

Units of commutative group algebra with involution

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Dedicated to the memory of Dr. Edit Szabó

Abstract. Let p be an odd prime, F the field of p elements and G a finite abelian p -group with an arbitrary involutory automorphism. Extend this automorphism to the group algebra FG and consider the unitary and the symmetric normalized units of FG . This paper provides bases and determines the invariants of the two subgroups formed by these units.

1. Introduction

Let p be an odd prime, F the field of p , G a finite abelian p -group with an arbitrary automorphism η of order 2 and $G_\eta = \{g \in G \mid \eta(g) = g\}$. Extending the automorphism η to the group algebra FG we obtain the involution

$$x = \sum_{g \in G} \alpha_g g \mapsto x^\oplus = \sum_{g \in G} \alpha_g \eta(g)$$

of FG which we will call as η -canonical involution. In particular, if $\eta(g) = g^{-1}$ or $\eta(g) = g$ for all $g \in G$ then the involution is called involutory; if $\eta(g) = g^{-1}$ for all $g \in G$ then it is called canonical and denote by $*$.

Mathematics Subject Classification: Primary: 16S34, 16U60; Secondary: 20C05.

Key words and phrases: group algebra, group of units, unitary unit, symmetric unit.

Supported by OTKA No. T037202 and by FAPESP Brasil (proc. 06/56203-3).

Let $V(FG) = \{\sum_{g \in G} \alpha_g g \mid \sum_{g \in G} \alpha_g = 1\}$ be the group of normalized units of the group algebra FG and consider the subgroups of symmetric and unitary units

$$S_{\otimes}(FG) = \{x \in V(FG) \mid x^{\otimes} = x\},$$

$$V_{\otimes}(FG) = \{x \in V(FG) \mid x^{\otimes} = x^{-1}\}$$

respectively.

Our goal is to study the unitary subgroup $V_{\otimes}(FG)$ and the group of symmetric units $S_{\otimes}(FG)$. The problem of determining the invariants and the basis of $V_{\otimes}(FG)$ had been raised by S. P. Novikov. Its solution for the canonical involution was given in [1]; here this result extended to arbitrary involutory involution.

2. Invariants

We start with some remarks about the invariants of unitary and symmetric subgroups to give a bases.

Since $V(FG)$ has an odd order and every $u \in V(FG)$ can be written as $u = (v^{\otimes})^2$, so $u = (v^{\otimes}v^{-1})(vv^{\otimes})$, where $v^{\otimes}v^{-1}$ is unitary and $vv^{\otimes}$ is a symmetric unit. But every $x \in S_{\otimes}(FG) \cap V_{\otimes}(FG)$ is such that $x = x^{\otimes} = x^{-1}$; since x is odd order it follows $x = 1$ and we have

$$V(FG) = S_{\otimes}(FG) \times V_{\otimes}(FG). \quad (1)$$

Define the mappings

$$\psi_1 : V(FG) \rightarrow V_{\otimes}(FG), \quad \psi_2 : V(FG) \rightarrow S_{\otimes}(FG)$$

given by $\psi_1(x) = x^{\otimes}x^{-1}$ and $\psi_2(x) = x^{\otimes}x$ respectively for $x \in V(FG)$. They are epimorphisms and as corollary conclude that

$$\begin{aligned} V_{\otimes}(FG) &= \{x^{\otimes}x^{-1} \mid x \in V(FG)\}, \\ S_{\otimes}(FG) &= \{x^{\otimes}x \mid x \in V(FG)\} \end{aligned} \quad (2)$$

and

$$S_{\otimes}(FG)^{p^i} = S_{\otimes}(FG^{p^i}), \quad V_{\otimes}(FG)^{p^i} = V_{\otimes}(FG^{p^i}), \quad (3)$$

which use for the description of the invariants of these groups. The subsets $\{g, \eta(g)\}$ with $g \in G \setminus G_\eta$ form a partition of the set $G \setminus G_\eta$ and let E be the system of representatives of these subsets. Clearly, $x \in S_\otimes(FG)$ can be uniquely written as

$$x = \sum_{g \in E} \alpha_g(g + \eta(g)) + \sum_{g \in G_\eta} \beta_g g$$

with $\alpha_g, \beta_g \in F$ and $\sum_{g \in E} 2\alpha_g + \sum_{g \in G_\eta} \beta_g = 1$, so the order of the group of symmetric units $S_\otimes(FG)$ equals $p^{\frac{1}{2}(|G|+|G_\eta|-2)}$. By (3) $S_\otimes(FG)^p = S_\otimes(FG^p)$, so as before, the order of $S_\otimes(FG)^p$ is $p^{\frac{1}{2}(|G^p|+|G_\eta^p|-2)}$. It follows that the p -rank of the group $S_\otimes(FG)$, that is the number of components in the decomposition of $S_\otimes(FG)$ into a direct product of cyclic groups, equals $\frac{1}{2}(|G| - |G^p| + |G_\eta| - |G_\eta^p|)$. Similarly, the p -rank of the group $S_\otimes(FG)^{p^{i-1}}$ equals $\frac{1}{2}(|G^{p^{i-1}}| - |G^{p^i}| + |G_\eta^{p^{i-1}}| - |G_\eta^{p^i}|)$.

We conclude that the number of components of order p^i in the decomposition of $S_\otimes(FG)$ into a direct product of cyclic groups equals

$$f_i(S_\otimes(FG)) = \frac{1}{2}(|G^{p^{i-1}}| - 2|G^{p^i}| + |G^{p^{i+1}}| + |G_\eta^{p^{i-1}}| - 2|G_\eta^{p^i}| + |G_\eta^{p^{i+1}}|). \quad (4)$$

We known from [3] that the equality $V(FG)^{p^i} = V(FG^{p^i})$ holds and the p -rank of $V(FG)^{p^{i-1}}$ equals $|G^{p^{i-1}}| - |G^{p^i}|$. Now (1) it yields that the p -rank of $V_\otimes(FG)^{p^{i-1}}$ equals $\frac{1}{2}(|G^{p^{i-1}}| - |G^{p^i}| - |G_\eta^{p^{i-1}}| + |G_\eta^{p^i}|)$. It immediately follows that the number of components of order p^i in the decomposition of the group $V_\otimes(FG)$ into a direct product of cyclic groups is equal to

$$f_i(V_\otimes(FG)) = \frac{1}{2}(|G^{p^{i-1}}| - 2|G^{p^i}| + |G^{p^{i+1}}| - |G_\eta^{p^{i-1}}| + 2|G_\eta^{p^i}| - |G_\eta^{p^{i+1}}|). \quad (5)$$

3. The bases

We will use the following well-known generators of $V(FG)$ (see [2]):

Lemma. *Let G be a finite abelian p -group, $I = I(FG)$ the augmentation ideal of FG and assume that $I^{s+1} = 0$. If*

$$v_{d1} + I^{d+1}, v_{d2} + I^{d+1}, \dots, v_{dr_d} + I^{d+1}$$

is a basis for I^d/I^{d+1} , then the units $1 + v_{dj}$ ($d = 1, 2, \dots, s$; $j = 1, 2, \dots, r_d$) generate $V(FG)$.

Let us return to the involutory automorphism η of G . Clearly, G has the decomposition

$$G = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_l \rangle \times \cdots \times \langle a_t \rangle$$

such that the elements $a_1, a_2, \dots, a_l$ inverted by η and if $t > l$ then η leaves fixed a_i for $i > l$.

Let q_i be the order of a_i and the set L consisting of those t -tuples $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_t)$ such that $\alpha_i \in \{0, 1, \dots, q_i - 1\}$ and at least one of α_j is not divisible by p ; the number of these elements $|G| - |G^p|$. Write L as the disjoint union $L = L_0 \cup L_1 \cup L_2$, where α belongs to L_0 , L_1 or L_2 according to whether $\alpha_1 + \alpha_2 + \cdots + \alpha_t$ is 0, odd or even and positive. The cardinality of L_1 is $\frac{1}{2}(|G| - |G^p| - |G_\eta| + |G_\eta^p|)$, and it is the p -rank of $V_\otimes(FG)$ and the cardinality of $L_0 \cup L_2$ is $\frac{1}{2}(|G| - |G^p| + |G_\eta| - |G_\eta^p|)$, which is the p -rank of $S_\otimes(FG)$.

Put $u_\alpha := 1 + (a_1 - 1)^{\alpha_1} (a_2 - 1)^{\alpha_2} \cdots (a_t - 1)^{\alpha_t}$ for $\alpha \in L$.

Theorem. *Let G be a finite abelian p -group of odd order with an involutory automorphism η and F is field of p elements. Then*

1. *The invariants of the unitary subgroup $V_\otimes(FG)$ are indicated in (5) and the set $\{u_\alpha^\otimes u_\alpha^{-1} \mid \alpha \in L_1\}$ is basis for it, that is $V_\otimes(FG) = \prod_{\alpha \in L_1} \langle u_\alpha^\otimes u_\alpha^{-1} \rangle$.*
2. *The invariants of the group of symmetric units $S_\otimes(FG)$ are indicated in (4) and the set $\{u_\alpha^\otimes u_\alpha \mid \alpha \in L_2\} \cup \{u_\alpha \mid \alpha \in L_0\}$ is basis for it, that is*

$$S_\otimes(FG) = \prod_{\alpha \in L_2} \langle u_\alpha^\otimes u_\alpha \rangle \times \prod_{\alpha \in L_0} \langle u_\alpha \rangle.$$

PROOF. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_t)$ with $\alpha_i \in \{0, 1, \dots, q_i - 1\}$; define $d(\alpha) = \alpha_1 + \alpha_2 + \cdots + \alpha_t$ and put $z_\alpha := (a_1 - 1)^{\alpha_1} (a_2 - 1)^{\alpha_2} \cdots (a_t - 1)^{\alpha_t}$. It is known from [2] that all z_α form a vector space basis for the augmentation ideal I and the elements $z_\alpha + I^{d+1}$ with $d(\alpha) = d$ constitute a basis for I^d/I^{d+1} with the property required for the application of Lemma. Therefore the elements $u_\alpha = 1 + z_\alpha$ generate $V(FG)$ and from [3] follows that $\{u_\alpha \mid \alpha \in L\}$ is a basis of $V(FG)$.

If $i \leq l$ then $((a_i - 1) + 1)^{-1} = ((a_i - 1) + 1)^{\otimes} = (a_i - 1)^{\otimes} + 1$ and from

$$\begin{aligned} (1 + (a_i - 1))(1 - (a_i - 1) + (a_i - 1)^2 - \cdots + (a_i - 1)^{q_i-1}) \\ = 1 + (a_i - 1)^{q_i} = 1 \end{aligned}$$

it follows that $(a_i - 1)^{\otimes} = -(a_i - 1) + (a_i - 1)^2 - \cdots + (a_i - 1)^{q_i-1}$. Note that for $i > l$ the equality $(a_i - 1)^{\otimes} = (a_i - 1)$ holds.

Let $\alpha \in L_1 \cup L_2$, $d = \alpha_1 + \alpha_2 + \cdots + \alpha_t$ and $k = \alpha_1 + \alpha_2 + \cdots + \alpha_l$. The above argument ensures that $z_\alpha^{\otimes} = (-1)^k z_\alpha + y$ for a suitable $y \in I^{d+1}$. It follows that if k is odd then

$$\begin{aligned} (1 + z_\alpha)^{-1}(1 + z_\alpha)^{\otimes} &= (1 - z_\alpha + z_\alpha^2 - \cdots)(1 - z_\alpha + y) \\ &\equiv 1 - 2z_\alpha \pmod{I^{d+1}} \end{aligned}$$

and for the even k we have

$$(1 + z_\alpha)(1 + z_\alpha)^{\otimes} = (1 + z_\alpha)(1 + z_\alpha + y) \equiv 1 + 2z_\alpha \pmod{I^{d+1}}.$$

Recall that $u_\alpha = 1 + z_\alpha$ and define

$$z'_\alpha = \begin{cases} u_\alpha^{-1}u_\alpha^{\otimes} - 1, & \text{if } \alpha \in L_1; \\ u_\alpha u_\alpha^{\otimes} - 1, & \text{if } \alpha \in L_2; \\ z_\alpha, & \text{if } \alpha \in L_0. \end{cases}$$

As a consequence of the foregoing argument, we obtain, modulo I^{d+1} ,

$$z'_\alpha \equiv \begin{cases} 2z_\alpha, & \text{if } \alpha \in L_1; \\ -2z_\alpha, & \text{if } \alpha \in L_2; \\ z_\alpha, & \text{if } \alpha \in L_0. \end{cases}$$

Now Lemma applies to the z'_α , yielding that the $u'_\alpha = 1 + z'_\alpha$ with $\alpha \in L$ also generate $V(FG)$. We claim that in fact they form a basis for $V(FG)$. To this end, it now suffices to show that the products of their orders is no larger than order $V(FG)$. Clearly that if $u'_\alpha \neq u_\alpha$ then such u'_α is the image of u_α either under the endomorphism $v \mapsto v^{\otimes}v^{-1}$ or under

the endomorphism $v \mapsto v^{\otimes} v$ of $V(FG)$ according to whether α belongs to L_1 or L_2 . It follows that $|u'_\alpha| \leq |u_\alpha|$, where $|u_\alpha|$ is the order of u_α . But [3] asserts that $\prod_{\alpha \in L} |u_\alpha|$ is the order of $V(FG)$ and this gives that $\prod_{\alpha \in L} |u'_\alpha| \leq |V(FG)|$, so $\{u'_\alpha \mid \alpha \in L\}$ is a basis. It follows from the definition of the u'_α with $\alpha \in L$ that each of them is either fixed or inverted by the involution $\otimes$. Accordingly, this basis is the disjoint union of bases for the subgroups of symmetric and unitary normalized units, and these are the bases in the theorem. $\square$

ACKNOWLEDGEMENT. The authors would like to thank the referee for their valuable comments and suggestions for clarifying the exposition.

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(Received January 4, 2006; revised September 1, 2006)