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On the derived length of Lie solvable group algebras

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Dedicated to Professor Adalbert Bovdi on his 70th birthday

Abstract. Let G be a nilpotent group with cyclic commutator subgroup of order p^n and let F be a field of characteristic p. It is shown here that the Lie derived length of the group algebra FG is at most $\lceil \log_2(p^n+1)\rceil$. Furthermore, this bound is achived if and only if one of the following conditions is satisfied: (i) p is odd; (ii) $p=2$ and $n\le 2$; (iii) $p=2$, $n\ge 3$ and the nilpotency class of G is at most n.

1. Introduction

Let G be a group and F a field. The group algebra FG may be considered as a Lie algebra, with the usual bracket operation. Define the Lie derived series and the strong Lie derived series of the group algebra FG respectively as follows: let $\delta^{[0]}(FG)=\delta^{(0)}(FG)=FG$ and

$$\delta^{[n+1]}(FG)=\left[\delta^{[n]}(FG),\delta^{[n]}(FG)\right],$$

$$\delta^{(n+1)}(FG)=\left[\delta^{(n)}(FG),\delta^{(n)}(FG)\right]FG,$$

where $[X,Y]$ is the additive subgroup generated by all Lie commutators $[x,y]=xy-yx$ with $x\in X$ and $y\in Y$. We say that FG is Lie solvable if there exists $m\in\mathbb{N}$ such that $\delta^{[m]}(FG)=0$ and the number

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$\mathrm{dl}_L(FG) = \min\{m \in \mathbb{N} \mid \delta^{[m]}(FG) = 0\}$ is called the Lie derived length of FG. Similarly, FG is said to be strongly Lie solvable of derived length $\mathrm{dl}^L(FG) = m$ if $\delta^{(m)}(FG) = 0$ and $\delta^{(m-1)}(FG) \neq 0$. According to a result of Passi, Passman and Sehgal [6] a group algebra FG is Lie solvable if and only if one of the following conditions holds: (i) G is abelian; (ii) $\mathrm{char}(F) = p$ and the commutator subgroup G' of G is a finite p-group; (iii) $\mathrm{char}(F) = 2$ and G has a subgroup H of index 2 whose commutator subgroup H' is a finite 2-group. It is easy to check that a group algebra FG is strongly Lie solvable if either G is abelian or $\mathrm{char}(F) = p$ and G' is a finite p-group.

Let G be a group with commutator subgroup of order p^n and $\mathrm{char}(F) = p$. Shalev [8] showed that

$$\mathrm{dl}_L(FG) \leq \lceil \log_2(2t(G')) \rceil,$$

where $t(G')$ denotes the nilpotent index of the augmentation ideal of FG' and $\lceil r \rceil$ the upper integral part of a real number r. Moreover, Lemma 2.2 in [8] states that if G is nilpotent of class 2 then $\mathrm{dl}_L(FG) \leq \lceil \log_2(t(G')+1) \rceil$. In particular, according to Proposition 2.3 in [8], if G is nilpotent of class 2 and G' is cyclic of order p^n, then

$$\mathrm{dl}_L(FG) = \lceil \log_2(p^n + 1) \rceil.$$

In this paper our goal is to generalize the above results of Shalev for the case when the nilpotency class of G is not necessary 2. We obtain the following.

Theorem 1. *Let G be a nilpotent group with cyclic commutator subgroup of order p^n and let F be a field of characteristic p. Then $\mathrm{dl}_L(FG) \leq \lceil \log_2(p^n + 1) \rceil$ with equality if and only if one of the following conditions holds:*

(i) *p is odd;*

(ii) *$p = 2$ and G' is of order less than 8;*

(iii) *$p = 2$, $n \geq 3$ and G has nilpotency class at most n.*

Moreover, if $\mathrm{char}(F) = 2$ we can extend our result as follows.

Corollary 1. *Let G be a nilpotent group with commutator subgroup of order 2^n and let F be a field of characteristic 2. Then $\mathrm{dl}_L(FG) = n+1$ if and only if one of the following conditions holds:*

(i) *G' is the noncyclic group of order 4 and $\gamma_3(G) \neq 1$;*

(ii) *G' is cyclic of order less than 8;*

(iii) *G' is cyclic, $n \geq 3$ and G has nilpotency class at most n.*

In this paper $\omega(FG)$ denotes the augmentation ideal of FG; for a normal subgroup $H \subseteq G$ we understand by $\mathfrak{I}(H)$ the ideal $FG \cdot \omega(FH)$. For $x, y, x_1, x_2, \dots, x_n \in G$ let $x^y = y^{-1}xy$, $(x,y) = x^{-1}x^y$, and the commutator $(x_1, x_2, \dots, x_n)$ is defined inductively to be $\big((x_1, x_2, \dots, x_{n-1}), x_n\big)$. By $\zeta(G)$ we mean the center of the group G, by $\gamma_n(G)$ the n-th term of the lower central series of G with $\gamma_1(G) = G$. Furthermore, denote by C_n the cyclic group of order n. The n-th term of the upper Lie power series of FG is denoted by $(FG)^{(n)}$ which is the associative ideal generated by all Lie commutators $[x,y]$ with $x \in FG^{(n-1)}$ and $y \in FG$, where $FG^{(1)} = FG$.

We shall use freely the identities

$$[x, yz] = [x,y]z + y[x,z], \qquad [xy, z] = x[y,z] + [x,z]y,$$

and for units a, b, c the commutator identities

$$(a, bc) = (a,c)(a,b)^c = (a,c)(a,b)(a,b,c);$$

$$(ab, c) = (a,c)^b(b,c) = (a,c)(a,c,b)(b,c),$$

and that $[a,b] = ba((a,b)-1)$.

2. Preliminaries

We begin with a statement of independent interest about the strong Lie derived length of group algebras which generalizes the Corollary 4 of Bagiński's paper [1].

Proposition 1. *Let G be a nilpotent group whose commutator subgroup G' is a finite p-group and let $\mathrm{char}(F) = p$. If $\gamma_3(G) \subseteq (G')^p$ then*

$$\mathrm{dl}^L(FG) = \lceil \log_2(t(G') + 1) \rceil.$$

Proof. We show by induction on n that

$$\delta^{(n)}(FG) \subseteq (FG)^{(2^n)} \quad \text{for all } n \geq 0.$$

Evidently, $\delta^{(0)}(FG) = (FG)^{(1)}$ and assume that $\delta^{(n)}(FG) \subseteq (FG)^{(2^n)}$ for some n. By elementary properties of upper Lie power series,

$$\begin{aligned}\delta^{(n+1)}(FG) &= \big[\delta^{(n)}(FG), \delta^{(n)}(FG)\big]FG \subseteq \big[(FG)^{(2^n)}, (FG)^{(2^n)}\big]FG \\ &\subseteq (FG)^{(2^{n+1})}FG = (FG)^{(2^{n+1})}.\end{aligned}$$

In view of $\gamma_3(G) \subseteq (G')^p$, Theorem 3.1(i) from [3] states that $(FG)^{(2^n)} = \mathfrak{I}(G')^{2^n-1}$. Furthermore, Lemma 2.2 in [7] asserts $\mathfrak{I}(G')^{2^n-1} \subseteq \delta^{(n)}(FG)$ for all $n \geq 1$ and we have $\delta^{(n)}(FG) = \mathfrak{I}(G')^{2^n-1}$. It is easy to see that $\delta^{(n)}(FG) = 0$ if and only if $2^n - 1 \geq t(G')$, therefore $n \geq \log_2(t(G') + 1)$, which implies the statement. $\square$

Remark 1. (i) Since $\delta^{[n]}(FG) \subseteq \delta^{(n)}(FG)$ for all n, Proposition 1 yields an upper bound on the Lie derived length. Furthermore, if G is nilpotent with cyclic commutator subgroup of order p^n, then the condition $\gamma_3(G) \subseteq (G')^p$ holds and thus

$$\mathrm{dl}_L(FG) \leq \lceil \log_2(p^n + 1) \rceil.$$

But, as we will see, the equality does not always hold.

(ii) As the following examples show, Proposition 1 breaks down without the condition $\gamma_3(G) \subseteq (G')^p$.

- Let G be a group with $G' = C_2 \times C_2$ such that $\gamma_3(G) \neq 1$ and let $\mathrm{char}(F) = 2$. Then $\gamma_3(G) \nsubseteq (G')^2$ and, by Theorem 3 and Theorem 6 from [5], $\mathrm{dl}^L(FG) > 2$. So $\mathrm{dl}^L(FG) \neq \lceil \log_2(t(G') + 1) \rceil$, because now $\lceil \log_2(t(G') + 1) \rceil = 2$.
- Let G be a group with $G' = C_3 \times C_3 \times C_3$ such that $\gamma_3(G) \neq 1$ and let $\mathrm{char}(F) = 3$. Then $\gamma_3(G) \nsubseteq (G')^3$ and, by Theorem 2.3 from [7], $\mathrm{dl}^L(FG) > 3$. It follows that $\mathrm{dl}^L(FG) \neq \lceil \log_2(t(G') + 1) \rceil$, because $\lceil \log_2(t(G') + 1) \rceil = 3$.

The next lemma will be used in the proof of the theorem.

Lemma 1. *Let G be a nilpotent group with cyclic commutator subgroup of order p^n and let* $\operatorname{char}(F) = p$. *Then for all $m, k \geq 1$*

(i) $\big[\omega^m(FG'), \omega(FG)\big] \subseteq \mathfrak{I}(G')^{m+p-1}$;

(ii) $\big[\mathfrak{I}(G')^m, \mathfrak{I}(G')^k\big] \subseteq \mathfrak{I}(G')^{m+k+1}$.

Proof. (i) We use induction on m. For every $y \in G'$ and $g \in G$ we have

$$[y-1, g-1] = [y, g] = gy\big((y,g)-1\big) \in \mathfrak{I}\big(\gamma_3(G)\big) \subseteq \mathfrak{I}(G')^p.$$

This shows that the statement (i) holds for $m = 1$, because the elements of the form $g - 1$ with $1 \neq g \in G$ constitute an F-basis of $\omega(FG)$.

Now, assume that $\big[\omega^m(FG'), \omega(FG)\big] \subseteq \mathfrak{I}(G')^{m+p-1}$ for some m. Then

$$\begin{aligned}
&\big[\omega^{m+1}(FG'), \omega(FG)\big] \\
&\qquad \subseteq \omega^m(FG')\big[\omega(FG'), \omega(FG)\big] + \big[\omega^m(FG'), \omega(FG)\big]\omega(FG') \\
&\qquad \subseteq \omega^m(FG')\mathfrak{I}(G')^p + \mathfrak{I}(G')^{m+p-1}\omega(FG') \subseteq \mathfrak{I}(G')^{m+p},
\end{aligned}$$

and the proof of (i) is complete.

(ii) The statement (ii) is a consequence of (i), because

$$\mathfrak{I}(G') = \omega(FG)\omega(FG') + \omega(FG'). \qquad \square$$

Let G be a group with commutator subgroup $G' = \langle x \mid x^{2^n} = 1\rangle$, where $n \geq 3$. It is well known that the automorphism group $\operatorname{aut}(G')$ of G' is a direct product of the cyclic group $\langle\alpha\rangle$ of order 2 and the cyclic group $\langle\beta\rangle$ of order 2^{n-2} where the action of these automorphisms on G' is given by $\alpha(x) = x^{-1}$, $\beta(x) = x^5$. For $g \in G$, let τ_g denote the restriction to G' of the inner automorphism $h \mapsto h^g$ of G. The map $G \to \operatorname{aut}(G)$, $g \mapsto \tau_g$ is a homomorphism whose kernel coincides with the centralizer $C = C_G(G')$. Clearly, the map $\varphi : G/C \to \operatorname{aut}(G')$ given by $\varphi(gC) = \tau_g$ is a monomorphism.

The subset

$$G_\beta = \{g \in G \mid \varphi(gC) \in \langle\beta\rangle\}$$

of G will play an important role in the sequel. It is easy to check that G_β is a subgroup of index not greater than 2 and $g \in G_\beta$ if and only if $x^g = x^{5^i}$ for some $i \in \mathbb{Z}$.

Lemma 2. *Let G be a group with cyclic commutator subgroup of order 2^n, where $n \geq 3$ and let* $\mathrm{char}(F) = 2$. *Then*

(i) $(y, g) \in (G')^4$ *for all* $y \in G'$ *and* $g \in G_\beta$;

(ii) $\big[\omega^m(FG'), \omega(FG_\beta)\big] \subseteq \Im(G')^{m+3}$.

Proof. Let $g \in G_\beta$ and $y \in G'$.

(i) Clearly, $(y, g) = y^{-1}y^g = y^{-1+5^i}$ for some $i \geq 0$ and $-1 + 5^i \equiv 0 \pmod 4$. Therefore, $(y, g) \in (G')^4$.

(ii) Using (i) we have that

$$[y - 1, g - 1] = [y, g] = gy\big((y, g) - 1\big) \in \Im(G')^4,$$

from which (ii) follows for $m = 1$. One can now finish the proof by induction, as in Lemma 1(i). □

Lemma 3. *Let G be a group with commutator subgroup $G' = \langle x \mid x^{2^n} = 1\rangle$, where $n \geq 3$. Then the following are equivalent:*

(i) $G_\beta = G$.

(ii) *G has nilpotency class at most n.*

Proof. First of all, note that G is a nilpotent group of class at most $n + 1$.

(i) ⇒ (ii) By Lemma 2(i), $\gamma_3(G) \subseteq (G')^4$, so $|\gamma_2(G)/\gamma_3(G)| \geq 4$ and the class of G is at most n.

(ii) ⇒ (i) Suppose that G has nilpotency class at most n, but $G_\beta \neq G$. We claim that $x^{2^{k-2}} \in \gamma_k(G)$ for all $k \geq 2$. Indeed, this is clear for $k = 2$ and assume its truth for some $k \geq 2$. If $g \in G \setminus G_\beta$ then $(x^{2^{k-2}}, g) \in \gamma_{k+1}(G)$ and $(x^{2^{k-2}}, g) = x^{2^{k-2}(-1-5^i)} = (x^{2^{k-1}})^j$ with some i and odd j. This means that $x^{2^{k-1}} \in \gamma_{k+1}(G)$, as desired. Therefore, $\gamma_{n+1}(G) \neq 1$, which is a contradiction. □

Lemma 4. *Let G be a group with commutator subgroup $G' = \langle x \mid x^{2^n} = 1\rangle$, where $n \geq 3$. If G has nilpotency class $n + 1$ then $(g, h) \in (G')^2$ for all $g, h \in G_\beta$.*

Proof. If the lemma were not true we could choose the elements $g, h \in G_\beta$ so that $(g, h) = x$. By definition of G_β we may additionally assume that $(g,x)=1$. Lemma 3 states that $G\setminus G_\beta\neq\emptyset$; let y be in $G\setminus G_\beta$. Evidently, $(g,y)=x^i$ for some i. Using the equalities

$$g^h=gx,\quad g^{h^{-1}}=g(x^{-1})^{h^{-1}},\quad g^y=gx^i,\quad g^{y^{-1}}=g(x^{-i})^{y^{-1}}$$

it is easy to check

$$\begin{aligned}g&=g^{(h,y)}=g^{h^{-1}y^{-1}hy}=\big(g(x^{-1})^{h^{-1}}\big)^{y^{-1}hy}=g^{y^{-1}hy}x^{-1}\\&=\big(g(x^{-i})^{y^{-1}}\big)^{hy}x^{-1}=\big(gx(x^{-i})^{y^{-1}h}\big)^{y}x^{-1}=gx^ix^y(x^{-i})^{y^{-1}hy}x^{-1}\\&=g(x,y)(x^{-i},h^y),\end{aligned}$$

which is a contradiction. Indeed, keeping in mind that $y\in G\setminus G_\beta$ and $h\in G_\beta$ we have $(x,y)=x^{-1-5^j}\in\langle x^2\rangle\setminus\langle x^4\rangle$ and $(x^{-i},h^y)=x^{i(1-5^l)}\in\langle x^4\rangle$, thus $(x,y)(x^{-i},h^y)\neq 1$. $\square$

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Lemma 5. *Let G be a group with commutator subgroup $G'=\langle x\mid x^{2^n}=1\rangle$, where $n\geq 3$ and let $\mathrm{char}(F)=2$. If G has nilpotency class $n+1$ then $\mathrm{dl}_L(FG)\leq n$.*

Proof. Clearly, the set of the Lie commutators $[a,b]$ with $a,b\in G$ spans the F-space $\delta^{[1]}(FG)$. Since $[a,b]=g^h+g$ with $g=ba$ and $h=b$, while of course $g^h+g=[a,b]$ with $a=h^{-1}g$ and $b=h$ whenever $g,h\in G$, this spanning set for $\delta^{[1]}(FG)$ can also be described as the set of the elements g^h+g with $g,h\in G$. It follows that the Lie commutators $[{g_1}^{h_1}+g_1,{g_2}^{h_2}+g_2]$, where $g_1,g_2,h_1,h_2\in G$, span $\delta^{[2]}(FG)$. We shall compute these Lie commutators. It is easy to check that

$$\begin{aligned}[{g_1}^{h_1}+g_1,{g_2}^{h_2}+g_2]=g_2g_1\Big(&\big((g_1,g_2)+1\big)\big((g_2,h_2)+1\big)\big((g_1,h_1)+1\big)\\&+(g_2,h_2)\big((g_2,h_2,g_1)+1\big)\big((g_1,h_1)+1\big)\\&+(g_1,g_2)(g_1,h_1)\big((g_1,h_1,g_2)+1\big)\big((g_2,h_2)+1\big)\Big).\end{aligned}\tag{1}$$

Firstly, if neither g_1 nor g_2 are in G_β then

$$[g_1^{h_1}+g_2,{g_2}^{h_2}+g_2]=b\varrho_3\tag{2}$$

for some $b \in G_\beta$ and $\varrho_3 \in \omega^3(FG')$. Indeed, it is clear from the definition of G_β that then $g_2 g_1 \in G_\beta$. Furthermore, the second factor on the right-hand side of (1) always belongs to $\omega^3(FG')$, because $\gamma_3(G) \subseteq (G')^2$.

Secondly, if g_1 or g_2, say g_1, belongs to G_β, then we claim that

$$[g_1^{h_1} + g_1, {g_2}^{h_2} + g_2] = g\varrho_4 \tag{3}$$

for some $\varrho_4 \in \omega^4(FG')$ and $g \in G$.

For $g_1 \in G_\beta$, Lemma 2(i) asserts $(g_2, h_2, g_1) \in (G')^4$, therefore the right-hand side of (1) can be written as

$$\begin{aligned}[{g_1}^{h_1} + g_1, {g_2}^{h_2} + g_2] = g_2 g_1 \Big(&\big((g_1, g_2) + 1\big)\big((g_1, h_1) + 1\big) \\ &+ (g_1, g_2)(g_1, h_1)\big((g_1, h_1, g_2) + 1\big)\Big)\big((g_2, h_2) + 1\big) + g_2 g_1 \varrho_4\end{aligned} \tag{4}$$

for some $\varrho_4 \in \omega^4(FG')$. In order to prove (3) it will be sufficient to show that the element for some $\varrho_4 \in \omega^4(FG')$. In order to prove (3) it will be sufficient to show that the element

$$\begin{aligned}\vartheta = &\big((g_1, g_2) + 1\big)\big((g_1, h_1) + 1\big) \\ &+ (g_1, g_2)(g_1, h_1)\big((g_1, h_1, g_2) + 1\big)\end{aligned}$$

from the right-hand side of (4) belongs to $\omega^3(FG')$.

This is clear if g_2 also belongs to G_β, because then by Lemma 4 and Lemma 2(i) both summands of ϑ are in $\omega^3(FG')$. Furthermore, if $g_2 \notin G_\beta$, then $x^{g_2} = x^{-5^l}$ for some l and we distinguish the following three cases:

Case 1: $(g_1, h_1) \in (G')^2$. Then $(g_1, h_1, g_2) = (g_1, h_1)^{-1-5^l} \in (G')^4$ and $\vartheta \in \omega^3(FG')$.

Case 2: $(g_1, g_2) \in (G')^2$. By the well-known Hall–Witt identity,

$$(g_1, h_1, g_2)^{h_1^{-1}} (h_1^{-1}, g_2^{-1}, g_1)^{g_2} (g_2, g_1^{-1}, h_1^{-1})^{g_1} = 1.$$

Lemma 2(i) ensures that the second factor on the left-hand side belongs to $(G')^4$ and this is true for the last factor too, because

$$\begin{aligned}(g_2, g_1^{-1}, h_1^{-1}) &= \Big((g_1, g_2)^{g_1^{-1}}, h_1^{-1}\Big) \\ &= \Big((g_1, g_2)^{g_1^{-1}}\Big)^{-1} (g_1, g_2)^{g_1^{-1} h_1^{-1}} = (g_1, g_2)^{2i}\end{aligned}$$

for some i. This means that $(g_1,h_1,g_2)\in (G')^4$, which proves $\vartheta\in\omega^3(FG')$.

Case 3: $(g_1,h_1)\notin (G')^2$ and $(g_1,g_2)\notin (G')^2$. Then $\langle (g_1,h_1)\rangle = \langle (g_1,g_2)\rangle = G'$ and $(g_1,g_2)=(g_1,h_1)^k$ for some odd k. With the notation $y=(g_1,h_1)$ ϑ can be written as

$$\vartheta = (y^k+1)(y+1)+y^{k+1}(y^{-5^l-1}+1) = y^{k-5^l}+1+y(y^{k-1}+1).$$

Of course, if $k\equiv 1 \pmod 4$ then $y^{-5^l-1}+1$ and $y(y^{k-1}+1)$ are in $\omega^4(FG')$, therefore $\vartheta\in\omega^4(FG')$. Otherwise, if $k\equiv 3 \pmod 4$ then $y^{k-3}+1\in\omega^4(FG')$ which implies that

$$\begin{aligned} y(y^{k-1}+1) &= y\Big((y^{k-3}+1)(y^2+1)+(y^{k-3}+1)+(y^2+1)\Big)\\ &\equiv y(y^2+1)\equiv y^2+1 \pmod{\omega^3(FG')}. \end{aligned}$$

Similarly, we can obtain that

$$\begin{aligned} y^{k-5^l}+1 &= (y^{k-5^l-2}+1)(y^2+1)+(y^{k-5^l-2}+1)+(y^2+1)\\ &\equiv y^2+1 \pmod{\omega^3(FG')}. \end{aligned}$$

Hence

$$\vartheta = y^{k-5^l}+1+y(y^{k-1}+1)\equiv 2(y^2+1)\equiv 0 \pmod{\omega^3(FG')},$$

which completes the checking of (3).

Let S be the additive subgroup generated by all elements of the form $g\varrho_4$ and $b\varrho_3$, where $g\in G$, $b\in G_\beta$ and $\varrho_3\in\omega^3(FG'), \varrho_4\in\omega^4(FG')$. We claim that $[S,S]\subseteq \Im(G')^8$. Indeed, the additive subgroup $[S,S]$ can be spanned by some Lie commutators of the forms $[g\varrho_4,h\varrho_3]$ and $[b_1\varrho_3,b_2\eta_3]$ with $g\in G, b_1,b_2\in G_\beta$, $\varrho_3,\eta_3\in\omega^3(FG'), \varrho_4\in\omega^4(FG')$. Furthermore, by Lemma 1(i),

$$\begin{aligned} [g\varrho_4,h\varrho_3] &= g[\varrho_4,h\varrho_3]+[g,h\varrho_3]\varrho_4\\ &= g[\varrho_4,h+1]\varrho_3+hg\big((g,h)+1\big)\varrho_3\varrho_4+h[g+1,\varrho_3]\varrho_4\in\Im(G')^8, \end{aligned}$$

and by Lemma 2(ii) and Lemma 4,

$$\begin{aligned} [b_1\varrho_3,b_2\eta_3] &= b_1[\varrho_3,b_2\eta_3]+[b_1,b_2\eta_3]\varrho_3\\ &= b_1[\varrho_3,b_2+1]\eta_3+b_2[b_1+1,\eta_3]\varrho_3+b_1b_2\big((b_1,b_2)+1\big)\eta_3\varrho_3 \end{aligned}$$

also belongs to $\Im(G')^8$. Therefore, $[S,S]\subseteq \Im(G')^8$.

From (2) and (3) we get $\delta^{[2]}(FG)\subseteq S$, so we have

$$\delta^{[3]}(FG)=\big[\delta^{[2]}(FG),\delta^{[2]}(FG)\big]\subseteq [S,S]\subseteq \Im(G')^8.$$

Now, we use induction on k to show that

$$\delta^{[k]}(FG)\subseteq \Im(G')^{2^k}\quad \text{for all}\quad k\geq 3. \tag{5}$$

Indeed, assuming the validity of (5) for some $k\geq 3$ we have

$$\delta^{[k+1]}(FG)=\big[\delta^{[k]}(FG),\delta^{[k]}(FG)\big]\subseteq \big[\Im(G')^{2^k},\Im(G')^{2^k}\big]\subseteq \Im(G')^{2^{k+1}}$$

and this proves the truth of (5) for every $k\geq 3$.

Keeping in mind that G' has order 2^n, (5) implies that $\delta^{[n]}(FG)=0$. Hence $\mathrm{dl}_L(FG)\leq n$ and the proof is complete. $\square$

Lemma 6. *Let G be a nilpotent group with commutator subgroup $G'=\langle x\mid x^{p^n}=1\rangle$, $\mathrm{char}(F)=p$ and assume that one of the following conditions holds:*

(i) *$p=2$, $n\geq 3$ and G has nilpotency class at most n;*

(ii) *p is odd.*

Then $\mathrm{dl}_L(FG)=\lceil\log_2(p^n+1)\rceil$.

Proof. Since G' is cyclic of order p^n, we can choose $a,b\in G$ such that $(a,b)=x$. First of all, we claim that

$$[b^la^m,b^sa^t]\equiv (ms-lt)b^{l+s}a^{m+t}(x-1)\pmod{\Im(G')^2} \tag{6}$$

for every $l,s,m,t\in\mathbb{Z}$. Indeed, an easy computation yields

$$\begin{aligned}\big[b^la^m,b^sa^t\big]&=b^sa^tb^la^m\big((b^la^m,b^sa^t)-1\big)\\&=b^{l+s}a^{m+t}(a^t,b^l)^{a^m}\big((b^la^m,b^sa^t)-1\big)\\&\equiv b^{l+s}a^{m+t}\big((b^la^m,b^sa^t)-1\big)\pmod{\Im(G')^2},\end{aligned} \tag{7}$$

and

$$\begin{aligned}(b^la^m,b^sa^t)&\equiv (b^l,a^t)(a^m,b^s)\equiv (b,a)^{lt}(a,b)^{ms}\\&\equiv x^{ms-lt}\pmod{(G')^p},\end{aligned}$$

because $\gamma_3(G) \subseteq (G')^p$. Thus $(b^l a^m, b^s a^t) = x^{ms-lt+pi}$ for some i. In view of the identity $uv - 1 = (u-1)(v-1) + (u-1) + (v-1)$, we have

$$\begin{aligned}(b^l a^m, b^s a^t) - 1 &\equiv (ms - lt + pi)(x-1)\\ &\equiv (ms-lt)(x-1) \pmod{\mathfrak{I}(G')^2}\end{aligned}$$

and putting this into (7) we obtain (6).

Now, let $k \geq 1$, $l, m, s, t \in \mathbb{Z}$, $z_1, z_2 \in \mathfrak{I}(G')^{2^k}$ and set

$$f_k(l,m,s,t,z_1,z_2) = \big[b^l a^m (x-1)^{2^k-1} + z_1, b^s a^t (x-1)^{2^k-1} + z_2\big].$$

We shall show that

$$\begin{aligned}&f_k(l,m,s,t,z_1,z_2)\\ &\qquad\equiv (ms-lt) b^{l+s} a^{m+t} (x-1)^{2^{k+1}-1} \pmod{\mathfrak{I}(G')^{2^{k+1}}}.\end{aligned} \tag{8}$$

Lemma 1(ii) ensures that the elements $\big[b^l a^m (x-1)^{2^k-1}, z_2\big]$, $\big[z_1, z_2\big]$ and $\big[z_1, b^s a^t (x-1)^{2^k-1}\big]$ belong to $\mathfrak{I}(G')^{2^{k+1}}$, thus

$$\begin{aligned}&f_k(l,m,s,t,z_1,z_2)\\ &\qquad\equiv \big[b^l a^m (x-1)^{2^k-1}, b^s a^t (x-1)^{2^k-1}\big] \pmod{\mathfrak{I}(G')^{2^{k+1}}}.\end{aligned}$$

In the case $p = 2$, Lemma 3 forces $b^l a^m, b^s a^t \in G_\beta$, so we may apply Lemma 2(ii) to obtain that

$$\big[b^l a^m, (x-1)^{2^k-1}\big], \big[(x-1)^{2^k-1}, b^s a^t\big] \in \mathfrak{I}(G')^{2^k+1}.$$

Furthermore, for $p > 2$ the above inclusion follows from Lemma 1(i). This implies that

$$f_k(l,m,s,t,z_1,z_2) \equiv [b^l a^m, b^s a^t](x-1)^{2^{k+1}-2} \pmod{\mathfrak{I}(G')^{2^{k+1}}},$$

which, together with (6), proves (8).

Define the following three series inductively by:

$$u_0 = a, \quad v_0 = b, \quad w_0 = b^{-1} a^{-1},$$

and, for $k > 0$,

$$u_{k+1} = [u_k, v_k], \quad v_{k+1} = [u_k, w_k], \quad w_{k+1} = [w_k, v_k].$$

Obviously, the k-th elements of these series belong to $\delta^{[k]}(FG)$. By induction on k we show for odd k that

$$\begin{aligned} u_k &\equiv \pm ba(x-1)^{2^k-1} \pmod{\Im(G')^{2^k}}; \\ v_k &\equiv \pm b^{-1}(x-1)^{2^k-1} \pmod{\Im(G')^{2^k}}; \\ w_k &\equiv \pm a^{-1}(x-1)^{2^k-1} \pmod{\Im(G')^{2^k}}, \end{aligned} \tag{9}$$

and if k is even then

$$\begin{aligned} u_k &\equiv \pm a(x-1)^{2^k-1} \pmod{\Im(G')^{2^k}}; \\ v_k &\equiv \pm b(x-1)^{2^k-1} \pmod{\Im(G')^{2^k}}; \\ w_k &\equiv \pm b^{-1}a^{-1}(x-1)^{2^k-1} \pmod{\Im(G')^{2^k}}. \end{aligned} \tag{10}$$

Evidently, $u_1 = [a, b] = ba(x-1)$, and by (6) we have

$$v_1 = [a, b^{-1}a^{-1}] \equiv -b^{-1}(x-1) \pmod{\Im(G')^2},$$

and $w_1 = [b^{-1}a^{-1}, b] \equiv -a^{-1}(x-1) \pmod{\Im(G')^2}$. Therefore (9) holds for $k = 1$.

Now, assume that (9) is true for some odd k. According to (8) the congruences

$$\begin{aligned} u_{k+1} &= \pm f_k(1, 1, -1, 0, u_k{}', v_k{}') \\ &\equiv \pm(-1)a(x-1)^{2^{k+1}-1} \pmod{\Im(G')^{2^{k+1}}}; \\ v_{k+1} &= \pm f_k(1, 1, 0, -1, u_k{}', v_k{}') \\ &\equiv \pm b(x-1)^{2^{k+1}-1} \pmod{\Im(G')^{2^{k+1}}}; \\ w_{k+1} &= \pm f_k(0, -1, -1, 0, u_k{}', v_k{}') \\ &\equiv \pm b^{-1}a^{-1}(x-1)^{2^{k+1}-1} \pmod{\Im(G')^{2^{k+1}}} \end{aligned}$$

hold, where $u_k{}'$, $v_k{}'$, $w_k{}'$ are suitable elements from $\Im(G')^{2^k}$. Similarly, supposing the truth of (10) for some even k we see

$$\begin{aligned} u_{k+1} &= \pm f_k(0,1,1,0,u_k{}',v_k{}') \\ &\equiv \pm ba(x-1)^{2^{k+1}-1} \pmod{\mathfrak{I}(G')^{2^{k+1}}}; \\ v_{k+1} &= \pm f_k(0,1,-1,-1,u_k{}',v_k{}') \\ &\equiv \pm(-1)b^{-1}(x-1)^{2^{k+1}-1} \pmod{\mathfrak{I}(G')^{2^{k+1}}}; \\ w_{k+1} &= \pm f_k(-1,-1,1,0,u_k{}',v_k{}') \\ &\equiv \pm(-1)a^{-1}(x-1)^{2^{k+1}-1} \pmod{\mathfrak{I}(G')^{2^{k+1}}}. \end{aligned}$$

So, (9) and (10) are valid for any $k > 0$.

Assume that $k < \lceil \log_2(p^n+1) \rceil$. Then $2^k - 1 < p^n$ and the elements u_k, v_k, w_k are nonzero in $\delta^{[k]}(FG)$, thus $\mathrm{dl}_L(FG) \geq \lceil \log_2(p^n+1) \rceil$.

At the same time, Remark 1(i) says that $\mathrm{dl}_L(FG) \leq \lceil \log_2(p^n+1) \rceil$. □

3. Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1. For $p = 2$ and $n < 3$ the statement is a consequence of Remark 1(i) and Theorem 3 in [5]. In the other cases Lemma 5 and Lemma 6 state the required result. The proof is complete. □

Proof of Corollary 1. Clearly, if G' is cyclic the statement immediately follows from Theorem 1. Now, assume that G' is noncyclic and $\delta^{[n]}(FG) \neq 0$. We know from [2] that FG is Lie nilpotent, and as we have already seen, $\delta^{[n]}(FG) \subseteq (FG)^{(2^n)}$. Thus $(FG)^{(2^n)} \neq 0$ and Theorem 1 of [4] states that $G' = C_2 \times C_2$ and $\gamma_3(G) \neq 1$. Conversely, if $G' = C_2 \times C_2$ then $t(G') = 3$ and $\mathrm{dl}_L(FG) \leq \lceil \log_2(2 \cdot 3) \rceil = 3$. Furthermore, when $\gamma_3(G) \neq 1$, Theorem 3 in [5] says that $\mathrm{dl}_L(FG) \neq 2$. Therefore $\mathrm{dl}_L(FG) = 3$ and the corollary is proved. □

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