

On skew 2-groups

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To the memory of Edith Szabó

Abstract. We study 2-groups whose non-linear irreducible characters are of the third kind, i.e. real but not afforded by a real representation.

The purpose of this short note is to draw attention to an interesting class of finite 2-groups, and to make a start in studying them. Our results are far from definitive. First, let us recall the definition of the *Frobenius–Schur indicator* $\nu(\chi)$ of an irreducible character χ of a finite group G . $\nu(\chi) = 1$ if χ is afforded by a real representation, $\nu(\chi) = -1$ if χ is real, but is not afforded by a real representation, and $\nu(\chi) = 0$ if χ is not real-valued. χ is said to be of the *first, second, or third kind*, if $\nu(\chi) = 1, 0$, or -1 , respectively. For the theory of this indicator, see, e.g., [JL, chapter 23]. Here we denote by $\text{Irr}(G)$ the set of irreducible characters of G , by $X = X(G)$ the set of non-linear irreducible characters, and by $t(x)$, for $x \in G$, the number of elements y such that $x = y^2$. In particular $t(1) = t + 1$, where t is the number of involutions of G . We need the

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following two formulae:

$$\nu(\chi) = \frac{\sum_{x \in G} \chi(x^2)}{|G|}, \quad (1)$$

$$t(x) = \sum_{\chi \in \text{Irr}(G)} \nu(\chi) \chi(x). \quad (2)$$

We also need the fact that irreducible characters of third kind have even degree.

Recall that G is *real*, if all its irreducible characters are real. This is equivalent to all elements being *real*, i.e. conjugate to their inverses. In [CM], D. CHILLAG and the present author considered groups in which all non-linear characters are real. Here we consider a more restricted class.

Definition. A group is termed *skew*, if it is non-abelian, and all its non-linear characters have Frobenius–Schur indicator -1 .

Thus in a skew group all non-linear characters are real. Among the linear characters, the real ones are the ones of order 2 (or 1), i.e. the characters of G/G^2 , and they are of the first kind.

Theorem 1 (W. Willems). *A skew group G is a 2-group.*

PROOF. This is proved by W. WILLEMS in [W], under the extra hypothesis that G is real. However, he relies on a paper (his reference [3]) which seems not to have been published. We will indicate here a different argument, which avoids the reality assumption, and also avoids the application of the Feit–Thompson theorem, which was used in [W]. That argument embodies a considerable simplification, suggested by the referee, of my original argument. Let G be a minimal counter example to the theorem. First, since all non-linear irreducible characters have even degree, a theorem of J. G. THOMPSON [T] shows that G has a normal 2-complement K . Let T be a Sylow 2-subgroup of G . Then $T \cong G/K$, therefore G and T have the same number of linear characters of order 2, and all irreducible characters of T can be considered as characters of G . Formula (2), for $x = 1$, shows that T has at least as many involutions as G has. This is possible only if G and T have the same number of involutions, and then (2) implies that all non-linear irreducible characters of G

are characters of T , which means that K is contained in the kernels of all these characters. But then $K = 1$, and G is a 2-group. $\square$

A similar argument establishes the following variation:

Theorem 2. *Let G be a finite group, in which all non-linear irreducible characters have even degree, and they are not of the first kind. Then G is a direct product of a 2-group and an abelian group.*

PROOF. As in the previous proof, G has a normal 2-complement K , and if T is a Sylow 2-subgroup, then G and T have the same number of involutions. This means that all involutions of G lie in T , and they generate a normal 2-subgroup S of G . By induction G/S is a direct product of a 2-group and an abelian group. We may assume that $K \neq 1$. Since $K \cap S = 1$, and G/K is a 2-group, G is also a direct product of a 2-group and an abelian group. $\square$

Examples. The quaternion group $\mathbf{Q}$ (of order 8) is skew, while in the dihedral 2-groups $\mathbf{D}_n$ of order 2^n all characters are of the first kind. More generally, for each n one of the two extraspecial groups of order 2^{2n+1} is skew, namely the one that is a central product of $\mathbf{Q}$ and several dihedral groups of order 8. The other extraspecial group has all its characters of the first kind. A direct product of a skew group and an elementary abelian 2-group is skew.

Two further examples, of order 64, will be noted below.

Recall that in a 2-group $G^2 = \Phi(G)$, the Frattini subgroup.

Proposition 3. *Let G be a skew group, and write $|G : G^2| = 2^d$. Then $|G| \leq 2^{2d-1}$. Equality holds only for $G \cong \mathbf{Q}$.*

PROOF. Write $|G : G'| = 2^k$, recall that X is the set of non-linear irreducible characters of G , and let $A = \sum_{\chi \in X} \chi(1)$. Then $t(1) = 2^d - A$, implying $A < 2^d$. Let $m = \max_{\chi \in X} \chi(1)$. Then $m \leq A$, therefore $m \leq 2^{d-1}$. Thus $|G| = 2^k + \sum_{\chi \in X} \chi(1)^2 \leq 2^k + Am < 2^k + 2^{2d-1}$. If $k \geq 2d-1$, we obtain $|G| \leq 2^k$, which is impossible. Thus $k < 2d-1$ and $|G| \leq 2^{2d-1}$.

Suppose that equality holds. Then $m = 2^{d-1}$, therefore $|G : Z(G)| \geq 2^{2d-2}$, so $|Z(G)| = 2$. Moreover, there is only one character of degree m , and $A - m < 2^d - 2^{d-1} = 2^{d-1}$. Thus $|G| = 2^{2d-1} \leq 2^k + (A - m)2^{d-2} + m^2 <$

$2^k + 2^{2d-3} + 2^{2d-2}$, which does not hold for $k \leq 2d - 3$. Thus $k = 2d - 2$ i.e. $G' = Z(G)$. Since $|Z(G)| = 2$, that means that G is extraspecial, and then its order is 2^{d+1} . Thus $d = 2$, $|G| = 8$, and $G \cong \mathbf{Q}$. $\square$

Proposition 4. *Let G be a skew group, and $1 \neq z \in G^2$. Then $t(z) > t(1)$. In particular, z is a square.*

This follows immediately from the formula $t(z) = 2^d - \sum_{\chi \in X} \chi(z)$. On the other hand, if all non-linear characters are of the first kind, we have $t(z) < t(1)$, while if all non-linear characters are of the second kind, then $t(z)$ is constant on G^2 . The last property actually characterizes 2-groups in which all non-linear characters are of the second kind, by [CM, Proposition 4.1].

We quote some further results from [CM].

Proposition 5. *Let G be a non-real 2-group, in which all non-linear characters are real. Then G/G' has exponent 4, while all other factors of the lower central series, and also all factors of the upper central series, have exponent 2. Let R/G' be the subgroup consisting of the elements of order at most 2 in G/G' . Then R is the set of real elements of G , all non-linear characters of G vanish off R , and if $x \notin R$, then the conjugacy class of x is the coset xG' .*

This follows by specializing to 2-groups Theorems 1.3, 1.4, and Proposition 4.9 of [CM].

Note that if G is a real group, then all factors of either the lower or upper central series have exponent 2.

Lemma 6. *A faithful character of a group G vanishes on $Z_2(G) - Z(G)$.*

This is well known. See [I, proof of (2.31)].

Proposition 7. *Let G be a non-abelian 2-group such that each factor group H of G satisfies: if $1 \neq z \in H^2$, then $t(z) > t(1)$. If $cl(G) \leq 3$, then G is a skew group. If we assume that G is real, we can relax the inequality to $t(z) \geq t(1)$. Dually, if we assume the reverse inequality, $t(z) < t(1)$, or that G is real and $t(z) \leq t(1)$, then all non-linear characters of G are of the first kind.*

PROOF. Let χ be a non-linear character of G . We wish to prove that χ is of the third kind. We may assume that χ is faithful. Then $Z(G)$ is cyclic. Suppose that it has order 4 at least. Then the restriction of χ to $Z(G)$ is a multiple of a faithful linear character, and it is not real. If we assume that G is real, this is a contradiction. If we do not assume reality, then we obtain that all faithful characters are of the second kind. Let N be the subgroup of order 2 in $Z(G)$, and let $N = \{1, z\}$. Then N is the unique minimal normal subgroup of G , and thus lies in the kernels of all non-faithful characters. Since $\nu(\chi) = 0$ for the faithful characters, equation (2) shows that $t(z) = t(1)$, contradicting our assumptions. Thus $Z(G) = N$ has order 2. By induction, G/N is either abelian or a skew group. If it is abelian, then $N = G' = Z(G)$, and thus G is an extraspecial group. Then G has a unique non-linear irreducible character, which is real, and our claims follow easily by counting involutions. Now assume that G/N is a skew group. We have $\chi(z) = -\chi(1)$. If $cl(G) = 2$, then χ vanishes off $Z(G)$. If $cl(G) = 3$, then χ vanishes on $Z_2(G) - Z(G)$, and in particular on $G' - Z(G)$. If G is real, all squares are in G' , by the remark following Proposition 5. If G is not real, let $K = G/\gamma_3(G)$. Then K is a skew group by induction, and so $\exp(K') = 2$, by Proposition 5 and the remark following it. Therefore $K^2 \leq Z(K)$. That means that in G the squares are in $Z_2(G)$. Thus in either case χ vanishes on non-central squares. Thus $|G|\nu(\chi) = \sum_{x \in G} \chi(x^2) = (t(1) - t(z))\chi(1)$, and this number is, by assumption, non-positive, and either it is strictly negative, or χ is real, so in either case χ is of the third kind. $\square$

A similar proof establishes the dual statement.

As a rule, skewness is not inherited by subgroups, but there are exceptions.

Proposition 8. *Let G be a non-real skew 2-group, and write $G/G' = K \times L$, where K is cyclic of order 4. Let $M = K^2$, and write $M \times L = H/G'$. Then H is a skew group. Dually, if all non-linear irreducible characters of G are of the first kind, the same applies to H .*

PROOF. Let G be a non-real skew group, let λ be a character of G/G' with kernel L , considered as a character of G , and let χ be a non-linear character of G . If $x \in H$, then $\lambda(x^2) = 1$, and if $x \notin H$, then $\lambda(x^2) = -1$. We have $\sum \chi(x^2) = -|G| = \sum_{x \notin H} \chi(x^2) + \sum_{x \in H} \chi(x^2) = A + B$, say.

Similarly $\sum(\chi\lambda)(x^2) = -|G| = -A + B$. It follows that $A = 0$, and $\sum_{x \in H} \chi(x^2) = -2|H|$. Since $|G : H| = 2$, the character $\chi|_H$ is either irreducible or the sum of two irreducible characters of H , and the above equality shows that the only possibility is that $\chi|_H$ is the sum of two irreducible characters of the third kind. This shows in particular that H is not abelian, since abelian groups do not have characters of the third kind. Since each non-linear character of H occurs in $\chi|_H$, for some χ , we see that H is a skew group. $\square$

The dual statement is proved in the same way. Note that in that case H may be abelian.

Proposition 9. *Let G be a non-real skew group. Suppose that G/G' is the direct product of r cyclic subgroups of order 4 and s subgroups of order 2. Then $s \geq r + 2 \geq 3$, and all non-linear irreducible characters of G have degree at least 2^{r+1} . If H is a subgroup of G such that $|G : H| \leq 2^r$, then $H' = G'$, and G contains a real skew subgroup S of index 2^r such that $S' = G'$.*

PROOF. Let H and χ be as in the previous proposition, and let η be one of the irreducible characters of H that occur in $\chi|_H$. Then $\chi(1) = 2\eta(1)$, and η is not linear, because it is of the third kind. Thus the claim about the degrees follows by induction on r , and then all subgroups of small index have derived subgroup G' , by Theorem 1 of [M]. Also, repeatedly applying the process of passing from G to H shows that the subgroup S consisting of all elements of order 2 (or 1) (*modulo* G') is a skew group satisfying $S' = G'$, which has index 2^r . S is real, because $\exp(S/S') = 2$.

Let N be a normal subgroup of G which is maximal in G' , and write $T = G/N$. Then $|T| = 2^{2r+s+1}$, and the non-linear characters of T have degree at least 2^{r+1} . Therefore $|T : Z(T)| \geq 2^{2r+2}$. On the other hand Proposition 5 shows that $\exp(T/Z(T)) = 2$, and therefore $T^2 \leq Z(T)$ and $|T : Z(T)| \leq 2^{r+s}$. Combining the two inequalities yields $s \geq r + 2$. $\square$

Proposition 10. *Let G be a skew group, in which $|G : G^2| = 2^d$ and $|G| = 2^{2d-2}$. Then $d \leq 4$ and $|G| \leq 2^6$. There are three such groups.*

PROOF. We use the notations X , A , and k , as in the proof of Proposition 3, and recall the inequalities $A < 2^d$ and $|G| \leq 2^k + Am$. Obviously $m \leq 2^{d-2}$ and $k \leq 2d - 3$. If $m < 2^{d-2}$ we get $|G| < 2^k + 2^{2d-3} \leq$

2^{2d-2} . Therefore $m = 2^{d-2}$. This implies that $|Z(G)| \leq 4$. Let r be the number of irreducible characters of degree m . Then $r \leq 3$ and $|G| < 2^k + (A - rm)2^{d-3} + r \cdot 2^{2d-4}$.

Let $r = 1$. Then the inequality $|G| = 2^{2d-2} \leq 2^k + (A - m)2^{d-3} + m^2 < 2^k + (2^d - 2^{d-2})2^{d-3} + 2^{2d-4} = 2^k + 2^{2d-3} + 2^{2d-5}$ implies $k = 2d - 3$, i.e. $|G'| = 2$, and then $G' \leq Z(G)$. But G is not extraspecial, because its order is an even power of 2, and so we have $|Z(G)| = 4$, and since $|G'| = 2$, the non-central elements of G have two conjugates each. Writing $k(G)$ for the class number of $|G|$, we obtain $k(G) = 4 + (2^{2d-2} - 4)/2 = 2^{2d-3} + 2$. That means that G has just two non-linear irreducible characters, and writing $|G| = \sum_{\text{Irr}(G)} \chi(1)^2$ shows that both non-linear characters have the same degree 2^{d-2} , a contradiction.

Now assume that $r = 2$. Then the inequality for $|G|$ becomes $2^{2d-2} < 2^k + 2^{2d-3} + 2^{2d-4}$, and this again implies $k = 2d - 3$, $|G'| = 2$, and $|Z(G)| = 4$. Since $cl(G) = 2$, we have $G^2 \leq Z(G)$. But $|G^2| = 2^{d-2}$, so that $d - 2 \leq 2$, $d \leq 4$, and $|G| \leq 2^6$.

Finally, let $r = 3$. In this case we get that $k \geq 2d - 4$. If $k = 2d - 3$, then $k(G)$ is as above, and there are only two non-linear characters, contradicting $r = 3$. Thus $k = 2d - 4$. Since $2^k + 3 \cdot 2^{2d-4} = 2^{2d-2}$, we see that the three characters of degree m are all the non-linear characters of G , and $k(G) = 2^{2d-4} + 3$. On the other hand, since $|G'| = |Z(G)| = 4$, we have $k(G) \geq 4 + (2^{2d-2} - 4)/4 = 2^{2d-4} + 3$. But we know already that this inequality is an equality, and that means that each non-central element x has exactly four conjugates, which are the elements of xG' . Taking $x \in Z_2(G)$, we get $G' = [x, G] \leq Z(G)$. Thus again $cl(G) = 2$. Since $\exp(Z(G)) = 2$, by Proposition 5 and its remark, we have $G^2 \leq Z(G)$, yielding $|G| \leq 2^6$ as in the previous case.

Thus we have either $d = 3$ or $d = 4$. In the first case it is easy to see that the only possibility is $\mathbf{Q} \times C_2$. In the second case we have $|G| = 64$. Using the information gathered so far in the proof, and also the previous propositions and the HALL–SENIOR tables [HS], one can determine that the only possibilities are the groups numbered 187 and 108 in the tables. Of these the first one is real, the second one not. $\square$

Remark. It is easy to see that among the groups of order 64 at most, the only other skew groups are the direct products of $\mathbf{Q}$ by two or three

copies of C_2 , one extraspecial group of order 32, and the direct product of the latter group and C_2 .

Corollary 11. *Let G be as in Proposition 9, and assume that $s = 3$. Then G is the group number 108 in the Hall–Senior list.*

PROOF. If $s = 3$, then Proposition 9 shows that $r = 1$, and thus $d = 4$. Since $G \not\cong \mathbf{Q}$, Proposition 3 shows that $|G| \leq 64$, and the previous proposition, and the remark following it, apply. $\square$

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