

On the characterization of n -polyadditive functions

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Abstract. The paper investigates the class of arithmetical functions satisfying the equation $\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} (-1)^k f(a_{i_1} a_{i_2} \dots a_{i_k}) = 0$. The set of solutions contains the completely additive and the completely quadritive functions.

In 1999 IMRE RUZSA [1] introduced a new class of functions which is wider than the class of the additive functions:

Definition. $f : \mathbb{N}^+ \rightarrow \mathbb{R}$ is quadritive if

$$f(abc) = f(ab) + f(ac) + f(bc) - f(a) - f(b) - f(c)$$

for all a, b, c pairwise coprime numbers.

He also proved several theorems concerning quadritive functions. We continue the generalization towards further dimensions.

Definition. For a positive integer n the function $f : \mathbb{N}^+ \rightarrow \mathbb{R}$ is n -polyadditive if

$$F(a_1, \dots, a_n) := \sum_{k=0}^n \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} (-1)^k f(a_{i_1} a_{i_2} \dots a_{i_k}) = 0 \quad (1)$$

for all $a_1, a_2, \dots, a_n$ positive integers.

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Remark. If f is 1-polyadditive then $f = f(1)$ (a constant). If f is 2-polyadditive then $f = g + f(1)$ with a completely additive function g . If f is 3-polyadditive then $f = h + f(1)$ with a completely quadritive function h .

Notation. $A_n :=$ the set of n -polyadditive functions
 $\nabla_a f(m) := f(am) - f(m)$.

Theorem 1.

- (1a) If $f \in A_k$ then $f \in A_t$ for all $t \geq k$.
- (1b) $f \in A_n$ if and only if $\nabla_a f \in A_{n-1}$ for all $a \in \mathbb{N}^+$.
- (1c) $f \in A_n$ if and only if $\nabla_{a_1} \nabla_{a_2} \dots \nabla_{a_n} f = 0$ for all $a_1, a_2, \dots, a_n$ positive integers.
- (1d) For any $f, g \in A_n$ and $\alpha, \beta \in \mathbb{R}$, also $\alpha f + \beta g \in A_n$.
- (1e) If $f \in A_k, g \in A_t$ then $fg \in A_{k+t-1}$.

Theorem 2. If f is n -polyadditive and $f(1) = 0$ then

(2a)

$$f(m^k) = \sum_{s=1}^{n-1} (-1)^{s-1} \left\{ \binom{n-1}{s} f(m^{k-s}) + \binom{n-1}{s-1} f(m^{n-s}) \right\} \quad (2)$$

for all $k \geq n$ and $m \in \mathbb{N}^+$.

(2b)

$$f(m^k) = \sum_{t=1}^{n-1} \frac{(-1)^{n-1-t}}{(n-1-t)!t!} \prod_{i=0, i \neq t}^{n-1} (k-i) f(m^t) \quad (k, m \in \mathbb{N}^+). \quad (3)$$

(2c) $f(\prod_{i=1}^s p_i^{\alpha_i})$ with primes p_i is a linear combination of the elements in

$$T = T(p_1, \dots, p_s) = \left\{ f(p_1^{\beta_1} \dots p_s^{\beta_s}) \mid \sum_{i=1}^s \beta_i \leq n-1 \right\}$$

with integer coefficients.

(2d) If f is completely additive then f^j is n -polyadditive for $j < n$ and f^j is not n -polyadditive for $j \geq n$ if $f \neq 0$.

Corollaries. (2a) $\implies$ If $f \in A_n$ then the values of $f(m^z)$ for all $z \geq n$ are determined by $f(m), f(m^2), \dots, f(m^{n-1})$ (a linear combination with integer coefficients).

(1a) and (2d) $\implies A_{n-1} \subsetneq A_n$.

Theorem 3. *If f is n -polyadditive and $f(1) = 0$ then to any $m > 1$ there exist $\alpha_{m,t} \in \mathbb{R}$ ($t = 1, \dots, n-1$) such that for all $k > 0$*

$$f(m^k) = \sum_{t=1}^{n-1} \alpha_{m,t} \log^t(m^k). \quad (4)$$

PROOF OF THEOREM 1. (1a) The identity

$$\begin{aligned} & F(a_1, \dots, a_{n-1}, a_n, a_{n+1}) \\ &= F(a_1, \dots, a_{n-1}, a_n) + F(a_1, \dots, a_{n-1}, a_{n+1}) - F(a_1, \dots, a_{n-1}, a_n a_{n+1}) \end{aligned}$$

implies $f \in A_n \implies f \in A_{n+1}$.

(1c) It can be easily checked by induction that

$$\Delta(m) = \nabla_{a_1} \dots \nabla_{a_n}(f(m)) = \sum_{j=0}^n \sum_{1 \leq i_1 < i_2 < \dots < i_j \leq n} (-1)^{n-j} f(a_{i_1} a_{i_2} \dots a_{i_j} m).$$

Now $\Delta(1) = F(a_1, \dots, a_n) = 0 \implies f \in A_n$. On the other hand $f \in A_n \implies f \in A_{n+1}$ by (1a), i.e. $F(a_1, \dots, a_n, m) = 0 \implies \Delta(m) = C(a_1, \dots, a_n)$ by (1). $\Delta(1) = 0 \implies C(a_1, \dots, a_n) = 0$.

(1b) This follows immediately from (1c).

(1d) is a direct consequence of (1c) or (1).

(1e) We prove by induction. For $k = 1$ it is true with an arbitrary $t \in \mathbb{N}^+$. Let us assume that it is true for $k + t < n$. We use the identity

$$\nabla_a(fg) = f\nabla_a g + g\nabla_a f + \nabla_a f \nabla_a g.$$

As $f \in A_k$, considering (1b) $\nabla_a f \in A_{k-1}$. As $g \in A_t$ by the hypothesis of the induction $g\nabla_a f \in A_{k+t-2}$. Considering (1a) $g\nabla_a f$ is in A_{k+t-1} . $f\nabla_a g$ and $\nabla_a f \nabla_a g \in A_{k+t-1}$ can be proven similarly. Using the identity and (1d) $\nabla_a(fg) \in A_{k+t-1}$. $\square$

PROOF OF THEOREM 2. (2a) For $k \geq n$ we substitute $a_1 = m^{k-n+1}$, $a_i = m$ ($i = 2, \dots, n$) in (1):

$$f(m^k) = \sum_{t=1}^{n-1} (-1)^{n-1-t} \left\{ \binom{n-1}{t-1} f(m^{k-n+1} m^{t-1}) + \binom{n-1}{t} f(m^t) \right\}.$$

By the substitution $s = n - t$ we get

$$f(m^k) = \sum_{s=1}^{n-1} (-1)^{s-1} \left\{ \binom{n-1}{s} f(m^{k-s}) + \binom{n-1}{s-1} f(m^{n-s}) \right\}.$$

Especially for $k = n$ we have

$$f(m^n) = \sum_{t=1}^{n-1} (-1)^{n-1-t} \binom{n}{t} f(m^t).$$

(2b) For $1 \leq k < n$ the product coefficient of $f(m^t)$ in (3) differs from 0 (and is equal to 1) only for $t = k$.

For $k = n$ (3) is true as

$$\begin{aligned} f(m^n) &= \sum_{t=1}^{n-1} (-1)^{n-1-t} \frac{n!}{(n-1-t)!t!(n-t)} f(m^t) \\ &= \sum_{t=1}^{n-1} (-1)^{n-1-t} \binom{n}{t} f(m^t) \end{aligned}$$

what we proved in (2a).

Let us assume that (3) is true for $n \leq v < k$. To verify that (3) is valid also for $v = k$ we substitute (3) into (2). We have to prove that

$$\begin{aligned} &-f(m^k) + \sum_{s=1}^{n-1} (-1)^{s-1} \left\{ \binom{n-1}{s} f(m^{k-s}) + \binom{n-1}{s-1} f(m^{n-s}) \right\} \\ &= \sum_{s=0}^{n-1} (-1)^{s-1} \left\{ \binom{n-1}{s} \sum_{t=1}^{n-1} \frac{(-1)^{n-1-t}}{(n-1-t)!t!} \prod_{i=0, i \neq t}^{n-1} (k-s-i) f(m^t) \right\} \\ &\quad + \sum_{t=1}^{n-1} (-1)^{n-1-t} \binom{n-1}{n-t-1} f(m^t) = 0. \end{aligned} \tag{5}$$

The coefficient of $f(m^z)$ ($z = 1, \dots, n-1$) in (5) is

$$\frac{(-1)^{n-z-1}}{(n-z-1)!z!} \left\{ \sum_{s=0}^{n-1} (-1)^{s-1} \binom{n-1}{s} \prod_{i=0, i \neq z}^{n-1} (k-s-i) + (n-1)! \right\}. \quad \square$$

Lemma.

$$D_n = d_{n,k,z} = \sum_{s=0}^{n-1} (-1)^{s-1} \binom{n-1}{s} \prod_{i=0, i \neq z}^{n-1} (k-s-i) = -(n-1)!$$

for all $z = 1, \dots, n-1$ and $k > n$.

PROOF OF THE LEMMA. We prove by induction. For $n = 2$ we have $-k + (k-1) = -1$. We assume that $D_t = -(t-1)!$ for $t \leq n$. First we prove that $nd_{n,k,z} - d_{n+1,k,z} = 0$ for $z \leq n-1$ and $k > n+1$. We have that

$$\begin{aligned} & \sum_{s=0}^{n-1} (-1)^{s-1} \binom{n}{s} (n-s) \prod_{i=0, i \neq z}^{n-1} (k-s-i) - \sum_{s=0}^n (-1)^{s-1} \binom{n}{s} \prod_{i=0, i \neq z}^n (k-s-i) \\ &= \sum_{s=0}^{n-1} (-1)^{s-1} \binom{n}{s} (2n-k) \prod_{i=0, i \neq z}^{n-1} (k-s-i) + (-1)^n \binom{n}{n} \prod_{i=0, i \neq z}^n (k-n-i) \\ &= (2n-k) \sum_{s=0}^n (-1)^{s-1} \binom{n}{s} \prod_{i=0, i \neq z}^{n-1} (k-s-i) = (2n-k) \sum_{s=0}^{n-1} (-1)^{s-1} \binom{n-1}{s} \\ & \quad \cdot \prod_{i=0, i \neq z}^{n-1} (k-s-i) + (2n-k) \sum_{s=1}^n (-1)^{s-1} \binom{n-1}{s-1} \prod_{i=0, i \neq z}^{n-1} (k-s-i) \\ &= (2n-k)[d_{n,k,z} - d_{n,k-1,z}] = 0 \end{aligned}$$

using the substitution $s' = s-1$ and $k' = k-1$ in the second sum.

For $z = n$ we have

$$\begin{aligned} d_{n+1,k,n} &= (d_{n+1,k,n} - d_{n+1,k,n-1}) - n! = \sum_{s=0}^n (-1)^{s-1} \binom{n}{s} \prod_{i=0}^{n-2} (k-s-i) - n! \\ &= \sum_{s=0}^{n-1} (-1)^{s-1} \binom{n-1}{s} \prod_{i=0}^{n-2} (k-s-i) + \sum_{s=1}^n (-1)^{s-1} \binom{n-1}{s-1} \prod_{i=0}^{n-2} (k-s-i) - n! \\ &= d_{n,k,n-1} + \sum_{s=1}^n (-1)^{s-1} \binom{n-1}{s-1} \prod_{i=0}^{n-2} [(k-1) - (s-1) - i] - n! \\ &= d_{n,k,n-1} - d_{n,k-1,n-1} - n! = -n!. \end{aligned}$$

(2c) First we prove the existence of the evaluation with integer coefficients. For $\alpha_1 + \dots + \alpha_s \leq n-1$ the assertion is trivial. Let us assume that it is also true

for $\alpha_1 + \dots + \alpha_s \leq z$ with some $z \geq n$. If $\alpha_1 + \dots + \alpha_s = z + 1 : f(p_1^{\alpha_1} \dots p_s^{\alpha_s}) = f(a_1 \dots a_{z+1})$ with $a_i > 1$ ($i = 1, \dots, z + 1$). By (1a) $f \in A_{z+1}$, hence

$$f(a_1 \dots a_{z+1}) = \sum_{k=1}^z \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq z+1} (-1)^{n-k-1} f(a_{i_1} a_{i_2} \dots a_{i_k}).$$

All the terms on the right hand side are determined by the values of the elements in $T = T(p_1, \dots, p_s)$ by the inductual hypothesis and the coefficients are integers.

(2d) $f^j \in A_{j+1}$ is an easy consequence of (1e), hence by (1a) f^j is in A_n for all $j < n$.

Now assume indirectly that $j \geq n$ and $f^j \in A_n$. As $f \neq 0$ there exists a prime p such that $f(p) \neq 0$. For any k we have $f^j(p^k) = k^j f^j(p)$, since f is completely additive. Then by (3) we obtain

$$k^j f^j(p) = f^j(p^k) = \sum_{z=1}^{n-1} \frac{(-1)^{n-1-z}}{(n-1-z)!z!} \prod_{i=0, i \neq z}^{n-1} (k-i) z^i f^j(p).$$

Dividing by $f^j(p)$ we see that the right hand side consists of terms bounded by $k^{n-1} n^j$, hence the right hand side is bounded by $c(n) \cdot k^{n-1}$. But the left hand side is $k^j \geq k^n$, which is a contradiction, if k is large enough. $\square$

PROOF OF THEOREM 3. For any fixed $m > 1$ the substitution $k = 1, \dots, n-1$ in (4) forms a system of linear equations with determinant $(n-1)!V(1, \dots, n-1) \log^{\frac{n(n-1)}{2}} m \neq 0$ (V denotes the Vandermonde-determinant), i.e. the values $\alpha_{m,t}$ for $t < n$ are uniquely determined. We prove that these numbers satisfy (4) also for $k \geq n$. We prove by induction. By (2a) for $k \geq n$

$$f(m^k) = \sum_{s=1}^{n-1} (-1)^{s-1} \left\{ \binom{n-1}{s} f(m^{k-s}) + \binom{n-1}{s-1} f(m^{n-s}) \right\},$$

hence it is enough to prove that

$$\begin{aligned} \sum_{t=1}^{n-1} \alpha_{m,t} \log^t(m^k) &= \sum_{s=1}^{n-1} (-1)^{s-1} \left\{ \binom{n-1}{s} \sum_{t=1}^{n-1} \alpha_{m,t} \log^t(m^{k-s}) \right. \\ &\quad \left. + \binom{n-1}{s-1} \sum_{t=1}^{n-1} \alpha_{m,t} \log^t(m^{n-s}) \right\} \quad (k = n, n+1, \dots). \end{aligned}$$

It is enough to examine the coefficients of $\alpha_{m,t} \log^t m$, i.e. to prove that

$$-\sum_{s=0}^{n-1} (-1)^s \binom{n-1}{s} (k-s)^t + \sum_{s=0}^{n-2} (-1)^s \binom{n-1}{s} (n-1-s)^t = 0 \quad (6)$$

for $k = n, n + 1, \dots$

We show that each of the two sums in (6) is 0 for $t \leq n - 2$, and is $(n - 1)!$ for $t = n - 1$.

Denote by $H(r, n - 1, t)$ the number of those t -digit integers which contain all digits $0, 1, \dots, n - 2$ in an r -base number system ($r \geq n - 1$). Then the first sum of (6) is $H(k, n - 1, t)$, and the second sum is $H(n - 1, n - 1, t)$, if we perform the calculation by the logical sieve: the sifting properties are that the digit i ($i = 0, 1, \dots, n - 2$) is missing. On the other hand, clearly $H(r, n - 1, t) = 0$ for $t < n - 1$ and $H(r, n - 1, n - 1) = (n - 1)!$. $\square$

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