

Certain curvature restrictions on a quasi Einstein manifold

By GOPAL CHANDRA GHOSH (Kalyani), U. C. DE (Kalyani)
and T. Q. BINH (Debrecen)

Abstract. Quasi Einstein manifold is a simple and natural generalization of Einstein manifold. We prove that a quasi-conformally flat quasi Einstein manifold is of quasi-constant curvature, and that a conformally flat pseudo symmetric manifold is a quasi Einstein manifold. Also conditions are found for a quasi Einstein manifold to be quasi conformally conservative.

Introduction

The notion of quasi Einstein manifold was introduced by M. C. CHAKI and R. K. MAITY [1]. A non-flat Riemannian manifold (M^n, g) ($n > 2$) is defined to be a quasi Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$S(X, Y) = a g(X, Y) + b A(X)A(Y) \quad (1)$$

where a, b are scalars of which $b \neq 0$ and A is a non-zero 1-form such that

$$g(X, U) = A(X) \quad (2)$$

for all vector fields X ; U being a unit vector field. In such a case a, b are called associated scalars. A is called the associated 1-form and U is called

Mathematics Subject Classification: 53C25.

Key words and phrases: quasi Einstein manifold, quasi-constant curvature, quasi-conformal curvature tensor.

the generator of the manifold. An n -dimensional manifold of this kind is denoted by the symbol $(QE)_n$. If either the 1-form A or the associated scalar b , or both of them are zero, then the manifold reduces to an Einstein manifold.

A Riemannian manifold of quasi-constant curvature was given by B. Y. CHEN and K. YANO [2] as a conformally flat manifold with the curvature tensor $'R$ of type $(0, 4)$ which satisfies the condition

$$\begin{aligned} 'R(X, Y, Z, W) = & p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ & + q[g(X, W)T(Y)T(Z) - g(X, Z)T(Y)T(W) \\ & + g(Y, Z)T(X)T(W) - g(Y, W)T(X)T(Z)] \end{aligned} \quad (3)$$

where $'R(X, Y, Z, W) = g(R(X, Y)Z, W)$, R is the curvature tensor of type $(1, 3)$, p, q are scalar functions, T is a non-zero 1-form defined by

$$g(X, \tilde{\rho}) = T(X), \quad (4)$$

and $\tilde{\rho}$ is a unit vector field.

It can be easily seen that if the curvature tensor $'R$ is of the form (3), then the manifold is conformally flat. On the other hand, GH. VRANCEANU [3] defined the notion of almost constant curvature. Later A. L. MOCANU [4] pointed out that the manifold introduced by CHEN and YANO and the manifold introduced by GH. VRANCEANU are the same. If $q = 0$, then it reduces to a manifold of constant curvature.

The notion of quasi-conformal curvature tensor

$$\begin{aligned} C^*(X, Y)Z = & a_1R(X, Y)Z + b_1[S(Y, Z)X - S(X, Z)Y \\ & + g(Y, Z)QX - g(X, Z)QY] \\ & - \frac{r}{n} \left[\frac{a_1}{n-1} + 2b_1 \right] [g(Y, Z)X - g(X, Z)Y] \end{aligned} \quad (5)$$

was given by YANO and SAWAKI [5]. Here a_1 and b_1 are constants, R is the Riemannian curvature tensor of type $(1, 3)$, S is the Ricci tensor of type $(0, 2)$, Q is the Ricci operator and r is the scalar curvature of the manifold. If $a_1 = 1$ and $b_1 = -\frac{1}{n-2}$, then (5) takes the form

$$C^*(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y]$$

$$+ g(Y, Z)QX - g(X, Z)QY] \quad (6)$$

$$- \frac{r}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y] = C(X, Y)Z,$$

where C is the conformal curvature tensor [6]. Thus the conformal curvature tensor C is a particular case of the tensor C^* . For this reason C^* is called the quasi-conformal curvature tensor.

A manifold (M^n, g) ($n > 3$) shall be called quasi-conformally flat or quasi-conformally conservative according as $C^* = 0$ or $\text{div } C^* = 0$. It is known [7] that a quasi-conformally flat space is either conformally flat or Einstein. Since an Einstein manifold need not be conformally flat, a quasi-conformally flat manifold need not be conformally flat.

A non-flat Riemannian manifold (M^n, g) ($n \geq 2$) is said to be a pseudo symmetric manifold [8] if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y, Z)W = 2B(X)R(Y, Z)W + B(Y)R(X, Z)W + B(Z)R(Y, X)W \\ + B(W)R(Y, Z)X + g(R(Y, Z)W, X)\tilde{U} \quad (7)$$

where B is a non-zero 1-form,

$$g(X, \tilde{U}) = B(X) \quad \forall X \quad (8)$$

and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g . Such a manifold is denoted by $(PS)_n$ ($n \geq 2$). It may be mentioned that CHAKI's pseudo symmetric manifold is different from that of R. DESZCZ [9].

It is known [10, p. 93] that a conformally flat Einstein manifold is of constant curvature. In the present paper we have generalized this result to a quasi-conformally flat quasi Einstein manifold and we prove that a quasi-conformally flat $(QE)_n$ ($n > 3$) is a manifold of quasi-constant curvature. In Section 2 we look for a sufficient condition in order that a $(QE)_n$ ($n > 3$) may be quasi-conformally conservative. Next we study conformally flat pseudo symmetric manifolds and prove that such a manifold is a quasi Einstein manifold. Finally we obtain a sufficient condition for a pseudo symmetric manifold to be a quasi Einstein manifold.

1. Quasi-conformally flat quasi Einstein manifold

From (5) we get

$$\begin{aligned} {}'C^*(X, Y, Z, W) &= a_1 {}'R(X, Y, Z, W) + b_1 [S(Y, Z)g(X, W) \\ &\quad - S(X, Z)g(Y, W) + S(X, W)g(Y, Z) - S(Y, W)g(X, Z)] \\ &\quad - \frac{r}{n} \left[\frac{a_1}{n-1} + 2b_1 \right] [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)], \end{aligned} \quad (1.1)$$

where ${}'C^*(X, Y, Z, W) = g(C^*(X, Y)Z, W)$ and ${}'R(X, Y, Z, W) = g(R(X, Y)Z, W)$. If the manifold is quasi-conformally flat, then we have

$$\begin{aligned} {}'R(X, Y, Z, W) &= \frac{b_1}{a_1} [S(X, Z)g(Y, W) - S(Y, Z)g(X, W) \\ &\quad + S(Y, W)g(X, Z) - S(X, W)g(Y, Z)] \\ &\quad - \frac{r}{n a_1} \left[\frac{a_1}{n-1} + 2b_1 \right] [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \end{aligned} \quad (1.2)$$

Using (1) in (1.2) we have

$$\begin{aligned} {}'R(X, Y, Z, W) &= - \left[2b_1 a + \frac{r}{n} \left(\frac{a_1}{n-1} + 2b_1 \right) \right] [g(Y, Z)g(X, W) \\ &\quad - g(X, Z)g(Y, W)] - b_1 b [g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z) \\ &\quad + g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W)], \end{aligned} \quad (1.3)$$

which implies that the manifold is a manifold of quasi-constant curvature. Hence we can state that

Theorem 1. *A quasi-conformally flat quasi Einstein manifold $(QE)_n$ ($n > 3$) is a manifold of quasi-constant curvature.*

2. $(QE)_n$ ($n > 3$) with divergence free quasi-conformal curvature tensor

In this section we look for a sufficient condition in order that a $(QE)_n$ ($n > 3$) may be quasi-conformally conservative. Quasi-conformal curvature tensor is said to be conservative [11] if divergence of C^* vanishes, i.e., $\text{div } C^* = 0$.

In a $(QE)_n$ if both a and b are constant, then contracting (1) we have $r = an + b$, i.e. $r = \text{constant}$, where r is the scalar curvature, i.e., $dr = 0$. Using this from (5) we obtain

$$\begin{aligned} (\nabla_W C^*)(X, Y, Z) &= a_1(\nabla_W R)(X, Y)Z + b_1[(\nabla_W S)(Y, Z)X \\ &\quad - (\nabla_W S)(X, Z)Y + g(Y, Z)(\nabla_W Q)X - g(X, Z)(\nabla_W Q)Y]. \end{aligned} \quad (2.1)$$

We know that $(\text{div } R)(X, Y, Z) = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)$ [10], and from (1) we get $(\nabla_X S)(Y, Z) = b[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)]$, since both a and b are constant. Hence contracting (2.1) we obtain

$$\begin{aligned} (\text{div } C^*)(X, Y, Z) &= 2b(a_1 + b_1)[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y) \\ &\quad - (\nabla_Y A)(X)A(Z) - (\nabla_Y A)(Z)A(X)] + bb_1[(\nabla_U A)(X) \\ &\quad + A(X)\text{div } U]g(Y, Z) - bb_1[(\nabla_U A)(Y) + A(Y)\text{div } U]g(X, Z). \end{aligned} \quad (2.2)$$

Imposing the condition that the generator U of the manifold is a recurrent vector field [12] with associated 1-form A not being the 1-form of recurrence, gives $\nabla_X U = B(X)U$, where B is the 1-form of recurrence. Hence $g(\nabla_X U, Y) = g(B(X)U, Y)$, that is,

$$(\nabla_X A)(Y) = B(X)A(Y). \quad (2.3)$$

In view of (2.3), (2.2) is expressed as follows

$$\begin{aligned} (\text{div } C^*)(X, Y, Z) &= 2b(a_1 + b_1)[B(X)A(Y)A(Z) - B(Z)A(X)A(Y)] \\ &\quad + 2bb_1B(U)A(X)g(Y, Z) - 2bb_1g(X, Z)B(U)A(Y). \end{aligned} \quad (2.4)$$

Since $(\nabla_X A)(U) = 0$, it follows from (2.3) that $B(X) = 0$. Hence from (2.4) it follows that $(\text{div } C^*)(X, Y, Z) = 0$. Thus we can state the following:

Theorem 2. *If in a $(QE)_n$ ($n > 3$) the associated scalars are constants and the generator U of the manifold is a recurrent vector field with the associated 1-form A not being the 1-form of recurrence, then the manifold is quasi-conformally conservative.*

3. Conformally flat pseudo symmetric manifolds

It is known [8] that in a conformally flat $(PS)_n$ ($n \geq 3$)

$$(n-1)B(X)S(Y, Z) - (n-1)B(Y)S(X, Z) - rB(X)g(Y, Z) \\ + rB(Y)g(X, Z) + D(X)g(Y, Z) - D(Y)g(X, Z) = 0, \quad (3.1)$$

where D is a 1-form defined by

$$D(X) = B(QX), \quad (3.2)$$

Q denotes the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S , i.e. $g(QX, Y) = S(X, Y)$ for every vector fields X, Y . Putting $Z = \tilde{U}$ in (3.1), where $g(X, \tilde{U}) = B(X)$ we get

$$B(X)D(Y) - B(Y)D(X) = 0. \quad (3.3)$$

$$\text{Hence } D(X) = tB(X), \quad (3.4)$$

where t is a scalar. Using (3.4), it follows from (3.1) that

$$S(Y, Z) = \frac{r-t}{n-1}g(Y, Z) + \frac{nt-r}{(n-1)B(\tilde{U})}B(Y)B(Z) \quad (3.5)$$

which implies that the manifold is a quasi Einstein manifold. Thus we state

Theorem 3. *A conformally flat pseudo symmetric manifold $(PS)_n$ ($n \geq 3$) is a quasi Einstein manifold.*

4. Sufficient condition for a pseudo symmetric manifold to be a quasi Einstein manifold

Now contracting (7) we get

$$(\nabla_X S)(Y, Z) = 2B(X)S(Y, Z) + B(Y)S(X, Z) + B(Z)S(Y, X) \\ + B(R(X, Y)Z) + B(R(X, Z)Y). \quad (4.1)$$

In a Riemannian manifold, a vector field ρ defined by $g(X, \rho) = A(X)$ for any vector field X is said to be a concircular vector field [12] if

$$(\nabla_X A)(Y) = \alpha g(X, Y) + \omega(X)A(Y), \quad (4.2)$$

where α is a non-zero scalar and ω is a closed 1-form. If ρ is a unit vector, then the equation (4.2) can be written as

$$(\nabla_X A)(Y) = \alpha[g(X, Y) - A(X)A(Y)]. \quad (4.3)$$

We suppose that a $(PS)_n$ admits a unit concircular vector field defined by (4.3), where α is a non-zero constant. Applying the Ricci identity to (4.3) we obtain

$$A(R(X, Y)Z) = -\alpha^2[g(X, Z)A(Y) - g(Y, Z)A(X)]. \quad (4.4)$$

Putting $Y = Z = e_i$ in (4.4), and taking summation over i , $1 \leq i \leq n$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, we get

$$A(QX) = (n-1)\alpha^2 A(X),$$

where Q is the Ricci operator defined by $g(QX, Y) = S(X, Y)$, which implies

$$S(X, \rho) = (n-1)\alpha^2 A(X). \quad (4.5)$$

From (4.5) we have

$$(\nabla_Y S)(X, \rho) = (n-1)\alpha^3 g(X, Y) - \alpha S(X, Y). \quad (4.6)$$

Using (4.4) we obtain

$$g(R(X, Y)Z, \rho) = -\alpha^2[g(X, Z)A(Y) - g(Y, Z)A(X)]$$

$$\text{or, } g(R(Z, \rho)X, Y) = -\alpha^2[g(X, Z)g(Y, \rho) - g(Z, Y)A(X)]$$

$$\text{or, } R(Z, \rho)X = -\alpha^2[g(X, Z)\rho - A(X)Z],$$

which implies

$$B(R(Z, \rho)X) = -\alpha^2[g(X, Z)B(\rho) - A(X)B(Z)] \quad (4.7)$$

$$\text{i.e., } B(R(X, \rho)Y) = -\alpha^2[g(X, Y)B(\rho) - A(Y)B(X)].$$

Similarly we have

$$B(R(X, Y)\rho) = -\alpha^2[A(Y)B(X) - A(X)B(Y)]. \quad (4.8)$$

In (4.1) putting $Z = \rho$ and using (4.5), (4.6), (4.7) and (4.8) we have

$$\begin{aligned} -(\alpha + B(\rho))S(X, Y) &= -[\alpha^2 B(\rho) + (n-1)\alpha^3]g(X, Y) \\ &\quad + 2(n-1)\alpha^2 B(X)A(Y) + n\alpha^2 B(Y)A(X). \end{aligned} \quad (4.9)$$

Putting $Y = \rho$ in (4.9) and using (4.5) we have

$$\begin{aligned} B(\rho)A(X) + (n-1)A(X) &= 0 \quad \forall X \\ \text{i.e. } B(X) &= -\frac{B(\rho)}{n-1}A(X). \end{aligned} \quad (4.10)$$

Let us impose the condition

$$\alpha + B(\rho) \neq 0. \quad (4.11)$$

Putting (4.10) in (4.9) we obtain

$$\begin{aligned} S(X, Y) &= \frac{\alpha^2[B(\rho) + (n-1)\alpha]}{\alpha + B(\rho)}g(X, Y) + \frac{(3n-2)B(\rho)}{(\alpha + B(\rho))(n-1)}A(X)A(Y) \\ \text{i.e. } S(X, Y) &= ag(X, Y) + bA(X)A(Y), \end{aligned} \quad (4.12)$$

where $a = \frac{\alpha^2[B(\rho) + (n-1)\alpha]}{\alpha + B(\rho)}$ and $b = \frac{(3n-2)B(\rho)}{(\alpha + B(\rho))(n-1)}$.

Thus we can state

Theorem 4. *If a pseudo symmetric manifold admits a unit concircular vector field whose associated scalar is a non-zero constant and satisfying the condition (4.11), then the manifold reduces to a quasi Einstein manifold.*

ACKNOWLEDGEMENT. The first author is grateful to C. S. I. R. for providing a fellowship under sanction no. F. No. 9/106(72)/2003-EMR-I

References

- [1] M. C. CHAKI and R. K. MAITY, On quasi Einstein manifolds, *Publ. Math. Debrecen* **57** (2000), 297–306.
- [2] B. Y. CHEN and K. YANO, Hypersurfaces of a conformally flat space, *Tensor N.S.* **26** (1972), 318–322.
- [3] GH. VRANCEANU, Lecons des Geometrie Differential, Vol. 4, *Ed. de l'Academie, Bucharest*, 1968.
- [4] A. L. MOCANU, Les variétés a courbure quasi-constant de type Vranceanu, *Lucr. Conf. Nat. de. Geom. Si Top., Tirgoviste*, 1987.
- [5] K. YANO and S. SAWAKI, Riemannian manifolds admitting a Conformal transformation group, *J. Diff. Geom.* **2** (1968), 161–184.
- [6] K. YANO, Integral formulas in Riemannian Geometry, *Marcel Dekker, Inc., New York*, 1970.
- [7] K. AMUR and Y. B. MARALABHAVI, On quasi-conformally flat spaces, *Tensor N. S.* **31** (1977), 194–198.
- [8] M. C. CHAKI, On pseudo symmetric manifolds, *Ann. St. Univ. "Al. I. Cuza" Iasi* **33** (1987), 53–58.
- [9] R. DESZCZ, On pseudosymmetric spaces, *Bull. Belg. Math. Soc. Serie A* **44** (1992), 1–34.
- [10] L. P. EISENHART, Riemannian Geometry, *Princeton University Press*, 1949.
- [11] N. J. HICKS, Notes on differential geometry, *D. Van Nostrand Company, INC. Princeton, New Jersey*.
- [12] J. A. SCHOUTEN, Ricci Calculus, (2nd edn.), *Springer-Verlag, Berlin*, 1954.

GOPAL CHANDRA GHOSH
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KALYANI
KALYANI 741235, WEST BENGAL
INDIA

U. C. DE
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KALYANI
KALYANI 741235, WEST BENGAL
INDIA

E-mail: ucde@klyuniv.ernet.in

TRAN QUOC BINH
DEPARTMENT OF MAYHEMATIOCS
UIVERSITY OF DEBRECEN
H-4010 DEBRECEN, P O. BOX 12
HUNGARY

E-mail: binh@math.klte.hu

(Received January 4, 2005)