

A problem of Galambos on Oppenheim series expansions

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Abstract. In this paper, we investigate the Hausdorff dimension of exceptional sets in the metric properties of digits of Oppenheim series expansions and answer a question posed by Galambos.

1. Introduction

For any $x \in (0, 1]$, the algorithm

$$x = x_1, \quad d_n = [1/x_n] + 1, \quad x_n = 1/d_n + a_n/b_n \cdot x_{n+1}, \quad (1)$$

where $a_n = a_n(d_1, \dots, d_n)$ and $b_n = b_n(d_1, \dots, d_n)$ are positive integer valued functions and $[y]$ denotes the integer part of y , leads to the OPPENHEIM expansion [12]

$$x \sim \frac{1}{d_1} + \frac{a_1}{b_1} \frac{1}{d_2} + \dots + \frac{a_1 a_2 \dots a_n}{b_1 b_2 \dots b_n} \frac{1}{d_{n+1}} + \dots \quad (2)$$

By (1),

$$\frac{1}{d_n} < x_n \leq \frac{1}{d_n - 1}, \quad (3)$$

and hence by the last equality in (1),

$$d_{n+1} > \frac{a_n}{b_n} d_n (d_n - 1). \quad (4)$$

The expansion defined by (1) and (2) is convergent and its sum is equal to x . A sufficient condition for a series on the right hand side in (2) to be the expansion of its sum by the algorithm (1) is (see [12])

$$d_{n+1} \geq \frac{a_n}{b_n} d_n (d_n - 1) + 1 \quad \text{for all } n \geq 1. \quad (5)$$

Definition 1.1. We call the expansion (2) (obtained by the algorithm (1)) restricted Oppenheim expansion of x if a_n and b_n depend on the last denominator d_n only and if the function

$$h_n(j) = \frac{a_n(j)}{b_n(j)} j(j-1) \quad (6)$$

is integer-valued, for all $n \geq 1$ and $j \geq 2$.

In the present paper, we deal with restricted Oppenheim expansions only. In this case, (4) and (5) are equivalent.

The representation (2) under (1) was first studied by OPPENHEIM [12], including LÜROTH ([11]), Engel, Sylvester expansions ([2]) and Cantor infinite product ([13]) as special cases. Oppenheim established the arithmetical properties, including the question of rationality of the expansion. The foundations of the metric theory of such expansions were laid down by GALAMBOS [5], [6], [7], [9], see also the monographs of GALAMBOS [8], SCHWEIGER [14], VERVAAT [15], DAJANI and KRAAIKAMP [1]. From [8], Chapter 6, it can be seen that the integer approximations $T_n(x)$ to the ratios $d_n(x)/h_{n-1}(d_{n-1}(x))$ defined by

$$T_n(x) < \frac{d_n(x)}{h_{n-1}(d_{n-1}(x))} \leq T_n(x) + 1, \quad n \geq 1, \quad (7)$$

where $h_0(x) \equiv 1$, plays an important role in the metric theory of Oppenheim expansions, see GALAMBOS [8] Chapter VI. Moreover, they are stochastically independent and are distributed as the denominators in the Lüroth expansion. GALAMBOS, see [8] Page 132, posed the question to calculate the Hausdorff dimension of the set

$$B_m = \{x \in (0, 1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1\}, \quad m \geq 2,$$

and compare this with the Lüroth case. In [16], the second author concerned this problem under the condition $h_n(j)$ is of order t ($t \geq 1$), see [16] for the definition. In this paper, we continue to consider this problem. Under more natural conditions, we obtain the Hausdorff dimension of B_m and thus answer

the question of Galambos. To obtain the lower bound of the Hausdorff dimension of a fractal set, a mass distribution is needed, which is a necessary (and sufficient) tool for this. The mass distribution constructed here is quite technical and subtle.

We use $|\cdot|$ to denote the diameter of a subset of $(0, 1]$, $\dim_H$ to denote the Hausdorff dimension and ‘cl’ the closure of a subset of $(0, 1]$ respectively.

2. Hausdorff dimension of B_m

For any $m \geq 2$, let

$$B_m = \{x \in (0, 1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1\}.$$

By (7), it is easy to check that

$$B_m = \left\{x \in (0, 1] : 1 < \frac{d_n(x)}{h_{n-1}(d_{n-1}(x))} \leq m + 1 \text{ for all } n \geq 1\right\}, \quad (8)$$

where $h_0(n) \equiv 1$. Thus in order to calculate the Hausdorff dimensions of B_m , $m \geq 2$, it is sufficient to consider the following sets

$$C_m = \left\{x \in (0, 1] : 1 < \frac{d_n(x)}{h_{n-1}(d_{n-1}(x))} \leq m \text{ for all } n \geq 1\right\}, \quad m \geq 3.$$

From now on, we fix $m \geq 3$ be a positive integer.

Lemma 2.1. *For any integer $a \geq 1$, let $S(a)$ be determined by the following equation*

$$\sum_{a < b \leq ma} \left(\frac{a}{b(b-1)}\right)^{S(a)} = 1. \quad (9)$$

Then

$$\lim_{a \rightarrow +\infty} S(a) = 1.$$

PROOF. Since

$$\sum_{a < b \leq ma} \left(\frac{a}{b(b-1)}\right) = 1 - \frac{1}{m} < 1,$$

we have $S(a) \leq 1$ for all $a \geq 1$.

On the other hand, for any $1/2 < s < 1$,

$$\sum_{a < b \leq ma} \left(\frac{a}{b(b-1)}\right)^s \geq \sum_{a \leq b \leq ma} \left(\frac{a}{b(b-1)}\right)^s - \left(\frac{1}{a-1}\right)^s$$

$$\begin{aligned}
&\geq \int_a^{ma} \frac{a^s}{x^{2s}} dx - \left(\frac{1}{a-1} \right)^s \\
&= \frac{1}{1-2s} ((ma)^{1-2s} - a^{1-2s}) \cdot a^s - \left(\frac{1}{a-1} \right)^s \\
&= \frac{(1-m^{1-2s}) \cdot a^{1-s}}{2s-1} - \left(\frac{1}{a-1} \right)^s > 1, \quad a \text{ is large enough.}
\end{aligned}$$

Thus when a is large enough, $S(a) > s$. The proof of Lemma 2.1 is finished. $\square$

We now state the mass distribution principle, see [4] Proposition 2.3, that will be used later.

Lemma 2.2. *Let $E \subset (0, 1]$ be a Borel set and μ be a measure with $\mu(E) > 0$. If for any $x \in E$,*

$$\liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \geq s,$$

where $B(x, r)$ denotes the open ball with center at x and radius r , then $\dim_H E \geq s$.

Now we are in the position to prove the main result of this paper.

Theorem 2.3. *Suppose $h_j(d) \geq d-1$ for all $j \geq 1$ and $d \geq 2$, then for each $m \geq 3$,*

$$\dim_H C_m = 1.$$

PROOF. For any $j \geq 1$ and $d \geq 2$, define

$$G_j(d) = m \cdot h_j(d);$$

$$M_j(m) = G_{j-1} \circ G_{j-2} \circ \cdots \circ G_1(m), \quad M_1(m) := m.$$

From the assumption on $h_j(d)$, it is easy to check that

$$M_j(m) \geq m^j - m^{j-1} - \cdots - m^2 - m \quad \text{for each } j \geq 1,$$

thus

$$\lim_{j \rightarrow \infty} M_j(m) = +\infty. \tag{10}$$

For any $0 < s < 1$, from Lemma 2.1, since $\lim_{a \rightarrow \infty} S(a) = 1$, there exists $a_0 \in \mathbb{N}$ such that for any $a \geq a_0$, $S(a) > s$. By (10), there exists $k_0 \geq 1$ such that for any $k \geq k_0$,

$$M_k(m) \geq a_0 + 1. \tag{11}$$

Define

$$E_m = \left\{ x \in (0, 1] : d_j(x) = M_j(m) \text{ for all } 1 \leq j \leq k_0, \right. \\ \left. \text{and } 1 < \frac{d_{j+1}(x)}{h_j(d_j(x))} \leq m \text{ for all } j \geq k_0 \right\}.$$

It is clear that $E_m \subset C_m$. Now we estimate the Hausdorff dimension of E_m .

For any $x \in E_m$, since $h_j(d) \geq d - 1$ for all $j \geq 1$ and $d \geq 2$, by (5), we have, for any $k \geq k_0$,

$$\begin{aligned} d_k(x) &\geq h_{k-1}(d_{k-1}(x)) + 1 \geq d_{k-1}(x) \geq \cdots \geq d_{k_0+1}(x) \\ &\geq h_{k_0}(d_{k_0}(x)) + 1 \geq d_{k_0}(x) = M_{k_0}(m) \geq a_0 + 1, \end{aligned} \quad (12)$$

and

$$h_k(d_k(x)) \geq d_k(x) - 1 \geq a_0. \quad (13)$$

Now we introduce a symbolic space defined as follows:

For any $k \geq k_0$, let

$$D_k = \left\{ \sigma = (\sigma_1, \dots, \sigma_k) \in \mathbb{N}^k : \sigma_j = M_j(m) \text{ for all } 1 \leq j \leq k_0, \right. \\ \left. \text{and } 1 < \frac{\sigma_{j+1}}{h_j(\sigma_j)} \leq m \text{ for all } k_0 \leq j \leq k-1 \right\},$$

and define

$$D = \bigcup_{k=k_0}^{\infty} D_k.$$

For any $k \geq k_0$ and $\sigma = (\sigma_1, \dots, \sigma_k) \in D_k$, let J_σ and I_σ denote the following closed subintervals of $(0, 1]$:

$$J_\sigma = \bigcup_{h_k(\sigma_k) < d \leq m h_k(\sigma_k)} \text{cl}\{x \in (0, 1] : d_1(x) = \sigma_1, d_2(x) = \sigma_2, \dots, d_k(x) = \sigma_k, d_{k+1}(x) = d\},$$

$$I_\sigma = \text{cl}\{x \in (0, 1] : d_1(x) = \sigma_1, d_2(x) = \sigma_2, \dots, d_k(x) = \sigma_k\},$$

and each J_σ is called an interval of k th-order. Finally, define

$$E = \bigcap_{k=k_0}^{\infty} \bigcup_{\sigma \in D_k} J_\sigma.$$

It is obvious that

$$E = E_m.$$

From the proof of Theorem 6.1 in [8], we have, for any $k \geq k_0$ and $\sigma \in D_k$,

$$|I_\sigma| = \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_2(\sigma_2)}{b_2(\sigma_2)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{(\sigma_k - 1)\sigma_k}, \quad (14)$$

thus by (6), we have

$$\begin{aligned} |J_\sigma| &= \sum_{h_k(\sigma_k) < d \leq m h_k(\sigma_k)} \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{(d-1)d} \\ &= \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \left(\frac{1}{h_k(\sigma_k)} - \frac{1}{m h_k(\sigma_k)} \right) \\ &= \left(1 - \frac{1}{m} \right) \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{h_k(\sigma_k)} \\ &= \left(1 - \frac{1}{m} \right) \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{(\sigma_k - 1)\sigma_k}. \end{aligned} \quad (15)$$

For any $k \geq k_0$, $\sigma \in D_k$, define

$$\mu(J_\sigma) = \prod_{i=k_0}^{k-1} \left(\frac{h_i(\sigma_i)}{\sigma_{i+1}(\sigma_{i+1} - 1)} \right)^{S(h_i(\sigma_i))}, \quad \text{if } k \geq k_0 + 1, \quad (16)$$

and

$$5\mu(J_\sigma) = 1, \quad \text{if } \sigma \in D_{k_0}.$$

μ is a probability mass distribution supported on E_m , because

$$\begin{aligned} &\sum_{\sigma_{k+1}=h_k(\sigma_k)+1}^{m h_k(\sigma_k)} \mu(J_{\sigma_1 \sigma_2 \dots \sigma_{k+1}}) \\ &= \sum_{\sigma_{k+1}=h_k(\sigma_k)+1}^{m h_k(\sigma_k)} \prod_{i=k_0}^k \left(\frac{h_i(\sigma_i)}{\sigma_{i+1}(\sigma_{i+1} - 1)} \right)^{S(h_i(\sigma_i))} = \mu(J_{\sigma_1 \sigma_2 \dots \sigma_k}), \end{aligned}$$

and

$$\sum_{\sigma_{k_0+1}=h_{k_0}(\sigma_{k_0})+1}^{m h_{k_0}(\sigma_{k_0})} \mu(J_{\sigma_1 \sigma_2 \dots \sigma_{k_0+1}})$$

$$\begin{aligned}
&= \sum_{\sigma_{k_0+1}=h_{k_0}(\sigma_{k_0})+1}^{mh_{k_0}(\sigma_{k_0})} \left(\frac{h_{k_0}(\sigma_{k_0})}{\sigma_{k_0+1}(\sigma_{k_0+1}-1)} \right)^{S(h_{k_0}(\sigma_{k_0}))} \\
&= 1 = \mu(J_{\sigma_1\sigma_2\ldots\sigma_{k_0}}).
\end{aligned}$$

For any $x \in E_m$, we prove that

$$\liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \geq s. \quad (17)$$

If (17) is proved, by Lemma 2.2, we have $\dim_H E_m \geq s$. Since $0 < s < 1$ is arbitrary, this implies $\dim_H C_m = 1$.

Now we prove (17).

For any $x \in E_m$, there exists $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n, \dots)$ such that for any $k \geq k_0$, $(\sigma|k) := (\sigma_1, \sigma_2, \dots, \sigma_k) \in D_k$ and $d_j(x) = \sigma_j$ for each $j \geq 1$. Thus

$$x \in J_{\sigma_1\sigma_2\ldots\sigma_k} \quad \text{for all } k \geq k_0.$$

From the proof of Theorem 6.1 in [8], we have, for any $k \geq k_0$, the right endpoint of the interval $J_{\sigma_1\sigma_2\ldots\sigma_k}$, i.e., $\max\{y \in (0, 1] : y \in J_{\sigma_1\sigma_2\ldots\sigma_k}\}$, is

$$\begin{aligned}
&\frac{1}{\sigma_1} + \sum_{j=2}^k \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{j-1}(\sigma_{j-1})}{b_{j-1}(\sigma_{j-1})} \cdot \frac{1}{\sigma_j} + \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{h_k(\sigma_k)} \\
&= \frac{1}{\sigma_1} + \sum_{j=2}^k \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{j-1}(\sigma_{j-1})}{b_{j-1}(\sigma_{j-1})} \cdot \frac{1}{\sigma_j} \\
&\quad + \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{\sigma_k(\sigma_k-1)} \\
&= \frac{1}{\sigma_1} + \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{1}{\sigma_2} + \cdots + \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{\sigma_k-1}. \quad (18)
\end{aligned}$$

The left endpoint of the interval $J_{\sigma_1\sigma_2\ldots\sigma_k}$, i.e., $\min\{y \in (0, 1] : y \in J_{\sigma_1\sigma_2\ldots\sigma_k}\}$, is

$$\begin{aligned}
&\frac{1}{\sigma_1} + \sum_{j=2}^k \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{j-1}(\sigma_{j-1})}{b_{j-1}(\sigma_{j-1})} \cdot \frac{1}{\sigma_j} + \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{mh_k(\sigma_k)} \\
&= \frac{1}{\sigma_1} + \sum_{j=2}^k \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{j-1}(\sigma_{j-1})}{b_{j-1}(\sigma_{j-1})} \cdot \frac{1}{\sigma_j} \\
&\quad + \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{m\sigma_k(\sigma_k-1)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sigma_1} + \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{1}{\sigma_2} + \dots \\
&\quad + \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \dots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \left(\frac{1}{\sigma_k} + \frac{1}{m\sigma_k(\sigma_k - 1)} \right). \tag{19}
\end{aligned}$$

If $\sigma_k - 1 > h_{k-1}(\sigma_{k-1})$, from (18), (19), we know the gap between $J_{\sigma_1\sigma_2\dots\sigma_k}$ and $J_{\sigma_1\dots\sigma_{k-1}\sigma_{k-1}}$ is

$$\frac{a_1(\sigma_1)}{b_1(\sigma_1)} \dots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{m(\sigma_k - 1)(\sigma_k - 2)}. \tag{20}$$

In the same way, if $\sigma_k + 1 \leq mh_{k-1}(\sigma_{k-1})$, from (18), (19), we know the gap between $J_{\sigma_1\sigma_2\dots\sigma_k}$ and $J_{\sigma_1\dots\sigma_{k-1}\sigma_k+1}$ is

$$\frac{a_1(\sigma_1)}{b_1(\sigma_1)} \dots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{m\sigma_k(\sigma_k - 1)}. \tag{21}$$

For any $0 < r < \frac{1}{m}|I_{M_1(m)M_2(m)\dots M_{k_0}(m)}|$, since

$$(\sigma|k_0) = (M_1(m), M_2(m), \dots, M_{k_0}(m))$$

and $|I_{(\sigma|k)}| \rightarrow 0$ as $k \rightarrow \infty$, there exists k (depends on x) such that

$$\frac{1}{m}|I_{(\sigma|k+1)}| < r \leq \frac{1}{m}|I_{(\sigma|k)}|,$$

that is,

$$\begin{aligned}
&\frac{1}{m} \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \dots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{\sigma_{k+1}(\sigma_{k+1} - 1)} \\
&\quad < r \leq \frac{1}{m} \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \dots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{\sigma_k(\sigma_k - 1)}. \tag{22}
\end{aligned}$$

By (14), (20) and (21), $B(x, r)$ can intersect only one k th-order interval $J_{\sigma_1\sigma_2\dots\sigma_k}$.

On the other hand, for every $h_k(\sigma_k) < j \leq mh_k(\sigma_k)$, from (14), we have

$$|I_{\sigma_1\sigma_2\dots\sigma_k j}| \geq \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \dots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{mh_k(\sigma_k)(mh_k(\sigma_k) - 1)}.$$

Thus $B(x, r)$ can intersect at most

$$\frac{4r(mh_k(\sigma_k))^2}{\frac{a_1(\sigma_1)}{b_1(\sigma_1)} \dots \frac{a_k(\sigma_k)}{b_k(\sigma_k)}} := l$$

$(k+1)$ -th-order intervals. Therefore

$$\mu(B(x, r)) \leq \min \left\{ \mu(J_{\sigma_1 \sigma_2 \dots \sigma_k}), \sum_i \mu(J_{\sigma_1 \sigma_2 \dots \sigma_k i}) \right\},$$

where the sum is over all i such that $\max\{\sigma_{k+1} - l, h_k(\sigma_k) + 1\} \leq i \leq \sigma_{k+1} + l$.

By (16), we have

$$\begin{aligned} \mu(B(x, r)) &\leq \mu(J_{\sigma_1 \sigma_2 \dots \sigma_k}) \min \left\{ 1, \sum_i \left(\frac{h_k(\sigma_k)}{i(i-1)} \right)^{S(h_k(\sigma_k))} \right\} \\ &\leq \mu(J_{\sigma_1 \sigma_2 \dots \sigma_k}) \min \left\{ 1, 2l \left(\frac{1}{h_k(\sigma_k)} \right)^{S(h_k(\sigma_k))} \right\} \\ &= \mu(J_{\sigma_1 \sigma_2 \dots \sigma_k}) \min \left\{ 1, \frac{8r(mh_k(\sigma_k))^2}{\frac{a_1(\sigma_1)}{b_1(\sigma_1)} \dots \frac{a_k(\sigma_k)}{b_k(\sigma_k)}} \left(\frac{1}{h_k(\sigma_k)} \right)^{S(h_k(\sigma_k))} \right\} \\ &\leq \mu(J_{\sigma_1 \sigma_2 \dots \sigma_k}) \cdot 1^{1-s} \cdot \left(\frac{8r(mh_k(\sigma_k))^2}{\frac{a_1(\sigma_1)}{b_1(\sigma_1)} \dots \frac{a_k(\sigma_k)}{b_k(\sigma_k)}} \left(\frac{1}{h_k(\sigma_k)} \right)^{S(h_k(\sigma_k))} \right)^s. \end{aligned}$$

From (13), we have, for any $n \geq k_0$,

$$h_n(\sigma_n) \geq a_0,$$

thus

$$S(h_n(\sigma_n)) \geq s \quad \text{for all } n \geq k_0. \quad (23)$$

Combining (6), (16) and (23), we have

$$\begin{aligned} \mu(B(x, r)) &\leq \left[\prod_{i=k_0}^{k-1} \frac{h_i(\sigma_i)}{\sigma_{i+1}(\sigma_{i+1} - 1)} \left(\frac{8r(mh_k(\sigma_k))^2}{\frac{a_1(\sigma_1)}{b_1(\sigma_1)} \dots \frac{a_k(\sigma_k)}{b_k(\sigma_k)}} \left(\frac{1}{h_k(\sigma_k)} \right)^{S(h_k(\sigma_k))} \right) \right]^s \\ &= \left(h_{k_0}(M_{k_0}(m)) \frac{a_{k_0+1}(\sigma_{k_0+1})}{b_{k_0+1}(\sigma_{k_0+1})} \dots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \frac{1}{\sigma_k(\sigma_k - 1)} \right)^s \\ &\quad \cdot \left(\frac{8r(mh_k(\sigma_k))^2}{\frac{a_1(\sigma_1)}{b_1(\sigma_1)} \dots \frac{a_k(\sigma_k)}{b_k(\sigma_k)}} \right)^s \cdot \left(\frac{1}{h_k(\sigma_k)} \right)^{sS(h_k(\sigma_k))} \\ &= \left(h_{k_0}(M_{k_0}(m)) \cdot \frac{b_1(M_1(m))}{a_1(M_1(m))} \dots \frac{b_{k_0}(M_{k_0}(m))}{a_{k_0}(M_{k_0}(m))} \right)^s \end{aligned}$$

$$\begin{aligned}
& \cdot \left(\frac{8r(mh_k(\sigma_k))^2}{\sigma_k(\sigma_k - 1)^{\frac{a_k(\sigma_k)}{b_k(\sigma_k)}}} \cdot \left(\frac{1}{h_k(\sigma_k)} \right)^{S(h_k(\sigma_k))} \right)^s \\
& \leq c_1^s \left(r \cdot h_k(\sigma_k) \left(\frac{1}{h_k(\sigma_k)} \right)^{S(h_k(\sigma_k))} \right)^s,
\end{aligned}$$

where c_1 is a positive constant which does not depend on x and r .

From the definition of $S(a)$, we have

$$\begin{aligned}
1 &= \sum_{a < b \leq ma} \left(\frac{a}{b(b-1)} \right)^{S(a)} \\
&\geq (m-1)a \left(\frac{a}{ma(ma-1)} \right)^{S(a)} \geq (m-1)a \left(\frac{a}{ma \cdot ma} \right)^{S(a)} \\
&= (m-1)a \left(\frac{1}{m^2 a} \right)^{S(a)},
\end{aligned}$$

thus

$$\frac{a}{a^{S(a)}} \leq \frac{m^{2S(a)}}{m-1} \leq \frac{m^2}{m-1},$$

and this implies

$$h_k(\sigma_k) \left(\frac{1}{h_k(\sigma_k)} \right)^{S(h_k(\sigma_k))} \leq \frac{m^2}{m-1}.$$

Therefore

$$\mu(B(x, r)) \leq c_2^s \cdot r^s, \quad (24)$$

where c_2 is a positive constant which does not depend on x and r .

From (24), we know (17) holds. This completes the proof of Theorem 2.3. $\square$

From (8) and Theorem 2.3, we have

Corollary 2.4. *Suppose $h_j(d) \geq d-1$ for all $j \geq 1$ and $d \geq 2$, then for each $m \geq 2$, we have $\dim_H B_m = 1$.*

Remark 2.5. Let $a_n(d_1, \dots, d_n) = 1$, $b_n(d_1, \dots, d_n) = d_n(d_n - 1)$, ($n = 1, 2, \dots$). Then the algorithm (1) leads to the Lüroth expansion of x ,

$$x = \frac{1}{d_1(x)} + \dots + \frac{1}{d_1(x)(d_1(x) - 1) \dots d_{n-1}(x)(d_{n-1}(x) - 1)d_n(x)} + \dots \quad (25)$$

Here $h_n(j) = 1$ and $T_n(x) = d_n(x) - 1$. For the Lüroth series, with the help of the theory of self similar set, see [3], Chapter 9, the Hausdorff dimension s of the B_m is determined by the following equation

$$\sum_{2 \leq b \leq m+1} \left(\frac{1}{b(b-1)} \right)^s = 1.$$

To some extent, Lüroth series expansion stands as a special case to say that the assumption on h_j in the main theorem is not superfluous. Moreover, we can obtain: if $l \leq h_j(d_j(x)) \leq L$, for all $x \in C_m = B_{m-1}$ and j larger than some fixed integer k_0 , then one can has

$$0 < \inf_{l \leq a \leq L} S(a) \leq \dim_H C_m \leq \sup_{l \leq a \leq L} S(a) < 1.$$

We now list some special cases which satisfy the assumption in Theorem 2.3.

Example 1. Engel expansion. Let $a_n(d_1, \dots, d_n) = 1$, $b_n(d_1, \dots, d_n) = d_n$, ($n = 1, 2, \dots$). Then (2), together with the algorithm (1), become Engel expansion of x ,

$$x = \frac{1}{d_1(x)} + \frac{1}{d_1(x)d_2(x)} + \dots + \frac{1}{d_1(x)d_2(x)\dots d_n(x)} + \dots \quad (26)$$

In this case, $h_n(j) = j - 1$ and $T_n(x) = \frac{d_n(x)}{d_{n-1}(x)-1} - 1$ if $\frac{d_n(x)}{d_{n-1}(x)-1}$ is an integer and $\left\lceil \frac{d_n(x)}{d_{n-1}(x)-1} \right\rceil$ otherwise. By Corollary 2.4, we have for each $m \geq 2$,

$$\dim_H \{x \in (0, 1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1\} = 1.$$

Example 2. Sylvester expansion. Choose $a_n(d_1, \dots, d_n) = 1$, $b_n(d_1, \dots, d_n) = 1$, ($n = 1, 2, \dots$). We get the Sylvester expansion of x ,

$$x = \frac{1}{d_1(x)} + \frac{1}{d_2(x)} + \dots + \frac{1}{d_n(x)} + \dots \quad (27)$$

Here $h_n(j) = j(j-1)$ and $T_n(x) = \frac{d_n(x)}{d_{n-1}(x)(d_{n-1}(x)-1)} - 1$ if $\frac{d_n(x)}{d_{n-1}(x)(d_{n-1}(x)-1)}$ is an integer and $\left\lceil \frac{d_n(x)}{d_{n-1}(x)(d_{n-1}(x)-1)} \right\rceil$ otherwise. By Corollary 2.4, we have for each $m \geq 2$,

$$\dim_H \{x \in (0, 1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1\} = 1.$$

Example 3. Cantor product. Take $a_n(d_1, \dots, d_n) = d_n + 1$, $b_n(d_1, \dots, d_n) = d_n$, ($n = 1, 2, \dots$), the expansion (2) yields the Cantor product,

$$1 + x = \left(1 + \frac{1}{d_1(x)}\right) \left(1 + \frac{1}{d_2(x)}\right) \dots \left(1 + \frac{1}{d_n(x)}\right) \dots \quad (28)$$

Here $h_n(j) = j^2 - 1$ and $T_n(x) = \frac{d_n(x)}{d_{n-1}^2(x)-1} - 1$ if $\frac{d_n(x)}{d_{n-1}^2(x)-1}$ is an integer and $\left[\frac{d_n(x)}{d_{n-1}^2(x)-1}\right]$ otherwise. By Corollary 2.4, we have for each $m \geq 2$,

$$\dim_H\{x \in (0, 1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1\} = 1.$$

Example 4. Modified Engel expansion. Let $a_n(d_1, \dots, d_n) = 1$, $b_n(d_1, \dots, d_n) = d_n - 1$, ($n = 1, 2, \dots$). We get the modified Engel expansion of x ,

$$x = \frac{1}{d_1(x)} + \dots + \frac{1}{(d_1(x)-1)(d_2(x)-1) \dots (d_{n-1}(x)-1)d_n(x)} + \dots \quad (29)$$

Thus $h_n(j) = j$ and $T_n(x) = \frac{d_n(x)}{d_{n-1}(x)} - 1$ if $\frac{d_n(x)}{d_{n-1}(x)}$ is an integer and $\left[\frac{d_n(x)}{d_{n-1}(x)}\right]$ otherwise. By Corollary 2.4, we have for each $m \geq 2$,

$$\dim_H\{x \in (0, 1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1\} = 1.$$

Example 5. Daróczy–Kátai–Birthday expansion. Choose $a_n(d_1, \dots, d_n) = d_n$, $b_n(d_1, \dots, d_n) = 1$, ($n = 1, 2, \dots$), the resulting series expansion of x takes the form,

$$x = \frac{1}{d_1(x)} + \frac{d_1(x)}{d_2(x)} + \dots + \frac{d_1(x)d_2(x) \dots d_{n-1}(x)}{d_n(x)} + \dots \quad (30)$$

The Daróczy–Kátai–Birthday expansion was introduced for the first time in GALAMBOS [9]. Here $h_n(j) = j^2(j-1)$ and $T_n(x) = \frac{d_n(x)}{d_{n-1}^2(x)(d_{n-1}(x)-1)} - 1$ if $\frac{d_n(x)}{d_{n-1}^2(x)(d_{n-1}(x)-1)}$ is an integer and $\left[\frac{d_n(x)}{d_{n-1}^2(x)(d_{n-1}(x)-1)}\right]$ otherwise. By Corollary 2.4, we have for each $m \geq 2$,

$$\dim_H\{x \in (0, 1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1\} = 1.$$

Remark 2.6. A modification of (1) and (3) to the algorithm $0 < x \leq 1$, $x = x_1$, and

$$\frac{1}{D_n + 1} < x_n \leq \frac{1}{D_n}, \quad \frac{1}{D_n} - x_n = \frac{a_n}{b_n} \cdot x_{n+1}. \quad (31)$$

generates an alternating series representation

$$x \sim \frac{1}{D_1} - \frac{a_1}{b_1} \frac{1}{D_2} + \cdots + (-1)^n \frac{a_1 a_2 \cdots a_n}{b_1 b_2 \cdots b_n} \frac{1}{D_{n+1}} + \cdots, \quad (32)$$

called alternating Oppenheim expansion. The metric theory for the alternating Oppenheim expansion was studied recently in [10]. Using the same method, we can get the corresponding results of Theorem 2.3 and Corollary 2.4 for this expansion.

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