

The Dunford–Pettis property on spaces of polynomials

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Abstract. We give conditions on a Banach space X so that the spaces of scalar-valued homogeneous polynomials on X do not have the Dunford–Pettis property (DPP). This allows us to obtain new examples of Banach spaces with the DPP such that their duals fail it.

In [P2], PEŁCZYŃSKI asks whether the projective tensor product of two Banach spaces with the Dunford–Pettis property (DPP) has the DPP. The same question is raised in [D1, Question 11] for the injective tensor product.

An important counterexample was given by TALAGRAND [T1] who constructed a Banach space X such that X^* has the Schur property (so X and X^* have the DPP) and $C([0, 1], X) \equiv C[0, 1] \otimes_{\epsilon} X$ and $L_1(X^*) \equiv L_1[0, 1] \otimes_{\pi} X^*$ fail the DPP. Positive results have been obtained by many authors (see, for instance, [A], [B2], [Ry1]).

In many cases, the injective and the projective tensor products may be identified with spaces of operators [DF, 5.3, Proposition and 5.7, Corollary 1]. In all cases, the duals of these tensor products are also spaces of operators. In [BV] and [GG4] conditions are given on Banach spaces X and Y so that the projective tensor product $X \otimes_{\pi} Y$, the injective tensor

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product $X \otimes_{\epsilon} Y$, or the duals of these tensor products do not have the DPP.

As far as we know, little has been done concerning the DPP on spaces of scalar-valued polynomials (some easy cases may be seen in [CG, p. 233]). In the present paper, we give conditions on X so that the spaces of scalar-valued homogeneous polynomials on X fail the DPP, obtaining new examples of Banach spaces with the DPP such that their duals fail it.

Throughout, X and Y denote Banach spaces, X^* is the dual of X , and B_X stands for its closed unit ball. By $\mathbb{N}$ we represent the set of all natural numbers and by $\mathbb{K}$ the scalar field (real or complex). The notation $X \equiv Y$ (respectively, $X \cong Y$) means that X and Y are isometrically isomorphic (respectively, isomorphic). By an *operator* from X into Y we always mean a bounded linear mapping. We use $\mathcal{L}(X, Y)$ for the space of all operators from X into Y , and $\mathcal{K}(X, Y)$ for the subspace of compact operators. Given an operator $T \in \mathcal{L}(X, Y)$, its adjoint is denoted by $T^* \in \mathcal{L}(Y^*, X^*)$.

By $\mathcal{I}(X, Y)$ we denote the space of all (*Grothendieck*) *integral operators* from X into Y (see [DU, Definition VIII.2.6]).

An operator $T \in \mathcal{L}(X, Y)$ is *absolutely (q, p) -summing* ($1 \leq p, q < \infty$) if there is a constant $K > 0$ such that, no matter how we select finitely many vectors $x_1, \dots, x_n \in X$, we have

$$\left(\sum_{k=1}^n \|T(x_k)\|^q \right)^{1/q} \leq K \cdot \sup \left\{ \left(\sum_{k=1}^n |\langle x^*, x_k \rangle|^p \right)^{1/p} : x^* \in B_{X^*} \right\}.$$

A Banach space X has *cotype* q if there is a constant $K \geq 0$ such that, however we choose finitely many vectors $x_1, \dots, x_n \in X$, we have

$$\left(\sum_{k=1}^n \|x_k\|^q \right)^{1/q} \leq K \left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^2 dt \right)^{1/2},$$

where $r_k(t)$ are the Rademacher functions (see [DJT, Chapter 11]).

A Banach space X has the *Orlicz property* if the identity operator on X is absolutely $(2, 1)$ -summing. Every Banach space with cotype 2 has the Orlicz property (see [DPR, Definition 5.1]). The converse is not true [T2].

Given $m \in \mathbb{N}$, we denote by $\mathcal{P}^m(X, Y)$ the space of all m -homogeneous (continuous) polynomials from X into Y endowed with the supremum

norm. Recall that with each $P \in \mathcal{P}(^mX, Y)$ we can associate a unique symmetric m -linear (continuous) mapping $\widehat{P} : X \times \dots \times X \rightarrow Y$ so that

$$P(x) = \widehat{P}(x, \dots, x) \quad (x \in X).$$

For simplicity, we write $\mathcal{P}(^mX) := \mathcal{P}(^mX, \mathbb{K})$. For $1 \leq k \leq m$ ($k, m \in \mathbb{N}$), $\mathcal{P}(^kX)$ is isomorphic to a complemented subspace of $\mathcal{P}(^mX)$ [AS, Proposition 5.3].

A polynomial $P \in \mathcal{P}(^mX, Y)$ is of *finite type* if it is a finite sum of terms of the form $\gamma^m \otimes y$, with $\gamma \in X^*$ and $y \in Y$, where $(\gamma^m \otimes y)(x) := \gamma(x)^m y$ for all $x \in X$. A polynomial is *approximable* if it lies in the norm closure of the space of polynomials of finite type.

A polynomial $P \in \mathcal{P}(^mX, Y)$ is *compact* if $P(B_X)$ is relatively compact in Y . A polynomial $P \in \mathcal{P}(^mX, Y)$ is *weakly continuous on bounded subsets* if for each bounded net $(x_\alpha) \subset X$ weakly converging to x , $(P(x_\alpha))$ converges to $P(x)$ in norm. We denote by $\mathcal{P}_{\text{wb}}(^mX, Y)$ the space of all polynomials in $\mathcal{P}(^mX, Y)$ which are weakly continuous on bounded sets. Every polynomial in $\mathcal{P}_{\text{wb}}(^mX, Y)$ is compact ([AP, Lemma 2.2] and [AHV, Theorem 2.9]). An operator is weakly continuous on bounded sets if and only if it is compact [AP, Proposition 2.5].

For the general theory of multilinear mappings and polynomials on Banach spaces, we refer to [Di] and [M].

By $X \otimes_s X$ we denote the 2-fold symmetric tensor product of X , that is, the set of all elements of $X \otimes X$ of the form

$$u = \sum_{i=1}^m \lambda_i x_i \otimes x_i \quad (m \in \mathbb{N}, \lambda_i \in \mathbb{K}, x_i \in X, 1 \leq i \leq m).$$

By $X \otimes_{\pi, s} X$ (respectively, $X \otimes_{\epsilon, s} X$), we denote the closure of $X \otimes_s X$ in $X \otimes_{\pi} X$ (respectively, in $X \otimes_{\epsilon} X$). Given $x, y \in X$, we denote

$$x \otimes_s y := \frac{1}{2} (x \otimes y + y \otimes x).$$

See [DF] or [DU] for the theory of tensor products. For symmetric tensor products, we refer to [F]. It is well known that $\mathcal{P}(^mX) \cong (\otimes_{\pi, s}^m X)^*$ [F, Proposition 2.2].

A Banach space X has the *Dunford–Pettis property* (DPP, for short) if every weakly compact operator on X is *completely continuous*, i.e., takes weakly null sequences into norm null sequences [Gr]. Equivalently, X has the DPP if and only if, given arbitrary weakly null sequences $(x_n) \subset X$ and $(x_n^*) \subset X^*$, we have $\lim_n \langle x_n, x_n^* \rangle = 0$. If X^* has the DPP, then so does X , but the converse is not true [S]. The DPP is inherited by complemented subspaces. The spaces $C(K)$ and $L_1(\mu)$ enjoy the DPP. For more on the DPP, the reader is referred to [D1].

The *Banach–Mazur distance* $d(X, Y)$ between two isomorphic Banach spaces X and Y is defined by $\inf (\|T\| \|T^{-1}\|)$ where the infimum is taken over all isomorphisms T from X onto Y . Recall that a Banach space X is an $\mathcal{L}_p$ -space ($1 \leq p \leq \infty$) [LP] if there is $\lambda \geq 1$ such that every finite-dimensional subspace of X is contained in another subspace N with $d(N, \ell_p^n) \leq \lambda$ for some integer n . The $\mathcal{L}_1$ -spaces, the $\mathcal{L}_\infty$ -spaces, and all their duals have the Dunford–Pettis property [B1, Corollary 1.30].

We start with a preparatory result of independent interest.

Theorem 1. *Suppose that X has the Orlicz property, does not have the Schur property, and contains a complemented copy of ℓ_1 . Then the space $X \otimes_{\epsilon, s} X$ does not have the DPP.*

PROOF. Since X is isomorphic to a complemented subspace of $X \otimes_{\epsilon, s} X$ [AF, 3.5], we can assume that X has the DPP.

Let $(z_n) \subset X$ be a weakly null normalized, basic sequence. We can find a sequence $(z_n^*) \subset X^*$ with $\|z_n^*\| \leq C$ for all n , and $\langle z_n, z_n^* \rangle = \delta_{nm}$.

There are operators

$$\ell_1 \xrightarrow{j} X \xrightarrow{h} \ell_1$$

so that $h \circ j = I_{\ell_1}$, where I_{ℓ_1} is the identity map on ℓ_1 . Taking adjoints, we have

$$\ell_\infty \xrightarrow{h^*} X^* \xrightarrow{j^*} \ell_\infty,$$

and

$$\langle j(e_n), h^*(e_m^*) \rangle = \langle hj(e_n), e_m^* \rangle = \langle e_n, e_m^* \rangle = \delta_{nm},$$

where $(e_n) \subset \ell_1$ and $(e_n^*) \subset \ell_\infty$ are the canonical unit vector sequences.

Define $T : X \otimes_{\epsilon, s} X \rightarrow \ell_2$ by

$$T(x \otimes x) = (z_n^*(x)h^*(e_n^*)(x))_{n=1}^\infty.$$

Since $\sum h^*(e_n^*)$ is weakly unconditionally Cauchy, T is well defined. Moreover, for an arbitrary element $\sum_{i=1}^m \lambda_i x_i \otimes x_i \in X \otimes_s X$, we have

$$\left\| T\left(\sum_{i=1}^m \lambda_i x_i \otimes x_i\right) \right\| = \left\| \sum_{i=1}^m \lambda_i \left(z_n^*(x_i) h^*(e_n^*)(x_i) \right)_{n=1}^\infty \right\|.$$

Let $H \in \mathcal{L}(X^*, X)$ be given by

$$\langle H(y^*), x^* \rangle = \sum_{i=1}^m \lambda_i y^*(x_i) x^*(x_i).$$

Then

$$\|H\| = \left\| \sum_{i=1}^m \lambda_i x_i \otimes x_i \right\|_\epsilon$$

[DF, Examples 4.2]. So, by the Orlicz property of X ,

$$\begin{aligned} \left\| T\left(\sum_{i=1}^m \lambda_i x_i \otimes x_i\right) \right\| &= \left(\sum_{n=1}^\infty |\langle H(h^*(e_n^*)), z_n^* \rangle|^2 \right)^{1/2} \\ &\leq C \cdot \left(\sum_{n=1}^\infty \|H(h^*(e_n^*))\|^2 \right)^{1/2} \\ &\leq M \sup_{x^* \in B_{X^*}} \sum_{n=1}^\infty |\langle H(h^*(e_n^*)), x^* \rangle| \\ &= M \sup_{x^* \in B_{X^*}} \sum_{n=1}^\infty |\langle e_n^*, h^{**}(H^*(x^*)) \rangle| \\ &\leq M \|h\| \|H\| \sup_{\xi \in B_{\ell_\infty^*}} \sum_{i=1}^\infty |\langle e_n^*, \xi \rangle| \\ &= M \|h\| \|H\| \sup_{y \in B_{\ell_1}} \sum_{i=1}^\infty |\langle e_n^*, y \rangle| \\ &= M \|h\| \|H\| = M \|h\| \left\| \sum_{i=1}^m \lambda_i x_i \otimes x_i \right\|_\epsilon, \end{aligned}$$

where we have used the decomposition $\ell_\infty^* = \ell_1 \oplus c_0^\perp$ [Kö, §31.1(11)]. Hence, T is continuous for the injective topology.

Consider the symmetric tensor

$$z_n \otimes_s j(e_n) = \frac{1}{2} z_n \otimes j(e_n) + \frac{1}{2} j(e_n) \otimes z_n.$$

Clearly, the sequence $(z_n \otimes_s j(e_n))$ is weakly null in $X \otimes_{\epsilon, s} X$. We have

$$\begin{aligned} 2T(z_n \otimes_s j(e_n)) &= T((z_n + j(e_n)) \otimes (z_n + j(e_n))) \\ &\quad - T(z_n \otimes z_n) - T(j(e_n) \otimes j(e_n)) \\ &= (z_m^*(z_n)h^*(e_m^*)(j(e_n)))_{m=1}^\infty + (z_m^*(j(e_n))h^*(e_m^*)(z_n))_{m=1}^\infty \\ &= e_n + (z_m^*(j(e_n))h^*(e_m^*)(z_n))_{m=1}^\infty. \end{aligned}$$

By the DPP of X , we have $h^*(e_n^*)(z_n) \rightarrow 0$, so

$$\limsup_{n \rightarrow \infty} \|T(z_n \otimes_s j(e_n))\| \geq \frac{1}{2} \limsup_{n \rightarrow \infty} |1 + z_n^*(j(e_n))h^*(e_n^*)(z_n)| = \frac{1}{2},$$

and T is not completely continuous. $\square$

Examples 2. The following spaces X with the DPP satisfy the conditions of Theorem 1:

(a) Every infinite dimensional $\mathcal{L}_1$ -space X without the Schur property ([DJT, Corollary 11.7], [LP, Proposition 7.3]).

(b) The dual $X = A^*$ of the disc algebra A ([P3, Corollaries 8.1 and 8.4], [W, Corollary III.I.14]). Since A contains a copy of ℓ_1 [P3, Corollary 3.1], A^* does not have the Schur property.

(c) $X = L_1/H_1$ (see [P3, Corollary 8.1], [W, Corollary III.I.14]). Since L_1/H_1 contains a copy of L_1 [B4], it does not have the Schur property.

(d) $X = (H_\infty)^*$ (see [B3], [B5, Corollary 5.4], [B5, comment after Corollary 2.11]). Recall that H_∞ is the dual of L_1/H_1 ([P3, page 11], [Pi, page 84, Remark after Theorem 6.17]).

Note that A^* , $(H_\infty)^*$ and L_1/H_1 are not $\mathcal{L}_1$ -spaces, since they do not have local unconditional structure [Pi, Theorem 8.18 and page 110, Remarks].

(e) The predual X (with the DPP and without the Schur property) of a C^* -algebra $\mathcal{A}$ (see [CI] for the DPP of C^* -algebras and [To, Proposition 3.2] for the cotype 2 property). Note that $\mathcal{A}$ contains a copy of

c_0 [R, Proposition 2.19], so X contains a complemented copy of ℓ_1 [D2, Theorem V.10].

(f) If Z is a Banach space with the DPP and cotype 2 not having the Schur property, then $X = \ell_1(Z)$ has the Orlicz property [Pi, page 83, lines 1–2], the DPP, and does not have the Schur property [Bo, Corollary 2.4(c)].

(g) $X = L_1/S$ where S is a reflexive subspace of $L_1 = L_1(\mu)$ (see [D1, Theorem 9] for the DPP and [Pi, page 82] for the cotype 2 property). Note that, by the next Lemma, X contains a complemented copy of ℓ_1 . Note also that, if S is infinite dimensional, X is not an $\mathcal{L}_1$ -space [Pi, page 82].

Recall that a Banach space X has *property* (V^*) if whenever a set $K \subset X$ satisfies

$$\limsup_n \sup_{x \in K} \langle x, x_n^* \rangle = 0$$

for every weakly unconditionally Cauchy series $\sum x_n^*$ in X^* , K is weakly sequentially compact. Reflexive spaces and $L_1(\mu)$ -spaces have property (V^*) [P1].

Lemma 3. *Let X be a nonreflexive Banach space with property (V^*) . Let Y be a reflexive subspace of X . Then X/Y contains a complemented copy of ℓ_1 .*

PROOF. The space X/Y has property (V^*) [GS, Theorem III.3]. Since reflexivity is a three-space property [CasG, 4.1], X/Y is not reflexive. Therefore, by [GS, Proposition III.1], X/Y contains a complemented copy of ℓ_1 . $\square$

Corollary 4. *Suppose that X^* has the Orlicz property, does not have the Schur property, contains a complemented copy of ℓ_1 , and has the approximation property. Then the space $\mathcal{P}_{\text{wb}}(^2X)$ does not have the DPP.*

PROOF. By the approximation property of X^* , the space $\mathcal{P}_{\text{wb}}(^2X)$ coincides with the space of approximable polynomials [AP, Proposition 2.7], and the latter is isomorphic to $X^* \otimes_{\epsilon, s} X^*$ [F, Proposition 3.2]. Hence, it is enough to apply Theorem 1. $\square$

Examples 5. The following spaces X satisfy the conditions of Corollary 4:

- (a) Every $\mathcal{L}_\infty$ -space X containing a copy of ℓ_1 .
- (b) $X = A$, the disc algebra. Recall that $A^* = L_1/H_1^0 \oplus_1 V_{\text{sing}}$ [P3, (1.2)], where V_{sing} is an $L_1(\nu)$ -space, and L_1/H_1^0 has the approximation property [P3, §10]. Given two Banach spaces Y and Z with the approximation property, it is easy to see that $Y \oplus_1 Z$ has the approximation property (use, for instance, the definition given in [Ry2, Proposition 4.1(iii)]).
- (c) Every C^* -algebra X with the DPP (by [CIW], this implies that X^* has the DPP), containing a copy of ℓ_1 , such that X^* has the approximation property.

Recall that it is unknown if H_∞ has the approximation property [P3, §10].

Theorem 1 and Corollary 4 are also true if the Orlicz property is replaced by having finite cotype $q \geq 2$. The operator T in the proof of Theorem 1 would then take values in ℓ_q , and we should use the fact that the identity on X is absolutely $(q, 1)$ -summing [DJT, Corollary 11.17]. However, we do not know if there are Banach spaces with finite cotype and the DPP that fail the Orlicz property.

Theorem 6. *Suppose that X^* does not have the Schur property and contains a complemented copy of ℓ_1 , and X^{**} contains no complemented copy of ℓ_1 . Then the space $\mathcal{P}({}^2X)$ does not have the DPP.*

PROOF. Consider the operators

$$\ell_1 \xrightarrow{j} X^* \xrightarrow{h} \ell_1$$

such that $h \circ j = I_{\ell_1}$, and their adjoints

$$\ell_\infty \xrightarrow{h^*} X^{**} \xrightarrow{j^*} \ell_\infty.$$

Clearly, $\langle j(e_n), h^*(e_n^*) \rangle = 1$.

Let $(\phi_n) \subset X^*$ be a weakly null normalized, basic sequence, with coefficient functionals $(z_n) \subset X^{**}$ so that $\|z_n\| \leq C$.

Define $P_n \in \mathcal{P}_{\text{wb}}({}^2X)$ by

$$P_n(x) = \langle \phi_n, x \rangle \langle j(e_n), x \rangle.$$

Denote by $\tilde{P}_n$ the natural extension of P_n to X^{**} by weak-star continuity on bounded sets. For every $z \in X^{**}$, we have

$$|\tilde{P}_n(z)| = |\langle \phi_n, z \rangle \langle j(e_n), z \rangle| \leq \|j\| \cdot \|z\| \cdot |\langle \phi_n, z \rangle| \xrightarrow{n} 0.$$

Then, the sequence (P_n) is weakly null in $\mathcal{P}_{\text{wb}}(^2X)$ [GG2, Corollary 5], and so in $\mathcal{P}(^2X)$.

Consider

$$\Phi_n := z_n \otimes h^*(e_n^*) + h^*(e_n^*) \otimes z_n \in X^{**} \otimes_{\pi, s} X^{**}.$$

Let $Q \in (X^{**} \otimes_{\pi, s} X^{**})^* \subset \mathcal{L}(X^{**}, X^{***})$. Since X^{***} contains no copy of ℓ_∞ , the operator $Q \circ h^*$ is completely continuous, so

$$|Q(z_n \otimes h^*(e_n^*))| = |\langle Q(h^*(e_n^*)), z_n \rangle| \leq C \cdot \|Q(h^*(e_n^*))\| \xrightarrow[n]{} 0,$$

and (Φ_n) is weakly null.

Define

$$S : X^{**} \otimes_{\pi, s} X^{**} \longrightarrow \mathcal{P}(^2X)^*$$

by

$$\left\langle S\left(\sum_{i=1}^m \lambda_i w_i \otimes w_i\right), P \right\rangle = \sum_{i=1}^m \lambda_i \tilde{P}(w_i),$$

where $\tilde{P} \in \mathcal{P}(^2X^{**})$ is the Aron–Berner extension of P [DG, Theorem 3].

We have

$$\begin{aligned} & \left\| S\left(\sum_{i=1}^m \lambda_i w_i \otimes w_i\right) \right\| \\ &= \sup \left\{ \left| \sum_{i=1}^m \lambda_i \tilde{P}(w_i) \right| : P \in \mathcal{P}(^2X), \|P\| \leq 1 \right\} \leq \sum_{i=1}^m |\lambda_i| \cdot \|w_i\|^2, \end{aligned}$$

from which we obtain that S is continuous.

Since X^* is complemented in $\mathcal{P}(^2X)$ [AS, Proposition 5.3], if X^* does not have the DPP, neither does $\mathcal{P}(^2X)$ and the proof is finished. Assume now that X^* has the DPP, then

$$\begin{aligned} \langle S(z_n \otimes h^*(e_n^*) + h^*(e_n^*) \otimes z_n), P_n \rangle &= 2\widehat{\tilde{P}}_n(z_n, h^*(e_n^*)) \\ &= \langle \phi_n, z_n \rangle \langle j(e_n), h^*(e_n^*) \rangle + \langle \phi_n, h^*(e_n^*) \rangle \langle j(e_n), z_n \rangle \xrightarrow[n]{} 1, \end{aligned}$$

and the space $\mathcal{P}(^2X)$ does not have the DPP. $\square$

Remark 7. (a) Theorem 6 is also true for the space $\mathcal{P}_{\text{wb}}(^2X)$.
 (b) Theorem 6 is also true for the spaces $\mathcal{P}(^mX)$ and $\mathcal{P}_{\text{wb}}(^mX)$, where $m \in \mathbb{N}$ ($m \geq 2$).

Examples 8. The following spaces X satisfy the conditions of Theorem 6:

- (a) Every $\mathcal{L}_\infty$ -space X containing a copy of ℓ_1 .
- (b) $X = A$, the disc algebra.
- (c) $X = H_\infty$.
- (d) Every C^* -algebra X with the DPP containing a copy of ℓ_1 .

Corollary 9. *Let X be a Banach space with the DPP such that X^* contains a complemented copy of ℓ_1 and X^{**} contains no complemented copy of ℓ_1 . The following assertions are equivalent:*

- (a) X contains no copy of ℓ_1 ;
- (b) $\mathcal{P}_{\text{wb}}(^2X)$ has the Schur property;
- (c) $\mathcal{P}_{\text{wb}}(^2X)$ has the DPP;
- (d) $\mathcal{P}(^2X)$ has the Schur property;
- (e) $\mathcal{P}(^2X)$ has the DPP;
- (f) $\mathcal{P}(^2X) = \mathcal{P}_{\text{wb}}(^2X)$;
- (g) X^* has the Schur property.

PROOF. (a) $\Leftrightarrow$ (g) is well-known [D1, Theorem 3].

(g) $\Rightarrow$ (d). By [Ry1, Corollary 3.4], $(X \otimes_\pi X)^*$ has the Schur property. Therefore, its complemented subspace $(X \otimes_{\pi,s} X)^* \cong \mathcal{P}(^2X)$ has the Schur property.

(d) $\Rightarrow$ (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (e) are obvious.

(e) $\Rightarrow$ (g). Suppose that X^* does not have the Schur property. By Theorem 6, the space $\mathcal{P}(^2X)$ does not have the DPP.

(c) $\Rightarrow$ (g) by the same argument as in (e) $\Rightarrow$ (g), using Remark 7,(a).

(a) $\Rightarrow$ (f) by [GG1, Corollary 3.8].

(f) $\Rightarrow$ (a) by [Gu, Theorem 4]. □

Remark 10. (a) The infinite-dimensional $\mathcal{L}_\infty$ -spaces satisfy all the hypotheses of Corollary 9.

(b) Corollary 9 is also true for the spaces $\mathcal{P}_{\text{wb}}(^mX)$ and $\mathcal{P}(^mX)$ ($m \in \mathbb{N}$).

(c) It was shown in [BV, Theorem 2.8] that the space $H_\infty \otimes_{\pi,s} H_\infty$ does not have the DPP, so $\mathcal{P}(^2H_\infty)$ does not have the DPP either. By Corollary 9, neither does $\mathcal{P}_{\text{wb}}(^2H_\infty)$. For the properties of H_∞ , see [B3, B5]. The same is true for the disc algebra [B5].

(d) The C^* -algebras with the DPP satisfy the conditions of Corollary 9.

(e) If X is an $\mathcal{L}_\infty$ -space without the Schur property, all the assertions of Corollary 9 are equivalent to:

(*) $X \otimes_{\pi,s} X$ has the DPP.

Indeed, (e) $\Rightarrow$ (*) is obvious. Suppose now that X contains a copy of ℓ_1 . Then, by [BV, Theorem 2.8], $X \otimes_{\pi,s} X$ does not have the DPP. Hence, (*) implies (a).

We shall now study conditions so that the duals $\mathcal{P}_{\text{wb}}(^2X)^*$ and $\mathcal{P}(^2X)^*$ fail the DPP. We give two preliminary results which may be of independent interest.

Denote by $\mathcal{L}(^2X)$ the space of all bilinear forms on X , and by $\mathcal{L}_{\text{wb}}(^2X)$ the subspace of all bilinear forms which are weakly continuous on bounded sets. We shall use the following isometric equalities: $\mathcal{L}(^2X) \equiv \mathcal{L}(X, X^*)$ and $\mathcal{L}_{\text{wb}}(^2X) \equiv \mathcal{K}(X, X^*)$ (for the latter, see [GG3, Proposition 12]).

Proposition 11. *Suppose that X^* has the bounded approximation property. Then the space $\mathcal{P}_{\text{wb}}(^2X)^*$ is isomorphic to a complemented subspace of $\mathcal{P}(^2X)^*$.*

PROOF. Consider the operators

$$\mathcal{P}_{\text{wb}}(^2X) \xrightarrow{I} \mathcal{L}_{\text{wb}}(^2X) \xrightarrow{U} \mathcal{P}_{\text{wb}}(^2X)$$

such that $I(P) = \widehat{P}$ for $P \in \mathcal{P}_{\text{wb}}(^2X)$, and $U(A) = Q$ for $A \in \mathcal{L}_{\text{wb}}(^2X)$, where $Q(x) := A(x, x)$ ($x \in X$). Then UI is the identity map on $\mathcal{P}_{\text{wb}}(^2X)$. Analogously, we define the operators

$$\mathcal{P}(^2X) \xrightarrow{J} \mathcal{L}(^2X) \xrightarrow{V} \mathcal{P}(^2X)$$

where VJ is the identity map on $\mathcal{P}(^2X)$. Note that JV leaves the symmetric bilinear forms invariant.

Using the bounded approximation property of X^* , by the proof of [J, Lemma 1], there are operators

$$\mathcal{K}(X, X^*)^* \xrightarrow{L} \mathcal{L}(X, X^*)^* \xrightarrow{R} \mathcal{K}(X, X^*)^*$$

such that RL is the identity map on $\mathcal{K}(X, X^*)^*$, R is the restriction operator, and

$$\langle L(\Phi), K \rangle = \langle \Phi, K \rangle$$

for all $\Phi \in \mathcal{K}(X, X^*)^*$ and $K \in \mathcal{K}(X, X^*) \subseteq \mathcal{L}(X, X^*)$.

Given $\phi \in \mathcal{P}_{\text{wb}}(^2X)^*$ and $P \in \mathcal{P}_{\text{wb}}(^2X)$, we have

$$\begin{aligned} \langle I^*RV^*J^*LU^*(\phi), P \rangle &= \langle RV^*J^*LU^*(\phi), \widehat{P} \rangle = \langle V^*J^*LU^*(\phi), \widehat{P} \rangle \\ &= \langle LU^*(\phi), JV(\widehat{P}) \rangle = \langle L(\phi \circ U), \widehat{P} \rangle \\ &= \langle \phi \circ U, \widehat{P} \rangle = \langle \phi, U(\widehat{P}) \rangle = \langle \phi, P \rangle, \end{aligned}$$

hence $I^*RV^*J^*LU^*$ is the identity map on $\mathcal{P}_{\text{wb}}(^2X)^*$, and $J^*LU^*I^*RV^*$ is a projection on $\mathcal{P}(^2X)^*$ with range isomorphic to $\mathcal{P}_{\text{wb}}(^2X)^*$. $\square$

Recall now that $\mathcal{P}(^m c_0) = \mathcal{P}_{\text{wb}}(^m c_0)$ [Ar, Corollary, page 215]. Let X be a closed subspace of a Banach space Y ; we say that X is *locally complemented* in Y if X^{**} is complemented in Y^{**} under the natural embedding [K, Theorem 3.5]. It is shown in [CaG] that a Banach space has the Dunford–Pettis property if and only if all its locally complemented subspaces have it. In the same paper, it is proved that $\ell_\infty \otimes_{\pi, s} \ell_\infty$ is a locally complemented subspace of $(c_0 \otimes_{\pi, s} c_0)^{**}$. Since $\ell_\infty \otimes_{\pi, s} \ell_\infty$ does not have the Dunford–Pettis property [BV, Theorem 2.6], the space $(c_0 \otimes_{\pi, s} c_0)^{**} \cong \mathcal{P}(^2 c_0)^*$ does not have it either.

Denote by $\mathcal{P}_{w^*}(^m X^{**})$ the space of all scalar-valued m -homogeneous polynomials on X^{**} such that, for every bounded net $(z_\alpha) \in X^{**}$ weak-star converging to z , we have $P(z_\alpha) \rightarrow P(z)$. It is shown in [Mo, Proposition 3] that there is a surjective isometric isomorphism

$$L : \mathcal{P}_{\text{wb}}(^m X) \longrightarrow \mathcal{P}_{w^*}(^m X^{**})$$

such that $L(P)$ is an extension of $P \in \mathcal{P}_{\text{wb}}(^m X)$ to X^{**} .

Proposition 12. *Let $m \in \mathbb{N}$ and suppose that X^* contains a complemented copy of ℓ_1 . Then $\mathcal{P}_{\text{wb}}(^m c_0)$ is isomorphic to a complemented subspace of $\mathcal{P}_{\text{wb}}(^m X)$.*

PROOF. There are operators

$$\ell_1 \xrightarrow{i} X^* \xrightarrow{\pi} \ell_1$$

such that $\pi \circ i$ is the identity map on ℓ_1 . Taking adjoints, we have

$$\ell_\infty \xrightarrow{\pi^*} X^{**} \xrightarrow{i^*} \ell_\infty$$

where $i^* \circ \pi^*$ is the identity map on ℓ_∞ .

Let

$$\mathcal{P}_{w^*}(^m \ell_\infty) \xrightarrow{L} \mathcal{P}_{w^*}(^m X^{**}) \xrightarrow{S} \mathcal{P}_{w^*}(^m \ell_\infty)$$

be the operators given by $L(P) := P \circ i^*$ for $P \in \mathcal{P}_{w^*}(^m \ell_\infty)$, and $S(Q) := Q \circ \pi^*$ for $Q \in \mathcal{P}_{w^*}(^m X^{**})$. Since i^* and π^* are weak-star-to-weak-star continuous, L and S are well defined. Then

$$S(L(P)) = S(P \circ i^*) = P \circ i^* \circ \pi^* = P \quad \text{for } P \in \mathcal{P}_{w^*}(^m \ell_\infty),$$

so $S \circ L$ is the identity map on $\mathcal{P}_{w^*}(^m \ell_\infty)$ and $L \circ S$ is a projection. Hence, $\mathcal{P}_{w^*}(^m \ell_\infty) \equiv \mathcal{P}_{\text{wb}}(^m c_0)$ is isomorphic to a complemented subspace of $\mathcal{P}_{w^*}(^m X^{**}) \equiv \mathcal{P}_{\text{wb}}(^m X)$. $\square$

Theorem 13. *Suppose that X^* has the bounded approximation property and contains a complemented copy of ℓ_1 . Then the spaces $\mathcal{P}_{\text{wb}}(^2 X)^*$ and $\mathcal{P}(^2 X)^*$ do not have the DPP.*

PROOF. By Proposition 12, $\mathcal{P}(^2 c_0)^*$ is isomorphic to a complemented subspace of $\mathcal{P}_{\text{wb}}(^2 X)^*$. Since $\mathcal{P}(^2 c_0)^*$ does not have the DPP, the space $\mathcal{P}_{\text{wb}}(^2 X)^*$ does not have it either.

Assume first that X^* has the Schur property. Then $\mathcal{P}(^2 X) = \mathcal{P}_{\text{wb}}(^2 X)$ [GG1, Corollary 3.8]. So $\mathcal{P}(^2 X)^*$ does not have the DPP.

Assume now that X^* does not have the Schur property. By Proposition 11, $\mathcal{P}_{\text{wb}}(^2 X)^*$ is isomorphic to a complemented subspace of $\mathcal{P}(^2 X)^*$. Therefore, $\mathcal{P}(^2 X)^*$ does not have the DPP. $\square$

Note that the bounded approximation property of X^* is only used to prove that $\mathcal{P}(^2X)^*$ fails the DPP when X^* does not have the Schur property.

Examples 14. From Corollary 9 and Theorem 13, we obtain a class of Banach spaces X such that $\mathcal{P}_{\text{wb}}(^mX)$ and $\mathcal{P}(^mX)$ have the Schur property while $\mathcal{P}_{\text{wb}}(^mX)^*$ and $\mathcal{P}(^mX)^*$ do not have the DPP. The following spaces X belong to this class:

(a) $X = C(K)$ for K a dispersed compact Hausdorff space [PS, Theorem 2].

(b) The space X constructed in [BL] not isomorphic to a complemented subspace of a $C(K)$ space, such that $X^* \equiv \ell_1$.

(c) The somewhat reflexive $\mathcal{L}_\infty$ -space X containing no copy of c_0 , constructed in [BD], such that $X^* \cong \ell_1$.

(d) $X = Y \otimes_\pi Z$, where Y^* and Z^* have the Schur property, Y^{**} (or Z^{**}) has the bounded approximation property, and Y^* (or Z^*) contains a complemented copy of ℓ_1 . Indeed, X^* has the Schur property [Ry1, Corollary 3.4]. Since Y^{**} or Z^{**} contains a (complemented) copy of ℓ_∞ , so does $Y^{**} \otimes_\pi Z^{**}$. By the bounded approximation property of Y^* or Z^* , $Y^{**} \otimes_\pi Z^{**}$ is isomorphic to a subspace of $X^{**} = (Y \otimes_\pi Z)^{**}$ [CaG]. Therefore, X^* contains a complemented copy of ℓ_1 .

We can take as Y a subspace of c_0 which is not an $\mathcal{L}_\infty$ -space (for instance, Y may be a subspace of c_0 without the approximation property [LT, Theorem 2.d.6]). Since c_0 has the hereditary DPP [D1, Theorem 4], Y^* has the Schur property. Let $Z = c_0$. Then $X := Y \otimes_\pi Z$ belongs to our class.

We could also take as Y Hagler's space [H] whose dual has the Schur property, and $Z = c_0$. Since Y^{**} contains a complemented copy of ℓ_1 [H, Lemma 9], Y is not an $\mathcal{L}_\infty$ -space.

Recall that the first example of a Banach space with the DPP such that its dual fails the DPP was given by STEGALL [S].

It is easy to see [CG, page 233] that, if X is an $\mathcal{L}_1$ -space, then $\mathcal{P}_{\text{wb}}(^mX)$ and $\mathcal{P}(^mX)$ are $\mathcal{L}_\infty$ -spaces, and so these spaces and all their duals have the DPP.

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References

- [A] K. T. ANDREWS, Dunford–Pettis sets in spaces of Bochner integrable functions, *Math. Ann.* **241** (1979), 35–41.
- [AF] J. M. ANSEMIL and K. FLORET, The symmetric tensor product of a direct sum of locally convex spaces, *Studia Math.* **129** (1998), 285–295.
- [Ar] R. M. ARON, Compact polynomials and compact differentiable mappings between Banach spaces, *Sém. P. Lelong (Analyse)*, 1974/75, Vol. 524, Lecture Notes in Math., *Springer, Berlin*, 1976, 213–222.
- [AHV] R. M. ARON, C. HERVÉS, and M. VALDIVIA, Weakly continuous mappings on Banach spaces, *J. Funct. Anal.* **52** (1983), 189–204.
- [AP] R. M. ARON and J. B. PROLLA, Polynomial approximation of differentiable functions on Banach spaces, *J. Reine Angew. Math.* **313** (1980), 195–216.
- [AS] R. M. ARON and M. SCHOTTENLOHER, Compact holomorphic mappings on Banach spaces and the approximation property, *J. Funct. Anal.* **21** (1976), 7–30.
- [BL] Y. BENYAMINI and J. LINDENSTRAUSS, A predual of ℓ_1 which is not isomorphic to a $C(K)$ space, *Israel J. Math.* **13** (1973), 246–254.
- [Bo] F. BOMBAL, Distinguished subsets in vector sequence spaces, *Progress in Functional Analysis*, Vol. 170, North-Holland Math. Studies, (K. D. Bierstedt, J. Bonnet, J. Horváth, and M. Maestre, eds.), *North-Holland, Amsterdam*, 1992, 293–306.
- [BV] F. BOMBAL and I. VILLANUEVA, On the Dunford–Pettis property of the tensor product of $C(K)$ spaces, *Proc. Amer. Math. Soc.* **129** (2001), 1359–1363.
- [B1] J. BOURGAIN, New classes of $\mathcal{L}_p$ -spaces, Vol. 889, Lecture Notes in Math., *Springer, Berlin*, 1981.
- [B2] J. BOURGAIN, On the Dunford–Pettis property, *Proc. Amer. Math. Soc.* **81** (1981), 265–272.
- [B3] J. BOURGAIN, H^∞ is a Grothendieck space, *Studia Math.* **75** (1983), 193–216.
- [B4] J. BOURGAIN, Embedding L^1 in L^1/H^1 , *Trans. Amer. Math. Soc.* **278** (1983), 689–702.
- [B5] J. BOURGAIN, New Banach space properties of the disc algebra and H^∞ , *Acta Math.* **152** (1984), 1–48.
- [BD] J. BOURGAIN and F. DELBAEN, A class of special $\mathcal{L}_\infty$ -spaces, *Acta Math.* **145** (1980), 155–176.
- [CaG] F. CABELLO SÁNCHEZ and R. GARCÍA, The bidual of a tensor product of Banach spaces (*to appear in* *Rev. Math. Iberoamericana*).
- [CasG] J. M. F. CASTILLO and M. GONZÁLEZ, Three-space Problems in Banach Space Theory, Vol. 1667, Lecture Notes in Math., *Springer, Berlin*, 1997.
- [CI] C.-H. CHU and B. IOCHUM, The Dunford–Pettis property in C^* -algebras, *Studia Math.* **97** (1990), 59–64.

- [CIW] C.-H. CHU, B. IOCHUM, and S. WATANABE, C^* -algebras with the Dunford–Pettis property, *Function Spaces*, Vol. 136, Lecture Notes Pure Appl. Math., (K. Jarosz, ed.), *Marcel Dekker, New York*, 1992, 67–70.
- [CG] R. CILIA and J. M. GUTIÉRREZ, Extension and lifting of weakly continuous polynomials, *Studia Math.* **169** (2005), 229–241.
- [DG] A. M. DAVIE and T. W. GAMELIN, A theorem on polynomial-star approximation, *Proc. Amer. Math. Soc.* **106** (1989), 351–356.
- [DF] A. DEFANT and K. FLORET, *Tensor Norms and Operator Ideals*, Vol. 176, Math. Studies, *North-Holland, Amsterdam*, 1993.
- [D1] J. DIESTEL, A survey of results related to the Dunford–Pettis property, *Proc. Conf. on Integration, Topology and Geometry in Linear Spaces*, Chapel Hill 1979, Vol. 2, *Contemp. Math.*, (W. H. Graves, ed.), *American Mathematical Society, Providence RI*, 1980, 15–60.
- [D2] J. DIESTEL, *Sequences and Series in Banach Spaces*, Vol. 92, Graduate Texts in Math., *Springer, Berlin*, 1984.
- [DJT] J. DIESTEL, H. JARCHOW, and A. TONGE, *Absolutely Summing Operators*, Vol. 43, *Cambridge Stud. Adv. Math.*, *Cambridge University Press, Cambridge*, 1995.
- [DU] J. DIESTEL and J. J. UHL, JR., *Vector Measures*, Vol. 15, Math. Surveys Monographs, *American Mathematical Society, Providence RI*, 1977.
- [Di] S. DINEEN, *Complex Analysis on Infinite Dimensional Spaces*, Springer Monogr. Math., *Springer, Berlin*, 1999.
- [DPR] E. DUBINSKY, A. PEŁCZYŃSKI, and H. P. ROSENTHAL, On Banach spaces X for which $\Pi_2(\mathcal{L}_\infty, X) = B(\mathcal{L}_\infty, X)$, *Studia Math.* **44** (1972), 617–648.
- [F] K. FLORET, Natural norms on symmetric tensor products of normed spaces, *Note Mat.* **17** (1997), 153–188.
- [GS] G. GODEFROY and P. SAAB, Quelques espaces de Banach ayant les propriétés (V) ou (V*) de A. Pełczyński, *C. R. Acad. Sci. Paris Sér. I Math.* **303** (1986), 503–506.
- [GG1] M. GONZÁLEZ and J. M. GUTIÉRREZ, Weakly continuous mappings on Banach spaces with the Dunford–Pettis property, *J. Math. Anal. Appl.* **173** (1993), 470–482.
- [GG2] M. GONZÁLEZ and J. M. GUTIÉRREZ, Weak compactness in spaces of differentiable mappings, *Rocky Mountain J. Math.* **25** (1995), 619–634.
- [GG3] M. GONZÁLEZ and J. M. GUTIÉRREZ, Factorization of weakly continuous holomorphic mappings, *Studia Math.* **118** (1996), 117–133.
- [GG4] M. GONZÁLEZ and J. M. GUTIÉRREZ, The Dunford–Pettis property on tensor products, *Math. Proc. Cambridge Philos. Soc.* **131** (2001), 185–192.
- [Gr] A. GROTHENDIECK, Sur les applications linéaires faiblement compactes d’espaces du type $C(K)$, *Canad. J. Math.* **5** (1953), 129–173.
- [Gu] J. M. GUTIÉRREZ, Weakly continuous functions on Banach spaces not containing ℓ_1 , *Proc. Amer. Math. Soc.* **119** (1993), 147–152.

- [H] J. HAGLER, A counterexample to several questions about Banach spaces, *Studia Math.* **60** (1977), 289–308.
- [J] J. JOHNSON, Remarks on Banach spaces of compact operators, *J. Funct. Anal.* **32** (1979), 304–311.
- [K] N. J. KALTON, Locally complemented subspaces and $\mathcal{L}_p$ -spaces for $0 < p < 1$, *Math. Nachr.* **115** (1984), 71–97.
- [Kö] G. KÖTHE, Topological Vector Spaces I, Vol. 159, Grundlehren Math. Wiss., Springer, Berlin, 1983.
- [LP] J. LINDENSTRAUSS and A. PELCZYŃSKI, Absolutely summing operators in $\mathcal{L}_p$ -spaces and their applications, *Studia Math.* **29** (1968), 275–326.
- [LT] J. LINDENSTRAUSS and L. TZAFRIRI, Classical Banach Spaces I, Vol. 92, Ergeb. Math. Grenzgeb., Springer, Berlin, 1977.
- [Mo] L. A. MORAES, Extension of holomorphic mappings from E to E'' , *Proc. Amer. Math. Soc.* **118** (1993), 455–461.
- [M] J. MUJICA, Complex Analysis in Banach Spaces, Vol. 120, Math. Studies, North-Holland, Amsterdam, 1986.
- [P1] A. PELCZYŃSKI, Banach spaces on which every unconditionally converging operator is weakly compact, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* **10** (1962), 641–648.
- [P2] A. PELCZYŃSKI, On weakly compact polynomial operators on B -spaces with Dunford–Pettis property, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* **11** (1963), 371–378.
- [P3] A. PELCZYŃSKI, Banach Spaces of Analytic Functions and Absolutely Summing Operators, Vol. 30, CBMS Reg. Conf. Ser. Math., American Mathematical Society, Providence RI, 1977.
- [PS] A. PELCZYŃSKI and Z. SEMADENI, Spaces of continuous functions (III) (Spaces $C(\Omega)$ for Ω without perfect subsets), *Studia Math.* **18** (1959), 211–222.
- [Pi] G. PISIER, Factorization of Linear Operators and Geometry of Banach Spaces, Vol. 60, CBMS Reg. Conf. Ser. Math., American Mathematical Society, Providence RI, 1986.
- [R] H. P. ROSENTHAL, The complete separable extension property, *J. Operator Theory* **43** (2000), 329–374.
- [Ry1] R. A. RYAN, The Dunford–Pettis property and projective tensor products, *Bull. Polish Acad. Sci. Math.* **35** (1987), 785–792.
- [Ry2] R. A. RYAN, Introduction to Tensor Products of Banach Spaces, Springer Monogr. Math., Springer, Berlin, 2002.
- [S] C. STEGALL, Duals of certain spaces with the Dunford–Pettis property, *Notices Amer. Math. Soc.* **19** (1972), A-799.
- [T1] M. TALAGRAND, La propriété de Dunford–Pettis dans $C(K, E)$ et $L^1(E)$, *Israel J. Math.* **44** (1983), 317–321.
- [T2] M. TALAGRAND, Cotype of operators from $C(K)$, *Invent. Math.* **107** (1992), 1–40.

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- [To] N. TOMCZAK-JAEGERMANN, The moduli of smoothness and convexity and the Rademacher averages of trace classes S_p ($1 \leq p < \infty$), *Studia Math.* **50** (1974), 163–182.
- [W] P. WOJTASZCZYK, Banach Spaces for Analysts, Cambridge Studies in Adv. Math., Vol. 25, *Cambridge University Press, Cambridge*, 1991.

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