_j^2)^{\frac{1}{2}} + \frac{d}{4} \geq n^{\frac{1}{2}} + \frac{d}{4}$ and hence

$$k > \begin{cases} 1 + c_1(k-1)^2 & \text{if } d \geq 4c_1(k-1)^2, \\ (c_1)^{\frac{1}{2}}(k-1)^{\frac{3}{2}} + 1 & \text{if } n \geq c_1(k-1)^3 \end{cases}$$

which is not true for $k \geq 17$. $\square$

Lemma 6. *Equation (2) implies that either*

$$d \geq 4c_1(k-1)^2$$

or

$$r \geq \left\lceil \frac{\omega(d)}{3} \right\rceil.$$

PROOF. If $r+1 \leq \lceil \frac{\omega(d)}{3} \rceil$, then $\omega(d) \geq 3(r+1)$ giving $d \geq 4c_1(k-1)^2$ by (14). $\square$

Lemma 7. *Let $S \subseteq \{A_i \mid 0 \leq i < k\}$ and $\min_{A_h \in S} A_h \geq U$. Let $t \geq 1$. Assume that*

$$|S| > Q_t \left(\frac{P_1-1}{2} \right) \cdots \left(\frac{P_t-1}{2} \right) \quad (19)$$

where $Q_t \geq 1$ is an integer. Then

$$\max_{A_h \in S} A_h \geq 2^\delta Q_t P_1 \cdots P_t + U. \quad (20)$$

PROOF. For an odd $p \mid d$, we have

$$\left(\frac{A_h}{p} \right) = \left(\frac{A_h X_h^2}{p} \right) = \left(\frac{n}{p} \right)$$

where $(\cdot)$ is Legendre symbol, so that A_h belongs to at most $\frac{p-1}{2}$ distinct residue classes modulo p for each $0 \leq h < k$. If d is even, then A_h also belongs to a unique residue class modulo 2^δ for each $0 \leq h < k$. Hence by Chinese remainder theorem, A_h belongs to at most $\left(\frac{P_1-1}{2} \right) \cdots \left(\frac{P_j-1}{2} \right)$ distinct residue classes modulo $2^\delta P_1 \cdots P_j$ for each j , $1 \leq j \leq t$. Assume that (20) does not hold. Then

$$\max_{A_h \in S} A_h - (U-1) \leq 2^\delta Q_t P_1 \cdots P_t.$$

Therefore

$$|S| \leq \frac{2^\delta Q_t P_1 \cdots P_t}{2^\delta P_1 \cdots P_t} \left(\frac{P_1-1}{2} \right) \cdots \left(\frac{P_t-1}{2} \right),$$

contradicting (19). $\square$

Corollary 2. *Let S and U be as in Lemma 7. Let $|S| \geq s > \left(\frac{P_1-1}{2}\right) \dots \left(\frac{P_{t-1}-1}{2}\right)$, then*

$$\max_{A_h \in S} A_h \geq \frac{3}{4} 2^{t+\delta} s + U. \quad (21)$$

PROOF. Let $(f-1)\left(\frac{P_1-1}{2}\right) \dots \left(\frac{P_{t-1}-1}{2}\right) < s - Q_t\left(\frac{P_1-1}{2}\right) \dots \left(\frac{P_{t-1}-1}{2}\right) \leq f\left(\frac{P_1-1}{2}\right) \dots \left(\frac{P_{t-1}-1}{2}\right)$ where $Q_t \geq 1$ and $1 \leq f \leq \frac{P_t-1}{2}$ is an integer. To see this, write $s = Q\left(\frac{P_1-1}{2}\right) \dots \left(\frac{P_{t-1}-1}{2}\right) + Q'\left(\frac{P_1-1}{2}\right) \dots \left(\frac{P_{t-1}-1}{2}\right) + R$ where $0 \leq Q' < \frac{P_t-1}{2}$ and $0 \leq R < \left(\frac{P_1-1}{2}\right) \dots \left(\frac{P_{t-1}-1}{2}\right)$. If $R > 0$, then take $Q_t = Q$, $f-1 = Q'$; if $R = 0$ and $Q' > 0$, then take $Q_t = Q$, $f = Q'$; and if $R = Q' = 0$, then take $Q_t = Q-1$ and $f = \frac{P_t-1}{2}$. We arrange the elements of S in increasing order and let $S' \subseteq S$ be the first $(f-1)\left(\frac{P_1-1}{2}\right) \dots \left(\frac{P_{t-1}-1}{2}\right) + 1$ elements and S'' consist of the remaining set. Then we see from Lemma 7 with $t = t-1$ and $Q_t = f-1$ that

$$\max_{A_h \in S'} A_h \geq 2^\delta (f-1) P_1 P_2 \dots P_{t-1} + U = U'.$$

Now we apply Lemma 7 with $U = U'$ in S'' to derive

$$\max_{A_h \in S} A_h \geq 2^\delta Q_t P_1 P_2 \dots P_t + 2^\delta (f-1) P_1 P_2 \dots P_{t-1} + U.$$

Hence to derive (21), it is enough to prove

$$\begin{aligned} & Q_t P_1 \dots P_t + (f-1) P_1 \dots P_{t-1} \\ & \geq \frac{3}{4} \{Q_t (P_1 - 1) \dots (P_t - 1) + 2f (P_1 - 1) \dots (P_{t-1} - 1)\}. \end{aligned}$$

By observing that

$$\begin{aligned} Q_t (P_1 - 1) \dots (P_t - 1) & \leq Q_t P_1 \dots P_t - Q_t P_1 \dots P_{t-1}, \\ 2f (P_1 - 1) \dots (P_{t-1} - 1) & \leq 2f P_1 \dots P_{t-1} - 2f P_1 \dots P_{t-2}, \end{aligned}$$

it suffices to show that

$$Q_t + \frac{3(Q_t - 1) - (2f + 1)}{P_t} + \frac{6f}{P_t P_{t-1}} \geq 0$$

which is true since $Q_t \geq 1$ and $1 \leq f \leq \frac{P_t-1}{2}$. $\square$

Lemma 8. *Let s_i denote the i -th squarefree positive integer. Then*

$$s_i \geq 1.6i \quad \text{for } i \geq 78 \quad (22)$$

and

$$\prod_{i=1}^l s_i \geq (1.6)^l l! \quad \text{for } l \geq 286. \quad (23)$$

Further let t_i be i -th odd squarefree positive integer. Then

$$t_i \geq 2.4i \quad \text{for } i \geq 51 \quad (24)$$

and

$$\prod_{i=1}^l t_i \geq (2.4)^l l! \quad \text{for } l \geq 200. \quad (25)$$

PROOF. The proof is similar to that of [12, (6.9)]. For (22) and (24), we check that $s_i \geq 1.6i$ for $78 \leq i \leq 286$ and $t_i \geq 2.4i$ for $51 \leq i \leq 132$, respectively. Further we observe that in a given set of 144 consecutive integers, there are at most 90 squarefree integers and at most 60 odd squarefree integers by deleting multiples of 4, 9, 25, 49, 121 and 2, 9, 25, 49, respectively. Then we continue as in the proof of [12, (6.9)] to get (22) and (24). Further we check that (23) holds at $l = 286$ and (25) holds at $l = 200$. Then we use (22) and (24) to obtain (23) and (25), respectively. $\square$

Lemma 9. *Let $X > 1$ be a positive integer. Then*

$$\sum_{i=1}^{X-1} 2^{\omega(i)} \leq \eta(X) X \log X \quad (26)$$

where

$$\eta := \eta(X) = \begin{cases} 1 & \text{if } X = 1, \\ \frac{\sum_{i=1}^{X-1} 2^{\omega(i)}}{X \log X} & \text{if } 1 < X < 248, \\ 0.75 & \text{if } X \geq 248. \end{cases} \quad (27)$$

PROOF. We check that (26) holds for $1 < X < 11500$. Thus we may assume $X \geq 11500$. Let s_j be the largest squarefree integer $\leq X$. Then by Lemma 8, we have $1.6j \leq s_j \leq X$ so that $j \leq \left[\frac{X}{1.6}\right]$. We have $2^{\omega(i)} = \sum_{e|i} |\mu(e)|$. Therefore

$$\begin{aligned} \sum_{i=1}^{X-1} 2^{\omega(i)} &= \sum_{i=1}^{X-1} \sum_{e|i} |\mu(e)| \\ &\leq \sum_{1 \leq e < X} \left\lfloor \frac{X-1}{e} \right\rfloor |\mu(e)| \leq (X-1) \sum_{1 \leq e < X} \frac{|\mu(e)|}{e} \leq X \sum_{i=1}^{\left[\frac{X}{1.6}\right]} \frac{1}{s_i}. \end{aligned}$$

We check that there are 6990 squarefree integers upto 11500. By using (22), we have

$$\begin{aligned} \sum_{i=1}^{X-1} 2^{\omega(i)} &\leq X \left\{ \sum_{i=1}^{6990} \frac{1}{s_i} - \frac{1}{1.6} \sum_{i=1}^{6990} \frac{1}{i} + \frac{1}{1.6} \sum_{i=1}^{\left[\frac{X}{1.6}\right]} \frac{1}{i} \right\} \\ &\leq X \left\{ \sum_{i=1}^{6990} \frac{1}{s_i} - \frac{1}{1.6} \sum_{i=1}^{6990} \frac{1}{i} + \frac{1}{1.6} \left(1 + \log \frac{X}{1.6} \right) \right\} \\ &\leq \frac{3}{4} X \log X \left\{ \frac{4}{3} \frac{1.1658}{\log X} + \frac{4}{3} \frac{1}{1.6} \right\}, \end{aligned}$$

implying (26). □

Lemma 10. Let $c > 0$ be such that $c2^{\omega(d)-3} > 1$, $\mu \geq 2$ and

$$\mathfrak{C}_\mu = \left\{ A_i \mid \nu(A_i) = \mu, A_i > \frac{\rho 2^\delta k}{3c2^{\omega(d)}} \right\}.$$

Then

$$\begin{aligned} \mathfrak{C} &:= \sum_{\mu \geq 2} \frac{\mu(\mu-1)}{2} |\mathfrak{C}_\mu| \\ &\leq \frac{c}{8} \eta(c2^{\omega(d)-3}) 2^{\omega(d)} (2^{\omega(d)-\theta} - 1) (\log c2^{\omega(d)-3}). \end{aligned} \tag{28}$$

PROOF. Let $i_1 > i_2 > \dots > i_\mu$ be such that $A_{i_1} = A_{i_2} = \dots = A_{i_\mu}$. These give rise to $\frac{\mu(\mu-1)}{2}$ pairs of (i, j) , $i > j$ with $A_i = A_j$. Therefore the total number of pairs (i, j) with $i > j$ and $A_i = A_j$ is $\mathfrak{C}$.

We know that there is a unique partition of d corresponding to each pair (i, j) , $i > j$ such that $A_i = A_j$. Hence by Box Principle, there exists at least $\frac{\mathfrak{C}}{2^{\omega(d)-\theta}-1}$ pairs of (i, j) , $i > j$ with $A_i = A_j$ and a partition (d_1, d_2) of d corresponding to these pairs. For every such pair (i, j) , we write $X_i - X_j = d_1 r_{ij}$, $X_i + X_j = d_2 s_{ij}$. Then $\gcd(X_i - X_j, X_i + X_j) = 2$ and $24 \mid (X_i^2 - X_j^2)$. Let r'_{ij}, s'_{ij} be such that $r'_{ij} \mid r_{ij}$, $s'_{ij} \mid s_{ij}$, $\gcd(r'_{ij}, s'_{ij}) = 1$ and $r_{ij}s_{ij} = \frac{24}{\rho 2^\delta} r'_{ij}s'_{ij}$. Then

$$r'_{ij}s'_{ij} = \frac{\rho 2^\delta}{24} r_{ij}s_{ij} = \frac{\rho 2^\delta}{24} \frac{X_i^2 - X_j^2}{d} = \frac{\rho 2^\delta}{24} \frac{i - j}{A_i} < \frac{\rho 2^\delta}{24} \frac{k}{A_i} < c 2^{\omega(d)-3}$$

since $A_i > \frac{\rho 2^\delta k}{3c 2^{\omega(d)}}$. There are at most $\sum_{i=1}^{c 2^{\omega(d)-3}-1} 2^{\omega(i)}$ possible pairs of (r'_{ij}, s'_{ij}) , and hence an equal number of possible pairs of (r_{ij}, s_{ij}) . By Lemma 9, we estimate

$$\sum_{i=1}^{c 2^{\omega(d)-3}-1} 2^{\omega(i)} \leq \eta(c 2^{\omega(d)-3}) c 2^{\omega(d)-3} (\log c 2^{\omega(d)-3}).$$

Thus if we have

$$\frac{\mathfrak{C}}{2^{\omega(d)-\theta}-1} > \eta(c 2^{\omega(d)-3}) c 2^{\omega(d)-3} (\log c 2^{\omega(d)-3}),$$

then there exist distinct pairs $(i, j) \neq (g, h)$, $i > j$, $g > h$ with $A_i = A_j$, $A_g = A_h$ such that $r_{ij} = r_{gh}$, $s_{ij} = s_{gh}$ giving

$$X_i - X_j = d_1 r_{ij} = X_g - X_h \quad \text{and} \quad X_i + X_j = d_2 s_{ij} = X_g + X_h.$$

Thus $X_i = X_g$, $X_j = X_h$ implying $(i, j) = (g, h)$, a contradiction. Hence

$$\frac{\mathfrak{C}}{2^{\omega(d)-\theta}-1} \leq \eta(c 2^{\omega(d)-3}) c 2^{\omega(d)-3} (\log c 2^{\omega(d)-3}), s$$

implying (28). □

The following lemma is a refinement of [16, Lemma 2].

Lemma 11. *Let $i > j, g > h, 0 \leq i, j, g, h < k$ be such that*

$$a_i = a_j, \quad a_g = a_h \tag{29}$$

and

$$x_i - x_j = d_1 r_1, \quad x_i + x_j = d_2 r_2, \quad x_g - x_h = d_1 s_1, \quad x_g + x_h = d_2 s_2 \quad (30)$$

where (d_1, d_2) is a partition of d ; $r_1 \equiv s_1 \pmod{2}$, $r_2 \equiv s_2 \pmod{2}$ when d is even; and either $r_1 \equiv s_1 \pmod{2}$ and $a_i \equiv a_g \pmod{4}$ or $2 \mid \gcd(r_1, s_1)$ when d is odd. Then we have either

$$a_i = a_g, \quad r_1 = s_1 \quad \text{or} \quad a_i = a_g, \quad r_2 = s_2 \quad (31)$$

or (4) and (5) hold.

PROOF. We follow the proof of [16, Lemma 2]. Suppose that (31) does not hold. Then

$$a_i r_1^2 - a_g s_1^2 \neq 0, \quad a_i r_2^2 - a_g s_2^2 \neq 0. \quad (32)$$

We proceed as in [16, Lemma 2] to conclude from $d \mid (a_i x_i^2 - a_g x_g^2)$ that

$$\begin{aligned} d_1 d_2 = d \mid \frac{1}{4} \Big\{ (a_i r_1^2 - a_g s_1^2) d_1^2 + (a_i r_2^2 - a_g s_2^2) d_2^2 \\ + 2d(a_i r_1 r_2 - a_g s_1 s_2) \Big\}. \end{aligned} \quad (33)$$

Thus we have

$$(a_i r_1^2 - a_g s_1^2) d_1^2 = a_i (x_i - x_j)^2 - a_g (x_g - x_h)^2 \neq 0$$

and

$$(a_i r_2^2 - a_g s_2^2) d_2^2 = a_i (x_i + x_j)^2 - a_g (x_g + x_h)^2 \neq 0.$$

Since

$$n \leq a_j x_j^2 < a_i x_i x_j < a_i x_i^2 \leq n + (k-1)d$$

and

$$n \leq a_h x_h^2 < a_g x_g x_h < a_g x_g^2 \leq n + (k-1)d,$$

we have

$$|a_i x_i x_j - a_g x_g x_h| < (k-1)d. \quad (34)$$

Also

$$|a_i x_i^2 - a_g x_g^2| = |i - g|d \leq (k-1)d \quad (35)$$

$$|a_j x_j^2 - a_h x_h^2| = |j - h|d \leq (k - 1)d \quad (36)$$

and

$$n \leq \min \left\{ \frac{1}{4} a_i (x_i + x_j)^2, \frac{1}{4} a_g (x_g + x_h)^2 \right\}. \quad (37)$$

Hence we derive from (34), (35) and (37) that

$$|(a_i r_2^2 - a_g s_2^2) d_2^2| < 4(k - 1)d \quad (38)$$

$$n |(a_i r_1^2 - a_g s_1^2) d_1^2| < \frac{1}{4} (k - 1)^2 d^2 \quad (39)$$

and further considering the cases $\{a_i r_1^2 > a_g s_1^2, a_i r_2^2 > a_g s_2^2\}$, $\{a_i r_1^2 > a_g s_1^2, a_i r_2^2 < a_g s_2^2\}$, $\{a_i r_1^2 < a_g s_1^2, a_i r_2^2 > a_g s_2^2\}$ and $\{a_i r_1^2 < a_g s_1^2, a_i r_2^2 < a_g s_2^2\}$, we derive

$$G(i, g) = |a_i r_1^2 - a_g s_1^2| d_1^2 + |a_i r_2^2 - a_g s_2^2| d_2^2 < 4(k - 1)d. \quad (40)$$

Let $d = d_1 d_2$ be odd, $\gcd(d_1, d_2) = 1$. We have either r_1, s_1 are even and hence r_1, r_2, s_1, s_2 are even, or $a_i \equiv a_g \pmod{4}$ and $r_1 \equiv s_1 \pmod{2}$ and hence $r_2 \equiv s_2 \pmod{2}$. Then reading modulo d_1 and d_2 separately in (33), we have

$$d_1 \mid \frac{1}{4} (a_i r_2^2 - a_g s_2^2) \quad \text{and} \quad d_2 \mid \frac{1}{4} (a_i r_1^2 - a_g s_1^2). \quad (41)$$

Therefore

$$4dd_2 = 4d_1 d_2^2 \leq |a_i r_2^2 - a_g s_2^2| d_2^2 \quad (42)$$

and

$$4dd_1 = 4d_1^2 d_2 \leq |a_i r_1^2 - a_g s_1^2| d_1^2. \quad (43)$$

From (40), we have

$$4d(d_1 + d_2) \leq G(i, g) < 4(k - 1)d$$

so that

$$d = d_1 d_2 \leq \left(\frac{d_1 + d_2}{2} \right)^2 < \frac{(k - 1)^2}{4}.$$

This gives (4). Again from (43) and (39), we see that $4ndd_1 < \frac{1}{4}(k-1)^2 d^2$, i.e., $n < \frac{1}{16}(k-1)^2 d_2$. From (42) and (38), we have $4dd_2 < 4(k-1)d$, i.e., $d_2 < (k-1)$. Thus (5) is also valid.

Let $d = d_1 d_2$ be even with $\text{ord}_2(d) = 1$ and d_1 odd. Then the x_i 's are odd and therefore both r_1 and s_1 is even. We see from (33) that

$$4d_1 \mid (a_i r_2^2 - a_g s_2^2) d_2^2 \quad \text{and} \quad 4d_2 \mid (a_i r_1^2 - a_g s_1^2) d_1^2. \quad (44)$$

Since $r_1 \equiv s_1 \pmod{2}$, $r_2 \equiv s_2 \pmod{2}$, $\gcd(d_1, d_2) = 1$ and d_1 odd, we derive that

$$2d_1 \mid (a_i r_2^2 - a_g s_2^2), \quad 4d_2 \mid (a_i r_1^2 - a_g s_1^2).$$

Therefore

$$2dd_2 = 2d_1 d_2^2 \leq |a_i r_2^2 - a_g s_2^2| d_2^2, \quad 4dd_1 = 4d_1^2 d_2 \leq |a_i r_1^2 - a_g s_1^2| d_1^2.$$

Now we argue as above to conclude (4) and (5).

Let $d = d_1 d_2$ be even with $\text{ord}_2(d) \geq 2$, $\gcd(d_1, d_2) = 2$. Then we see from (33) that (44) holds. Since $\gcd(d_1, d_2) = 2$, $r_1 \equiv s_1 \pmod{2}$ and $r_2 \equiv s_2 \pmod{2}$, we derive that

$$2d_1 \mid (a_i r_2^2 - a_g s_2^2), \quad 2d_2 \mid (a_i r_1^2 - a_g s_1^2).$$

Therefore

$$2dd_2 = 2d_1 d_2^2 \leq |a_i r_2^2 - a_g s_2^2| d_2^2, \quad 2dd_1 = 2d_1^2 d_2 \leq |a_i r_1^2 - a_g s_1^2| d_1^2.$$

Now we argue as above to conclude (4) and (5). □

Lemma 12. For a prime $p < k$, let

$$\gamma_p = \text{ord}_p \left(\prod_{a_i \in R} a_i \right), \quad \gamma'_p = 1 + \text{ord}_p((k-1)!).$$

Let $m > 1$ by any real number. Then

$$\prod_{2 \leq p \leq m} p^{\gamma_p - \gamma'_p} \leq k^{1.5\pi(m)} \left(z_1 \prod_{2 < p \leq m} p^{\frac{2p}{p^2-1}} \right) \left(z_2 \prod_{2 < p \leq m} p^{\frac{2}{p^2-1}} \right)^{-k} \quad (45)$$

where $(z_1, z_2) = (2^{\frac{4}{3}}, 2^{\frac{2}{3}})$ if d is odd and $(z_1, z_2) = (4, 2)$ if d is even.

PROOF. The proof is the refinement of inequality [12, (6.4)]. Let $p^h \leq k-1 < p^{h+1}$ where h is a positive integer. Then

$$\gamma'_p - 1 = \left\lfloor \frac{k-1}{p} \right\rfloor + \left\lfloor \frac{k-1}{p^2} \right\rfloor + \cdots + \left\lfloor \frac{k-1}{p^h} \right\rfloor. \quad (46)$$

Let $p \nmid d$. Then we see that g_p is the number of terms in $\{n, n+d, \dots, n+(k-1)d\}$ divisible by p to an odd power. After removing a term to which p appears to a maximal power, the number of terms in the remaining set divisible by p to an odd power is at most

$$\begin{aligned} \left\lfloor \frac{k-1}{p} \right\rfloor - \left(\left\lfloor \frac{k-1}{p^2} \right\rfloor - 1 \right) + \left\lfloor \frac{k-1}{p^3} \right\rfloor - \left(\left\lfloor \frac{k-1}{p^4} \right\rfloor - 1 \right) + \cdots \\ + (-1)^\epsilon \left(\left\lfloor \frac{k-1}{p^h} \right\rfloor + (-1)^\epsilon \right) \end{aligned}$$

where $\epsilon = 1$ or 0 according as h is even or odd, respectively. We note that the above expression is always positive. Combining this with (46) and $\left\lfloor \frac{k-1}{p^i} \right\rfloor \geq \frac{k-1}{p^i} - 1 + \frac{1}{p^i} = \frac{k}{p^i} - 1$, we have

$$\begin{aligned} \gamma_p - \gamma'_p &\leq -2 \left\{ \left\lfloor \frac{k-1}{p^2} \right\rfloor + \cdots + \left\lfloor \frac{k-1}{p^{h-1+\epsilon}} \right\rfloor \right\} + \frac{h-1+\epsilon}{2} \\ &\leq -2 \left\{ \frac{k}{p^2} + \cdots + \frac{k}{p^{h-1+\epsilon}} - \frac{h-1+\epsilon}{2} \right\} + \frac{h-1+\epsilon}{2} \\ &= -\frac{2k}{p^2(1-\frac{1}{p^2})} \left(1 - \frac{1}{p^{h-1+\epsilon}} \right) + 1.5(h-1+\epsilon). \end{aligned}$$

Since $p^h \geq \frac{k}{p}$ and $h < \frac{\log k}{\log p}$, we get

$$\begin{aligned} \gamma_p - \gamma'_p &< -\frac{2k}{p^2-1} + \frac{1.5 \log k}{\log p} + \frac{2p^{2-\epsilon}}{p^2-1} + 1.5\epsilon - 1.5 \\ &\leq -\frac{2k}{p^2-1} + \frac{1.5 \log k}{\log p} + \frac{2p}{p^2-1}. \end{aligned}$$

When d is even, we have $\gamma_2 - \gamma'_2 = -1 - \text{ord}_2(k-1) < -k + \frac{\log k}{\log 2} + 2$ by Lemma 1 (v). Now (45) follows immediately. $\square$

Lemma 13. *Suppose that $n \geq c_1(k-1)^3$ or $d \geq 4c_1(k-1)^2$ or both. Let $1 \leq \varrho \leq 2^{\omega(d)-\theta}$ be the greatest integer such that $R_\varrho = \{a_i \mid \nu(a_i) = \varrho\} \neq \emptyset$. For $k \geq \kappa_0$, we have*

$$\mathfrak{r} = |\{(i, j) \mid a_i = a_j, i > j\}| \geq g(\varrho) := \begin{cases} 4\varrho(2^{\omega(d)} - 1) & \text{if } d \text{ is odd,} \\ 2\varrho(2^{\omega(d)-\theta} - 1) & \text{if } d \text{ is even.} \end{cases}$$

PROOF. We have

$$k = \sum_{\mu=1}^{\varrho} \mu r_\mu \quad \text{and} \quad |R| = \sum_{\mu=1}^{\varrho} r_\mu$$

where $r_\mu = |R_\mu| = |\{a_i \mid \nu(a_i) = \mu\}|$. Each R_μ gives rise to $\frac{\mu(\mu-1)}{2}r_\mu$ pairs of i, j with $i > j$ such that $a_i = a_j$. Then

$$\mathfrak{r} = \sum_{\mu=1}^{\varrho} \frac{\mu(\mu-1)}{2} r_\mu = k - |R| + \sum_{\mu=1}^{\varrho} \frac{(\mu-1)(\mu-2)}{2} r_\mu.$$

Suppose that the assertion of the Lemma 13 does not hold. Then $g(\varrho) > k - |R| + \sum_{\mu=1}^{\varrho} \frac{(\mu-1)(\mu-2)}{2} r_\mu$. We have

$$g(\varrho) - \sum_{\mu=1}^{\varrho} \frac{(\mu-1)(\mu-2)}{2} r_\mu \leq g(\varrho) - \frac{(\varrho-1)(\varrho-2)}{2} := g_0(\varrho).$$

We see that $g_0(\varrho)$ is an increasing function of ϱ . Since $\varrho \leq 2^{\omega(d)-\theta}$, we find that

$$k - |R| < g_0(2^{\omega(d)-\theta}) = (2^{\omega(d)-\theta} - 1)(z_3 2^{\omega(d)-\theta} + 1) := g_1$$

where $z_3 = \frac{7}{2}$ if d is odd and $\frac{3}{2}$ if d is even. Thus $|R| > k - g_1$. Since the a_i 's are squarefree, we have by Lemma 8 that

$$\prod_{a_i \in R} a_i \geq z_4^{k-g_1} (k - g_1)!$$

where $z_4 = 1.6$ if d is odd and 2.4 if d is even. Also, we have

$$\prod_{a_i \in R} a_i \mid (k-1)! \left(\prod_{p < k} p \right) \prod_{2 \leq p \leq m} p^{\gamma_p - \gamma'_p}$$

where γ_p, γ'_p and $\mathfrak{m}$ are as in Lemma 12. This with (45) and Lemma 1 (iv) gives

$$\prod_{a_i \in R} a_i < k! k^{1.5\pi(\mathfrak{m})-1} \left(z_1 \prod_{2 < p \leq \mathfrak{m}} p^{\frac{2p}{p^2-1}} \right) \left(\frac{z_2}{2.7205} \prod_{2 < p \leq \mathfrak{m}} p^{\frac{2}{p^2-1}} \right)^{-k}.$$

Comparing the lower and upper bounds, we have

$$\frac{z_4^{g_1} k!}{(k - g_1)!} > k^{-1.5\pi(\mathfrak{m})+1} \left(z_1 \prod_{2 < p \leq \mathfrak{m}} p^{\frac{2p}{p^2-1}} \right)^{-1} \left(\frac{z_2 z_4}{2.7205} \prod_{2 < p \leq \mathfrak{m}} p^{\frac{2}{p^2-1}} \right)^k. \quad (47)$$

By Lemma 2, we have

$$\frac{z_4^{g_1} k!}{(k - g_1)!} < z_4^{g_1} e^{-g_1} k^{g_1} \left(\frac{k}{k - g_1} \right)^{k-g_1+\frac{1}{2}} \frac{e^{\frac{1}{12k}}}{e^{\frac{1}{12(k-g_1)+1}}}.$$

Since $k \geq \kappa_0$, we find that $g_1 < \frac{k}{z_5}$ for $\omega(d) \geq 12$ where $z_5 = 37, 18$ for d odd and d even, respectively. Thus

$$\frac{z_4^{g_1} k!}{(k - g_1)!} < \begin{cases} \left(\frac{z_4(k - g_1)}{e} \right)^{g_1} \left(\frac{k}{k - g_1} \right)^{k+\frac{1}{2}} & \text{if } \omega(d) \leq 11, \\ \left(\frac{z_5}{z_5 - 1} \right)^{k+\frac{1}{2}} \left(\frac{z_4(z_5 - 1)k}{z_5 e} \right)^{g_1} & \text{if } \omega(d) \geq 12. \end{cases}$$

Hence we derive from (47) that

$$g_1 > \frac{k \log \left(\frac{z_2 z_4}{2.7205} \prod_{2 < p \leq \mathfrak{m}} p^{\frac{2}{p^2-1}} \right) + (k + \frac{1}{2}) \log(1 - \frac{g_1}{k})}{\log(k - g_1) - 1 + \log z_4} - \frac{(1.5\pi(\mathfrak{m}) - 1) \log k + \log \left(z_1 \prod_{2 < p \leq \mathfrak{m}} p^{\frac{2p}{p^2-1}} \right)}{\log(k - g_1) - 1 + \log z_4} \quad (48)$$

for $\omega(d) \leq 11$ and

$$g_1 > \frac{k \log \left(\frac{z_5 - 1}{z_5} \frac{z_2 z_4}{2.7205} \prod_{2 < p \leq \mathfrak{m}} p^{\frac{2}{p^2-1}} \right) - (1.5\pi(\mathfrak{m}) - 1) \log k - \log \left(\sqrt{\frac{z_5}{z_5 - 1}} z_1 \prod_{2 < p \leq \mathfrak{m}} p^{\frac{2p}{p^2-1}} \right)}{\log k - 1 + \log z_4(z_5 - 1) - \log z_5} \quad (49)$$

for $\omega(d) \geq 12$.

Let $\omega(d) \leq 11$. Taking $m = \min(1000, \sqrt{\kappa_0})$ in (48), we observe that the right hand side of (48) is an increasing function of k and the inequality does not hold at $k = \kappa_0$. Hence (48) is not valid for all $k \geq \kappa_0$. For instance, when $\omega(d) = 4$, d odd, we have $\kappa_0 = 15700$ and $g_1 = 855$. With these values, we see that the right hand side of (48) exceeds 855 at $k = 15300$, a contradiction. Hence (48) is not valid for all $k \geq 15300$.

Let $\omega(d) \geq 12$. Taking $m = 1000$ in (49), we derive that

$$g_1 > \begin{cases} 0.63104 \frac{k}{\log k} & \text{if } d \text{ is odd,} \\ 1.183 \frac{k}{\log k} & \text{if } d \text{ is even.} \end{cases}$$

For d odd, we see that

$$\begin{aligned} 0.63104 \frac{k}{\log k} &\geq 0.63104 \frac{\kappa_0}{\log \kappa_0} \\ &= \frac{0.63104 \times 11\omega(d)4^{\omega(d)}}{\omega(d) \log 4 + \log 11 + \log \omega(d)} > \frac{7}{2} 4^{\omega(d)} > g_1, \end{aligned}$$

a contradiction. Similarly, we get a contradiction for d even. $\square$

Lemma 14. Let $k \geq \kappa_0 = \kappa_0(\omega(d))$. Assume that $d < 4c_1(k-1)^2$. Let $T_1 = \{0 \leq i < k \mid X_i > 1\}$ defined in Section 2 be such that

$$|T_1| > C_1 := \begin{cases} \frac{k}{C_2} + \frac{k}{48} + C_3 + \frac{8}{3} & \text{if } \omega(d) = 2, \\ \frac{k}{C_2} + \frac{k}{12} + C_3 + \frac{2^{\omega(d)+1}}{3} & \text{if } \omega(d) = 3, 4, 5, \\ \frac{k}{C_2} + \frac{k}{12} + \frac{k}{9} + & \text{if } \omega(d) \geq 6, \end{cases}$$

where $C_2 \leq 2k^{\frac{1}{3}}$ and $C_3 = 39, 42, 195, 806$ for $\omega(d) = 2, 3, 4, 5$, respectively. Then

$$\max_{i \in T_1} A_i \geq 2^\delta C_0 \frac{k}{C_2} \text{ where } C_0 = C_0(\omega(d)) = \begin{cases} 1 & \text{if } \omega(d) = 2, \\ \frac{3}{2} 2^{\lfloor \frac{\omega(d)}{3} \rfloor} & \text{if } \omega(d) \geq 3. \end{cases} \quad (50)$$

PROOF. We see that for $\omega(d) \geq 6$,

$$\frac{k}{20 \cdot 2^{\omega(d)}} \geq (4c_1(k-1)^2)^{\frac{1}{\omega(d)}} > d^{\frac{1}{\omega(d)}}$$

where c_1 is given by Proposition 2. Hence there exists a partition $d = d_1 d_2$ of d with

$$d_1 < \frac{k}{20 \cdot 2^{\omega(d)}} \quad \text{with} \quad \omega(d_1) \geq 1 \quad \text{and} \quad \omega(d_2) \leq \omega(d) - 1.$$

Therefore

$$\nu(A_i) \leq 2^{\omega(d_2)} \leq 2^{\omega(d)-1} \quad \text{for } A_i \geq \frac{k}{20 \cdot 2^{\omega(d)}} \quad (51)$$

by Lemma 4.

Let

$$T_2 = \left\{ i \in T_1 \mid A_i > \frac{2^\delta \rho k}{3c2^{\omega(d)}} \right\}, \quad T_3 = T_1 - T_2, \quad (52)$$

where $c = 16$ if $\omega(d) = 2$, $c = 4$ if $\omega(d) = 3, 4, 5$ and $c = 2$ if $\omega(d) \geq 6$. Further let

$$S_2 = \{A_i \mid i \in T_2\}, \quad S_3 = \{A_i \mid i \in T_3\} \quad (53)$$

and $|S_3| = s$. Then considering residue classes modulo $2^\delta \rho$, we derive that

$$\frac{2^\delta \rho k}{3c \cdot 2^{\omega(d)}} \geq \max_{A_i \in S_3} A_i \geq 2^\delta \rho(s-1) + 1$$

so that $|S_3| = s \leq \frac{k}{3c2^{\omega(d)}} - \frac{1}{\rho} + 1 \leq \frac{k}{3c2^{\omega(d)}} + \frac{2}{3}$. We see from Corollary 1, (51), (52) and (53) that

$$\begin{aligned} |T_3| &\leq \frac{k}{20 \cdot 2^{\omega(d)}} 2^{\omega(d)} + \left(\frac{k}{6 \cdot 2^{\omega(d)}} - \frac{k}{20 \cdot 2^{\omega(d)}} + \frac{2}{3} \right) 2^{\omega(d)-1} \\ &\leq \frac{k}{20} + \left(\frac{k}{6} - \frac{k}{20} \right) 2^{-1} + \frac{2}{3} 2^{\omega(d)-1} \leq \frac{k}{12} + \frac{k}{40} + \frac{k}{6 \times 2^6} \leq \frac{k}{9} \end{aligned}$$

if $\omega(d) \geq 6$ and

$$|T_3| \leq \begin{cases} \left(\frac{k}{48 \cdot 2^{\omega(d)}} + \frac{2}{3} \right) 2^{\omega(d)} = \frac{k}{48} + \frac{8}{3} & \text{if } \omega(d) = 2, \\ \left(\frac{k}{12 \cdot 2^{\omega(d)}} + \frac{2}{3} \right) 2^{\omega(d)} = \frac{k}{12} + \frac{2^{\omega(d)+1}}{3} & \text{if } \omega(d) = 3, 4, 5. \end{cases}$$

Therefore

$$|T_2| > C_1 - |T_3| \geq C_4 := \begin{cases} \frac{k}{C_2} + C_3 & \text{if } \omega(d) = 2, 3, 4, 5, \\ \frac{k}{C_2} + \frac{k}{4} & \text{if } \omega(d) \geq 6. \end{cases}$$

Let $\mathfrak{C}, \mathfrak{C}_\mu$ be as in Lemma 10 with $c = 16$ if $\omega(d) = 2$, $c = 4$ if $\omega(d) = 3, 4, 5$ and $c = 2$ if $\omega(d) \geq 6$. Then $C_4 < |T_2| = |S_2| + \sum_{\mu \geq 2} (\mu - 1) |\mathfrak{C}_\mu|$. Now we apply Lemma 10 and use $k \geq \kappa_0 \geq \eta(2^{\omega(d)-2})(\log 2^{\omega(d)-2})2^{\omega(d)}(2^{\omega(d)-\theta} - 1)$ for $\omega(d) \geq 6$ to get

$$C_4 < \begin{cases} |S_2| + C_3 & \text{if } 2 \leq \omega(d) \leq 5, \\ |S_2| + \frac{k}{12} & \text{if } \omega(d) \geq 6. \end{cases}$$

Thus

$$|S_2| > \frac{k}{C_2}.$$

Let $\omega(d) = 2$. Then considering the A_i 's modulo 2^δ , we see that

$$\max_{A_i \in S_2} A_i \geq 2^\delta \left\lceil \frac{k}{C_2} \right\rceil + \frac{2^\delta k}{48 \times 4} \geq 2^\delta \frac{k}{C_2}$$

which gives (50). Now we take $\omega(d) \geq 3$. Since $d < 4c_1(k-1)^2$, we have $r \geq \lceil \frac{\omega(d)}{3} \rceil$ by Lemma 6. By (14), we have $\frac{k}{C_2} \geq \frac{k^{\frac{2}{3}}}{2} > \frac{1}{2^r}(4c_1(k-1)^2)^{\frac{1}{3}} > \prod_{j=1}^r \left(\frac{P_j-1}{2} \right)$. We now apply Corollary 2 with $s = \lceil \frac{k}{C_2} + 1 \rceil$ and $U = 1$ to get

$$\max_{A_i \in S_2} A_i \geq \frac{3}{4} 2^{r+\delta} \left\lceil \frac{k}{C_2} + 1 \right\rceil \geq \frac{3}{4} 2^{\lceil \frac{\omega(d)}{3} \rceil + \delta} \frac{k}{C_2}$$

which yields (50). □

4. Proof of Proposition 2

We assume that either $n \geq c_1(k-1)^3$ or $d \geq 4c_1(k-1)^2$. Then $\nu(a_{i_0}) \leq 2^{\omega(d)-\theta}$ for $0 \leq i_0 < k$ by Lemma 5. Let ϱ be as defined in the statement of Lemma 13. Then $\nu(a_{i_0}) \leq \varrho$. By Lemma 13, there are at least $z\varrho(2^{\omega(d)} - 1)$ distinct pairs (i, j) with $i > j$ and $a_i = a_j$, where $z = 4$ if d is odd and 2 if d is even. Since there can be at most $2^{\omega(d)-\theta} - 1$ possible partitions of d , by Box principle, there exists a partition (d_1, d_2) of d and at least $z\varrho$ pairs of (i, j) with $a_i = a_j$, $i > j$ corresponding to this partition. We write

$$x_i - x_j = d_1 r_1(i, j) \quad \text{and} \quad x_i + x_j = d_2 r_2(i, j).$$

Let d be odd. Suppose there are at least ϱ distinct pairs $(i_1, j_1), \dots, (i_\varrho, j_\varrho), \dots$ with the corresponding $r_1(i, j)$ even. Then $|\{i_1, \dots, i_\varrho, j_1, \dots, j_\varrho\}| > \varrho$. Hence we can find $1 \leq l, m \leq \varrho$ with $(i_l, j_l) \neq (i_m, j_m)$, $a_{i_l} = a_{j_l}$, $a_{i_m} = a_{j_m}$ and $a_{i_l} \neq a_{i_m}$. Now the result follows by Lemma 11. Thus we may assume that there are at most $\varrho - 1$ pairs (i, j) with $r_1(i, j)$ even. Then there are at least $3\varrho + 1$ distinct pairs (i, j) with $r_1(i, j)$ odd. Since $a_i \equiv 1, 2, 3 \pmod{4}$, we can find at least ϱ pairs with $a_i \equiv a_g \pmod{4}$ for any two such pairs $(i, j), (g, h)$. Then there exist two distinct pairs $(i, j), (g, h)$ with $a_i = a_j$, $a_g = a_h$ and $a_i \neq a_g$ from these pairs. Also $r_1(i, j) \equiv r_1(g, h) \pmod{2}$. This gives (4) and (5) by Lemma 11 which is a contradiction.

Let d be even. We observe that $8 \mid (x_i^2 - x_j^2)$ and $\gcd(x_i - x_j, x_i + x_j) = 2$. We claim that there are at least ϱ pairs with $r_1(i, j) \equiv r_1(g, h) \pmod{2}$ and $r_2(i, j) \equiv r_2(g, h) \pmod{2}$ for any two such distinct pairs (i, j) and (g, h) . If the claim is true, then there are two pairs $(i, j) \neq (g, h)$ with $i > j$, $g > h$, $a_i = a_j$, $a_g = a_h$ and $a_i \neq a_g$ since $\nu(a_i) \leq \varrho$. This implies (4) and (5) by Lemma 11, contradicting our assumption. Let $\text{ord}_2(d) = 1$. Then d_1 is odd, implying that $r_1(i, j)$ is even. We can choose at least ϱ pairs whose r_2 's are of the same parity. Thus the claim is true in this case. Let $\text{ord}_2(d) \geq 3$. Then we have either $\text{ord}_2(d_1) = 1$ implying that all the r_1 's are odd, or $\text{ord}_2(d_2) = 1$ implying that all the r_2 's are odd. Thus the claim follows. Finally, let $\text{ord}_2(d) = 2$. Then $2 \parallel d_1$ and $2 \parallel d_2$ so that r_1 and r_2 are of the opposite parity for any pair and hence the claim holds. $\square$

5. Proof of Proposition 3

In this section, we assume that $k \geq \kappa_0 = \kappa_0(\omega(d))$. In view of Proposition 2, we may assume that $d < 4c_1(k-1)^2$. We may also assume that X_i is a prime for each $i \in T_1$ in the proof of Proposition 3. Otherwise $n + (k-1)d \geq (k+1)^4$, which implies the assertion.

Since $d < 4c_1(k-1)^2$, d has at least one prime divisor $\leq k$ otherwise $d > k^{\omega(d)} \geq k^2$, giving a contradiction. Thus $\pi_d(k) \leq \pi(k) - 1$. Let $n + (k-1)d \geq L$ for some $L > 0$. By Lemma 3 and Lemma 1 (i), we have

$$|T_1| > k - \frac{(k-1)\log(k-1)}{\log L - \log 2} - \frac{k}{\log k} \left(1 + \frac{1.5}{\log k}\right). \quad (54)$$

We see from [5] that $n(n+d) \dots (n+(k-1)d)$ is divisible by at least $\pi(2k) - \pi_d(k) \geq \pi(2k) - \pi(k) + 1$ primes exceeding k . Hence we have $n + (k-1)d \geq 4k^2$. Thus by taking $L = 4k^2$ in (54), we get

$$|T_1| > k - \frac{(k-1)\log(k-1)}{\log(2k^2)} - \frac{k}{\log k} \left(1 + \frac{1.5}{\log k}\right).$$

The right hand side of the above inequality is an increasing function of k and

$$|T_1| > \begin{cases} \frac{k}{5} + \frac{k}{48} + C_3 + \frac{8}{3} & \text{if } \omega(d) = 2, \\ \frac{k}{6} + \frac{k}{12} + C_3 + \frac{16}{3} & \text{if } \omega(d) = 3, \\ \frac{5}{24}k + \frac{k}{12} + C_3 + \frac{2^{\omega(d)+1}}{3} & \text{if } \omega(d) = 4, 5, \\ \frac{5}{48}k + \frac{k}{12} + \frac{k}{9} & \text{if } \omega(d) \geq 6. \end{cases} \quad (55)$$

Now we see from Lemma 14 that (50) holds with

$$C_2 = \begin{cases} 5 & \text{if } \omega(d) = 2, \\ 6 & \text{if } \omega(d) = 3, \\ \frac{24}{5} & \text{if } \omega(d) = 4, 5, \\ \frac{48}{5} & \text{if } \omega(d) \geq 6. \end{cases}$$

This gives $n + (k - 1)d \geq \frac{C_0}{C_2}k^3$. Hence (7) is valid for $\omega(d) \geq 4$. Now we take $\omega(d) = 2, 3$. Putting $L = \frac{C_0}{5}k^3$ in (54), we derive that

$$|T_1| > \begin{cases} \frac{5k}{16} + \frac{k}{48} + C_3 + \frac{2^{\omega(d)+1}}{3} & \text{if } \omega(d) = 2, \\ \frac{5k}{24} + \frac{k}{12} + C_3 + \frac{2^{\omega(d)+1}}{3} & \text{if } \omega(d) = 3. \end{cases}$$

We apply Lemma 14 again to get $\max_{i \in T_1} A_i \geq 2^\delta \frac{5}{16}k$ so that $n + (k - 1)d \geq 2^\delta \frac{5}{16}k^3$, which implies (7). This completes the proof. $\square$

References

- [1] B. BRINDZA, L. HAJDU and I. Z. RUZSA, On the equation $x(x + d) \dots (x + (k - 1)d) = by^2$, *Glasgow. Math. J.* (2000), 255–261.
- [2] P. DUSART, Autour de la fonction qui compte le nombre de nombres premiers, Ph.D. thesis, *Université de Limoges*, 1998.
- [3] P. ERDŐS and J. L. SELFRIDGE, The product of consecutive integers is never a power, *Illinois J. Math.* **19** (1975), 92–301.
- [4] K. GYÖRY, Power values of products of consecutive integers and binomial coefficients, Number Theory and its Applications, *Kluwer Acad. Publ.*, 1999, 145–156.
- [5] S. LAISHRAM and T. N. SHOREY, Number of prime divisors in a product of terms of an arithmetic progression, *Indag Math.* **15**(4) (2004), (in press).
- [6] S. LAISHRAM, Topics in Diophantine Equations, *M.Sc. Thesis, University of Mumbai*, 2004.
- [7] R. MARSZALEK, On the product of consecutive elements of an arithmetic progression, *Monatsh. für Math.* **100** (1985), 215–222.
- [8] A. MUKHOPADHYAY and T. N. SHOREY, Almost squares in arithmetic progression (II), *Acta Arith.* **110** (2003), 1–14.
- [9] H. ROBBINS, A remark on Stirling's formula, *Amer. Math. Monthly* **62** (1955), 26–29.
- [10] J. B. ROSSER and L. SCHOENFELD, Approximate formulas for some functions of prime numbers, *Illinois Jour. Math.* **6** (1962), 64–94.
- [11] N. SARADHA, On perfect powers in products with terms from arithmetic progressions, *Acta Arith.* **82** (1997), 147–172.
- [12] N. SARADHA and T. N. SHOREY, Almost squares in arithmetic progression, *Compositio Math.* **138** (2003), 73–111.
- [13] L. SCHOENFELD, Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$, II, *Math. Comp.* **30** (1976), 337–360.

- [14] T. N. SHOREY, Exponential diophantine equations involving products of consecutive integers and related equations, Number Theory, (R. P. Bambah, V. C. Dumir and R. J. Hans-Gill, eds.), *Hindustan Book Agency*, 1999, 463–495.
- [15] T. N. SHOREY, Powers in arithmetic progression, A Panorama in Number Theory or The View from Baker’s Garden, (G. Wüstholz, ed.), *Cambridge University Press*, 2002, 325–336.
- [16] T. N. SHOREY and R. TIJDEMAN, Perfect powers in products of terms in an arithmetical progression, *Compositio Math.* **75** (1990), 307–344.

SHANTA LAISHRAM
SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
HOMI BHABHA ROAD, MUMBAI 400005
INDIA

E-mail: shanta@math.tifr.res.in

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