Publ. Math. Debrecen
70/1-2 (2007), 103–117

The extensibility of $D(-1)$-triples $\{1, b, c\}$

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Abstract. Let $b = 5, 10, 17, 26, 37$ or 50. In this paper, we show that for those integers c greater than 1 such that both $c - 1$ and $bc - 1$ are squares, the system of simultaneous Pell equations

$$z^2 - cx^2 = c - 1, \quad bz^2 - cy^2 = c - b$$

has only the trivial solutions $(x, y, z) = (0, \pm\sqrt{b-1}, \pm\sqrt{c-1})$. This implies that there do not exist integers $c, d\,(> 1)$ such that the product of any two distinct elements of the set $\{1, b, c, d\}$ diminished by 1 is a square. We also show that in case b is a positive integer with $\sqrt{b-1}$ a prime, if it is true for the smallest eight c's with $c > 1$ for which the set $\{1, b, c\}$ has the property above, then the same is true for all such c's.

1. Introduction

Diophantus raised the problem of finding four (positive rational) numbers a_1, a_2, a_3, a_4 such that $a_ia_j + 1$ is a square for each $1 \le i < j \le 4$ and gave a solution $\{1/16, 33/16, 68/16, 105/16\}$. The first set of four positive integers $\{1, 3, 8, 120\}$ with the property above was found by Fermat. Replacing "+1" by "$+n$" leads to the following definition.

Definition 1.1. Let n be a nonzero integer. A set of m distinct positive integers $\{a_1, \ldots, a_m\}$ is called a $D(n)$-m-tuple (or a Diophantine m-tuple with the property $D(n)$, or a P_n-set of size m) if $a_ia_j + n$ is a square for each $1 \le i < j \le m$.

Mathematics Subject Classification: 11D09, 11J68, 11J86.
Key words and phrases: simultaneous Pell equations, Diophantine tuples.

In 1993, Dujella showed that if $n \not\equiv 2 \pmod 4$ and $n \notin S := \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists a $D(n)$-quadruple ([7]), and conjectured that if $n \in S$, then there does not exist a $D(n)$-quadruple ([8]). In the case of $n = -1$, there are several results supporting the validity of this conjecture. It is known that the following $D(-1)$-triples can not be extended to $D(-1)$-quadruples:

$\{1, 5, 10\}$ (by Mohanty and Ramasamy [14]); $\{1, 2, 5\}$ (by Brown [5]); $\{1, 2, 145\}$, $\{1, 2, 4901\}$, $\{1, 5, 65\}$, $\{1, 5, 20737\}$, $\{1, 10, 17\}$, $\{1, 26, 37\}$ (by Kedlaya [13]); $\{1, 5, 422\}$ (by Abu Muriefah and Al-Rashed [2]).

Dujella, in 1998, proved that the pair $\{1, 2\}$ can not be extended to a $D(-1)$-quadruple ([9, Corollary 1]), and recently Dujella and Fuchs showed the following.

Theorem 1.2 ([10, Theorem 1b])**.** *There does not exist a $D(-1)$-quadruple $\{a, b, c, d\}$ with $2 \le a < b < c < d$.*

Thus, it is enough to examine the extensibility of $D(-1)$-triples $\{1, b, c\}$ with $b \ge 5$. In this paper, we show the following.

Theorem 1.3. *There does not exist a $D(-1)$-quadruple $\{1, b, c, d\}$ with $b = 5, 10, 17, 26, 37$ or 50.*

Our proof of this theorem goes along the lines of that of [9, Corollary 1] or of Theorem 1.2. For $b = 5$, Abu Muriefah and Al-Rashed have already had a partial result on the non-extensibility of the $D(-1)$-triples $\{1, 5, c\}$ ([1, Theorem 2.1]). We also obtained a similar but stronger result on the non-extensibility of $D(-1)$-triples $\{1, b, c\}$ with $\sqrt{b-1}$ a prime (or with $b = 17$ or 37) (see Corollary 3.10), together with which applying the reduction method due to Dujella and Pethő ([11]) to each case of $b = 5, 10, 17, 26, 37$ and 50 completes the proof of Theorem 1.3.

Remark 1.4. Dujella told us that Filipin [12] filled a gap in [1] to show the non-extensibility of the triples $\{1, 5, c\}$, and he further showed that of the triples $\{1, 10, c\}$; Filipin sent us the manuscript [12]. We would like to thank them.

2. The simultaneous Pell equations

Let $\{1, b, c\}$ be a $D(-1)$-triple. Then there exist positive integers r, s, t such that $b - 1 = r^2$, $c - 1 = s^2$, $bc - 1 = t^2$, and we have

$$t^2 - bs^2 = r^2. \tag{1}$$

Let (t_k, s_k) ($s_k < s_{k+1}$, $k = 0,1,2,\dots$) denote the positive solutions of (1). Then there exists an integer k such that $c = c_k := s_k^2 + 1$. Suppose $\{1,b,c,d\}$ is a $D(-1)$-quadruple. Then there exist integers x, y, z such that

$$d-1 = x^2, \quad bd-1 = y^2, \quad cd-1 = z^2.$$

Eliminating d, we obtain the simultaneous Pell equations

$$\begin{cases} z^2 - cx^2 = c-1, & (2) \\ bz^2 - cy^2 = c-b. & (3) \end{cases}$$

Thus, Theorem 1.3 is a corollary of the following.

Theorem 2.1. *Let $b = 5, 10, 17, 26, 37$ or 50 and $c = c_k$ as above. Then the simultaneous Pell equations* (2) *and* (3) *have only the trivial solutions* $(x,y,z) = (0, \pm\sqrt{b-1}, \pm\sqrt{c-1})$.

We will prove Theorem 2.1 in the forthcoming sections.

3. An upper bound for c

Throughout this section, let $b \geq 5$ and assume that $c = c_k$ is minimal for which the equations (2) and (3) have a nontrivial solution.

By Lemma 1 in [10], the positive solutions (z,x) of (2) and (z,y) of (3) are respectively given by

$$z + x\sqrt{c} = (z_0^{(i)} + x_0^{(i)}\sqrt{c})(s+\sqrt{c})^{2m} \qquad (i = 1,\dots,i_0,\ m \geq 0), \tag{4}$$

$$z\sqrt{b} + y\sqrt{c} = (z_1^{(j)}\sqrt{b} + y_1^{(j)}\sqrt{c})(t+\sqrt{bc})^{2n} \qquad (j = 1,\dots,j_0,\ n \geq 0), \tag{5}$$

where

$$0 < z_0^{(i)} < c, \quad 0 < z_1^{(j)} < c \tag{6}$$

for all i and j. (We call $(z_0^{(i)}, x_0^{(i)})$ $(1 \leq i \leq i_0)$ and $(z_1^{(j)}, y_1^{(j)})$ $(1 \leq j \leq j_0)$ the fundamental solutions of (2) and (3), respectively.) By (4), there exist i and m such that

$$\begin{gathered} z = v_m^{(i)}, \quad \text{where} \\ v_0^{(i)} = z_0^{(i)},\ v_1^{(i)} = (2c-1)z_0^{(i)} + 2scx_0^{(i)},\ v_{m+2}^{(i)} = 2(2c-1)v_{m+1}^{(i)} - v_m^{(i)}, \end{gathered} \tag{7}$$

and by (5), there exist j and n such that

$$
\begin{gathered}
z = w_n^{(j)}, \quad \text{where} \\
w_0^{(j)} = z_1^{(j)},\ w_1^{(j)} = (2bc-1)z_1^{(j)} + 2tcy_1^{(j)},\ w_{n+2}^{(j)} = 2(2bc-1)w_{n+1}^{(j)} - w_n^{(j)}.
\end{gathered} \tag{8}
$$

By induction we have

$$
v_m^{(i)} \equiv (-1)^m z_0^{(i)} \pmod{2c}, \quad w_n^{(j)} \equiv (-1)^n z_1^{(j)} \pmod{2c}. \tag{9}
$$

Further, the sequences $\{v_m^{(i)}\}$ and $\{w_n^{(j)}\}$ satisfy the following relations.

Lemma 3.1 (cf. [10, Lemma 2])**.**

$$
\begin{aligned}
v_m^{(i)} &\equiv (-1)^m (z_0^{(i)} - 2cm^2 z_0^{(i)} - 2csmx_0^{(i)}) \pmod{8c^2}, \\
w_n^{(j)} &\equiv (-1)^n (z_1^{(j)} - 2bcn^2 z_1^{(j)} - 2ctny_1^{(j)}) \pmod{8c^2}.
\end{aligned}
$$

In what follows, we will assume $v_m^{(i)} = w_n^{(j)}$ (and omit the superscripts (i) and (j)). Putting $d_0 := (z_0^2+1)/c$, we see from (6) that $d_0 \le c$ and from (2) and (3) that

$$
d_0 - 1 = x_0^2, \quad bd_0 - 1 = y_1^2, \quad cd_0 - 1 = z_0^2
$$

(see the proof of Lemma 3 in [10]). Since $d_0 = x_0^2 + 1$ is a positive integer, the minimality of $c = c_k$ implies the following.

Lemma 3.2 (cf. [10, Lemma 4])**.** *$z_0 = z_1 = \sqrt{c-1}\,(= s)$ and $x_0 = 0$, $y_1 = \pm\sqrt{b-1}(= \pm r)$.*

Lemma 3.3. (i) *$m \equiv n \pmod 2$.*
(ii) *$n \le m \le 2n$.*
(iii) *If $n \ne 0$ and $b < \sqrt{c}$, then we have $\sqrt[4]{c}/\sqrt{b-1} < n$.*

Proof. (i) It is obvious from (9).
(ii) It is exactly Lemma 6 in [10].
(iii) By (i), Lemmas 3.1 and 3.2, we have

$$
s - 2cm^2 s \equiv s - 2bcn^2 s \pm 2\sqrt{b-1}ctn \pmod{8c^2},
$$

which implies

$$
s(m^2 - bn^2) \equiv \pm\sqrt{b-1}tn \pmod{4c}. \tag{10}
$$

Since $s = \sqrt{c-1}$ and $t = \sqrt{bc-1}$, we have

$$(c-1)(m^2 - bn^2)^2 \equiv (b-1)(bc-1)n^2 \pmod{4c},$$

which implies

$$(m^2 - bn^2)^2 \equiv (b-1)n^2 \pmod{c}. \tag{11}$$

Now, suppose $n \le \sqrt[4]{c}/\sqrt{b-1}$. By (ii), we have

$$|s(m^2 - bn^2)| \le \sqrt{c-1}(b-1)n^2 < c, \quad (m^2 - bn^2)^2 \le (b-1)^2 n^4 \le c.$$

On the other hand, by the assumption $b < \sqrt{c}$ we know that

$$\sqrt{b-1}tn \le \sqrt{bc-1}\sqrt[4]{c} < c, \quad (b-1)n^2 \le \sqrt{c} < c.$$

It follows from (10) and (11) that

$$s(m^2 - bn^2) = -\sqrt{b-1}tn, \quad (m^2 - bn^2)^2 = (b-1)n^2.$$

Hence we have

$$s^2(m^2 - bn^2)^2 = (b-1)t^2n^2 = t^2(m^2 - bn^2)^2,$$

which together with $n \ne 0$ implies $t^2 = s^2$, which is a contradiction. $\square$

From this lemma, it is easy to see the following.

Proposition 3.4. *Let x, y, z be positive integer solutions of the simultaneous Pell equations* (2) *and* (3). *If $b < \sqrt{c}$, then we have*

$$\left(\frac{\sqrt[4]{c}}{\sqrt{b-1}} - 1\right)\log(4c-3) < \log y.$$

Proof. Let $z = v_m = w_n$. $x > 0$ implies $m > 0$. It follows from (4) and Lemma 3.2 that

$$x = \frac{s}{2\sqrt{c}}\left\{(s+\sqrt{c})^{2m} - (s-\sqrt{c})^{2m}\right\},$$

which together with $y^2 - bx^2 = b-1 > 0$ implies

$$\begin{aligned} y > x\sqrt{b} &= \frac{s\sqrt{b}}{2\sqrt{c}}\left\{(s+\sqrt{c})^{2m} - (s-\sqrt{c})^{2m}\right\} \\ &> (s+\sqrt{c})^{2(m-1)}\left\{(s+\sqrt{c})^2 - (s-\sqrt{c})^2\right\} \\ &> (s+\sqrt{c})^{2(m-1)} > (4c-3)^{m-1}. \end{aligned}$$

Hence the proposition follows from Lemma 3.3 (iii). $\square$

In order to get an upper bound for $\log y$, we need the following theorem, which is a slightly modified version of Rickert's theorem (or of a special case of Bennett's theorem).

Theorem 3.5 (cf. [5, Theorem 3.2], [15, Theorem] or [17, Theorem]). *Let b and N be integers with $b \geq 5$ and $N \geq 2.39b^7$. Then the numbers*

$$\theta_1 := \sqrt{1 + \frac{1-b}{N}} \quad \text{and} \quad \theta_2 := \sqrt{1 + \frac{1}{N}}$$

satisfy

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > \left\{32.1\,\frac{b^2(b-1)^2}{2b-1}\,N\right\}^{-1} q^{-1-\lambda} \tag{12}$$

for all integers p_1, p_2, q with $q > 0$, where

$$\lambda := \frac{\log \frac{16.1b^2(b-1)^2N}{2b-1}}{\log \frac{3.37N^2}{b^2(b-1)^2}} < 1.$$

Proof. Note that the condition $N \geq 2.39b^7$ implies $\lambda < 1$. All we have to do is find those real numbers satisfying the assumption in the following lemma.

Lemma 3.6 (cf. [5, Lemma 3.1], [15, Lemma 2.1]). *Let $\theta_1, \dots, \theta_m$ be arbitrary real numbers and $\theta_0 = 1$. Assume that there exist positive real numbers l, p, L, P and positive integers D, f with f dividing D and with $L > D$, having the following property. For each positive integer k, we can find rational numbers p_{ijk} $(0 \leq i, j \leq m)$ with nonzero determinant such that $f^{-1}D^k p_{ijk}$ $(0 \leq i, j \leq m)$ are integers and*

$$|p_{ijk}| \leq pP^k \quad (0 \leq i, j \leq m), \qquad \left|\sum_{j=0}^{m} p_{ijk}\theta_j\right| \leq lL^{-k} \quad (0 \leq i \leq m).$$

Then

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \dots, \left|\theta_m - \frac{p_m}{q}\right|\right\} > cq^{-1-\lambda}$$

holds for all integers $p_1, \dots, p_m, q$ with $q > 0$, where

$$\lambda = \frac{\log(DP)}{\log(L/D)} \quad \text{and} \quad c^{-1} = 2mf^{-1}pDP\left(\max\{1, 2f^{-1}l\}\right)^{\lambda}.$$

Note that l, p, L, P, p_{ijk} in [5, Lemma 3.1] denote $f^{-1}l$, $f^{-1}p$, L/D, DP, $f^{-1}D^k p_{ijk}$ in the lemma above (or in [15, Lemma 2.1]), respectively. In our situation, we take $m=2$ and θ_1, θ_2 as in Theorem 3.5. The only difference from Theorem 3.2 in [5] is that we may take $f=2$ and $D=2b^2(b-1)^2N$, whereas in [5] $f=1$ and $D=4b^2(b-1)^2N$ are taken (note that C_k in [5] denotes $f^{-1}D^k$ in our notation). The validity of this substitution follows from the fact that $b(b-1)$ is even. Indeed, let $p_{ij}(x)$ be those polynomials appearing in [15, Lemma 3.3], which have rational coefficients of degree at most k ([15, (3.7)]). Following [15], we take $p_{ijk}=p_{ij}(1/N)$ for varying values of k. Denoting $b(b-1)=2b'$ with an integer b', we see from the expression (3.7) in [15] of $p_{ij}(1/N)$ that

$$2^{l_1}(b')^{l_2}N^k p_{ij}(1/N)\in\mathbb{Z}$$

for some integers l_1, l_2; further, we see $l_1\le 3k-1$ in the same way just as the proof of Lemma 4.3 in [15] and $l_2\le 2k$ is easy to find. Hence we obtain

$$2^{-1}\cdot 2^k\{b(b-1)\}^{2k}N^k p_{ij}(1/N)\in\mathbb{Z}.$$

Thus, by exactly the same arguments as the ones following Lemma 3.1 in [5] (with $a_0=1-b$, $a_1=0$, $a_2=1$), the numbers

$$p=\left(1+\frac{1}{2N}\right)^{1/2},\qquad P=\frac{8}{2b-1}\left(1+\frac{3}{2N}\right),$$

$$l=\frac{27}{64}\left(1-\frac{b-1}{N}\right)^{-1},\qquad L=\frac{27}{4}\left(1-\frac{b-1}{N}\right)^{2}N^3$$

and $f=2$, $D=2b^2(b-1)^2N$, $p_{ijk}=p_{ij}(1/N)$ satisfy the assumption in Lemma 3.6. Since $N\ge 2.39$ and $b\ge 5$, we have

$$DP\le\frac{16.1b^2(b-1)^2N}{2b-1},\quad 2pDP\le\frac{32.1b^2(b-1)^2N}{2b-1},\quad \frac{L}{D}\ge\frac{3.37N^2}{b^2(b-1)^2}.$$

Therefore, Theorem 3.5 immediately follows from Lemma 3.6. □

The following is essentially the same as Lemma 6 in [9].

Lemma 3.7. *Let $N=t^2$ and θ_1, θ_2 be as in Theorem 3.5. Then all positive integer solutions x, y, z of the simultaneous Pell equations* (2) *and* (3) *satisfy*

$$\max\left\{\left|\theta_1-\frac{bsx}{ty}\right|,\left|\theta_2-\frac{bz}{ty}\right|\right\}<\frac{b-1}{y^2}.$$

Proof. Since $\theta_1 = s\sqrt{b}/t$ and $\theta_2 = \sqrt{bc}/t$, we have

$$\begin{aligned}\left|\theta_1 - \frac{bsx}{ty}\right| &= \frac{s\sqrt{b}}{t}\left|1 - \frac{x\sqrt{b}}{y}\right| \\ &= \frac{s\sqrt{b}}{t}\left|1 - \frac{bx^2}{y^2}\right| \cdot \left|1 + \frac{x\sqrt{b}}{y}\right|^{-1} < \frac{b-1}{y^2}\end{aligned}$$

and

$$\begin{aligned}\left|\theta_2 - \frac{bz}{ty}\right| = \frac{1}{t}\left|\sqrt{bc} - \frac{bz}{y}\right| &= \frac{b}{t}\left|c - \frac{bz^2}{y^2}\right| \cdot \left|\sqrt{bc} + \frac{bz}{y}\right|^{-1} \\ &< \frac{b}{t} \cdot \frac{c-b}{y^2} \cdot \frac{1}{2\sqrt{bc}} < \frac{1}{2y^2} \cdot \frac{bc-1}{\sqrt{bc(bc-1)}} < \frac{1}{2y^2}.\end{aligned}$$

These complete the proof of Lemma 3.7. □

Proposition 3.8. *If $b \geq 5$ and $c \geq 2.4b^6$, then we have*

$$\log y < \frac{2\log\dfrac{1.9c}{b-1} \cdot \log(17b^6c^2)}{\log\dfrac{0.41c}{b^6}}.$$

Proof. We apply Theorem 3.5 with $N = t^2$, $p_1 = bsx$, $p_2 = bz$ and $q = ty$. Note that $c \geq 2.4b^6$ implies $N = t^2 = bc-1 > 2.39b^7$. Theorem 3.5 and Lemma 3.7 together imply

$$\left\{32.1\frac{b^2(b-1)^2}{2b-1}t^2\right\}^{-1}(ty)^{-1-\lambda} < \frac{b-1}{y^2}.$$

Noting $\lambda < 1$, we have

$$y^{1-\lambda} < 32.1\frac{b^2(b-1)^3}{2b-1}t^{3+\lambda} < \frac{32.1b^2(b-1)^3(bc-1)^2}{2b-1}.$$

It follows from

$$\frac{1}{1-\lambda} = \frac{\log\frac{3.37(bc-1)^2}{b^2(b-1)^2}}{\log\frac{3.37(2b-1)(bc-1)}{16.1b^4(b-1)^4}}$$

that

$$\log y < \frac{\log\frac{3.37(bc-1)^2}{b^2(b-1)^2} \cdot \log\frac{32.1b^2(b-1)^3(bc-1)^2}{2b-1}}{\log\frac{3.37(2b-1)(bc-1)}{16.1b^4(b-1)^4}}.$$

Since

$$\begin{aligned}\log\frac{3.37(bc-1)^2}{b^2(b-1)^2}&<2\log\frac{1.9c}{b-1},\\ \log\frac{32.1b^2(b-1)^3(bc-1)^2}{2b-1}&<\log(17b^6c^2),\\ \log\frac{3.37(2b-1)(bc-1)}{16.1b^4(b-1)^4}&>\log\frac{0.418c}{b^6}\left(1-\frac{1}{bc}\right)\geq\log\frac{0.41c}{b^6},\end{aligned}$$

we obtain the proposition. □

We are now ready to bound above for c.

Theorem 3.9. *Let $b\geq 5$. If c is minimal for which the equations* (2) *and* (3) *have a nontrivial solution, then we have $c<200b^6$.*

Proof. Suppose $c\geq 200b^6$. Propositions 3.4 implies

$$\begin{aligned}\log y&>\left(\frac{200^{1/12}c^{1/4}}{c^{1/12}}-1\right)\log(4c-3)\\ &>200^{1/12}c^{1/6}\log(4c-3)\left(1-\frac{1}{200^{1/12}c^{1/6}}\right)>1.47c^{1/6}\log(4c-3)\end{aligned}$$

and Proposition 3.8 implies

$$\begin{aligned}\log y&<\frac{2\log(1.9c/4)\cdot\log(17c^3/200)}{\log(0.41\cdot 200)}\\ &<1.37\log(0.48c)\log(0.44c).\end{aligned}$$

Hence we have

$$f(c):=1.37\log(0.48c)\log(0.44c)-1.47c^{1/6}\log(4c-3)>0.$$

However, $f(c)$ is a decreasing function for $c(\geq 200b^6)\geq 200{\cdot}5^6$ with $f(200{\cdot}5^6)<0$, which is a contradiction. □

It is to be noted that in case $r=\sqrt{b-1}$ is an odd prime, the Pell equation (1) has exactly the three fundamental solutions

$$(t,s)=(r,0),\ (b-r,\pm(r-1)). \tag{13}$$

To see this, let $(t,s)=(\beta,\alpha)$ be a solution of (1) with $\gcd(\alpha,\beta)=1$, which is called primitive. Then Theorem 22 in [16] implies that there exists an integer j such that

$$\beta\equiv j\alpha\pmod{r^2}\quad\text{and}\quad j^2\equiv b\equiv 1\pmod{r^2}.$$

(In this case, we say that (β,α) belongs to j.) Since $\gcd(j+1, j-1)$ divides 2 and r is an odd prime, the latter relation implies $j \equiv \pm 1 \pmod{r^2}$. On the other hand, by Theorem 23 in [16] we see that the fundamental solutions of (1) which are primitive are the ones that belong to the values ± 1 of $j \bmod r^2$; $(\beta,\alpha) = (b-r, \pm(r-1))$ belong to ∓ 1 respectively. Since r is a prime, the fundamental solution of (1) independent of the ones above is only the trivial one $(r, 0)$. Therefore, the fundamental solutions of (1) are given by (13).

Since we may take the fundamental solutions of (1) with $0 < t < b$ (cf. (2) and (6)), it is easy to check in each case that the same is true for $b = 5, 17$ or 37. Hence for these b's, the positive solutions (t, s) of (1) are given by $t + s\sqrt{b} =$

$$\begin{aligned}\left\{b - r + (r-1)\sqrt{b}\right\}\left(2b - 1 + 2r\sqrt{b}\right)^{\nu} \quad &(\nu = 0, 1, 2, \dots),\\ \left\{b - r - (r-1)\sqrt{b}\right\}\left(2b - 1 + 2r\sqrt{b}\right)^{\nu} \quad &(\nu = 1, 2, 3, \dots),\\ r\left(2b - 1 + 2r\sqrt{b}\right)^{\nu} \quad &(\nu = 1, 2, 3, \dots).\end{aligned} \tag{14}$$

Since our numbering for the solutions takes as (t_0, s_0) the first solution above with $\nu = 0$, we have

$$t_8 + s_8\sqrt{b} = r\,(2b - 1 + 2r\sqrt{b}\,)^3,$$

which implies

$$\begin{aligned}c_8 = s_8^2 + 1 &= \left[(b-1)\left\{6(2b-1)^2 + 8b(b-1)\right\}\right]^2 + 1\\ &= 4(b^2 - 2b + 1)(256b^4 - 512b^3 + 352b^2 - 96b + 9) + 1\\ &> b^2\left(2 - \frac{4}{b}\right)\cdot 256b^4\left(2 - \frac{4}{b}\right) > 256b^6 > 200b^6.\end{aligned}$$

Consequently we obtain

Corollary 3.10. *Assume that $\sqrt{b-1}$ is a prime or that $b = 17$ or 37. If they have only the trivial solutions for $c = c_k$ with $0 \le k \le 7$, then the equations* (2) *and* (3) *have only the trivial solutions for $c = c_k$ with $k \ge 0$.*

Remark 3.11. In case $b = 65$, the Pell equation $t^2 - 65s^2 = 64$ of (1) has exactly the five fundamental solutions

$$(t, s) = (8, 0), (18, \pm 2), (57, \pm 7).$$

Since $t_{14} + s_{14}\sqrt{65} = 8(129 + 16\sqrt{65})^3$ and $c_{14} = s_{14}^2 + 1 > 200 \cdot 65^6$, if they have only the trivial solutions for $c = c_k$ with $0 \le k \le 13$, then the equations (2) and (3) with $b = 65$ have only the trivial solutions for $c = c_k$ with $k \ge 0$.

Since $(t,s)=(b-r,r-1)$ is a positive solution of (1) and the attached $c=b-(2r-1)<b$, c_0 is always less than b. Hence, the non-extensibility of $D(-1)$-triples $\{1,b,c\}$ for $b=2,5,10,17,26,37$ and for the attached c's with $c\ge c_1$ implies that of $\{1,b,c_0\}$ for $b=5,10,17,26,37,50$ and for the attached c_0's. Therefore, it is enough to show Theorem 2.1 for $k\ge 1$.

Corollary 3.12. *If it holds for $c=c_k$ with $1\le k\le 7$, then Theorem 2.1 holds for $c=c_k$ with $k\ge 0$.*

4. The reduction method

Throughout this section, let $b=5,10,17,26,37$ or 50 and assume that $(m\ge)$ $n\ge 1$ (that is, Theorem 2.1 is not valid) and that $c=c_k$ is minimal for which Theorem 2.1 is not valid. By the non-extensibility of those $D(-1)$-triples listed in Section 1, we may assume that $c\ge 26$. Moreover, since $t_1+s_1\sqrt{b}$ equals the middle expression of (14) with $\nu=1$, we may assume that

$$c\ge (c_1=)b+2r+1. \tag{15}$$

We will complete the proof of Theorem 2.1 by combining Corollary 3.12 with the reduction method ([11]) of Dujella and Pethő (based on that of Baker and Davenport).

By (7) and (8), we have

$$\begin{aligned}
v_m &= \frac{s}{2}\left\{(s+\sqrt{c}\,)^{2m}+(s-\sqrt{c}\,)^{2m}\right\},\\
w_n &= \frac{1}{2\sqrt{b}}\left\{(s\sqrt{b}\pm r\sqrt{c}\,)(t+\sqrt{bc}\,)^{2n}+(s\sqrt{b}\mp r\sqrt{c}\,)(t-\sqrt{bc}\,)^{2n}\right\}.
\end{aligned}$$

Putting

$$P=s(s+\sqrt{c}\,)^{2m},\quad Q=\frac{1}{\sqrt{b}}(s\sqrt{b}\pm r\sqrt{c}\,)(t+\sqrt{bc}\,)^{2n},$$

we have

$$P^{-1}=\frac{1}{s}(s-\sqrt{c}\,)^{2m},\quad Q^{-1}=\frac{\sqrt{b}}{c-b}(s\sqrt{b}\mp r\sqrt{c}\,)(t-\sqrt{bc}\,)^{2n}. \tag{16}$$

It follows from $v_m=w_n$ that

$$P+(c-1)P^{-1}=Q+\frac{c-b}{b}Q^{-1}.$$

Let us bound the linear form "$\log(Q/P)$" in logarithms.

First, we have

$$\begin{aligned}Q-P &= (c-1)P^{-1}-\frac{c-b}{b}Q^{-1}\\ &> (c-1)(P^{-1}-Q^{-1})=(c-1)(Q-P)P^{-1}Q^{-1}.\end{aligned}$$

Since we see from $b\geq 5$, $c\geq 26$, $m\geq n\geq 1$ and (15) that

$$\begin{aligned}P-(c-1) &= (c-1)\left\{\frac{(s+\sqrt{c}\,)^{2m}}{s}-1\right\}\\ &\geq (c-1)\left(s+2\sqrt{c}+\frac{c}{s}-1\right)>0,\\ Q &\geq \frac{1}{\sqrt{b}}(s\sqrt{b}-r\sqrt{c}\,)(t+\sqrt{bc}\,)^2\\ &> \frac{c-b}{2b\sqrt{c}}(4bc-3)=2\left(1-\frac{3}{4bc}\right)(c-b)\sqrt{c}\\ &> 1.9(2\sqrt{b-1}+1)\sqrt{c}\geq 9.5\sqrt{c}>1,\end{aligned}\tag{17}$$

we have $Q>P$.

Secondly, since (16) and (17) imply $P>Q-(c-1)P^{-1}>Q-1$, we have

$$\frac{Q-P}{Q}<Q^{-1}<\frac{1}{9.5\sqrt{c}}<\frac{1}{2}.\tag{18}$$

On the other hand, by $b\geq 5$ and (15) we have

$$\begin{aligned}Q^{-1} &\leq \frac{\sqrt{b}}{c-b}(s\sqrt{b}+r\sqrt{c}\,)(t+\sqrt{bc}\,)^{-2n}\leq\frac{2b\sqrt{c}}{c-b}(t+\sqrt{bc}\,)^{-2n}\\ &< \frac{b(\sqrt{b}+1)}{\sqrt{b-1}}(t+\sqrt{bc}\,)^{-2n}<1.62b(t+\sqrt{bc}\,)^{-2n}.\end{aligned}\tag{19}$$

It follows from (18) and (19) that

$$\begin{aligned}0<\log\frac{Q}{P} &= -\log\left(1-\frac{Q-P}{Q}\right)<-\log\left(1-Q^{-1}\right)\\ &< Q^{-1}+Q^{-2}=(1+Q^{-1})Q^{-1}\\ &< \left(1+\frac{1}{9.5\sqrt{c}}\right)1.62b(t+\sqrt{bc}\,)^{-2n}\\ &< 1.7b(t+\sqrt{bc}\,)^{-2n}.\end{aligned}$$

Therefore, we have

$$0<(\Lambda :=) n\log\alpha_1 - m\log\alpha_2 + \log\alpha_3 < 1.7b\alpha_1^{-n}, \tag{20}$$

where

$$\alpha_1 := 2bc-1+2\sqrt{bc(bc-1)}, \quad \alpha_2 := 2c-1+2\sqrt{c(c-1)},$$
$$\alpha_3 := \frac{\sqrt{b(c-1)}\pm\sqrt{(b-1)c}}{\sqrt{b(c-1)}}.$$

The following theorem of Baker and Wüstholz gives a lower bound for $\log\Lambda$.

Theorem 4.1 ([4, Theorem])**.** *For a linear form $\Lambda\neq 0$ in logarithms of l algebraic numbers $\alpha_1,\dots,\alpha_l$ with rational coefficients $\beta_1,\dots,\beta_l$, we have*

$$\log|\Lambda| \geq -18(l+1)!\, l^{l+1}(32d)^{l+2}h'(\alpha_1)\cdots h'(\alpha_l)\log(2ld)\log\beta,$$

where $\beta := \max\{|\beta_1|,\dots,|\beta_l|\}$, $d := [\mathbb{Q}(\alpha_1,\dots,\alpha_l):\mathbb{Q}]$ and

$$h'(\alpha) := \frac{1}{d}\max\{h(\alpha), |\log\alpha|, 1\}$$

with the standard logarithmic Weil height $h(\alpha)$ of α.

Applying Theorem 4.1 with $l=3$, $d=4$, $\beta=m\leq 2n$ and

$$h'(\alpha_1)=\frac{1}{2}\log\alpha_1<\frac{1}{2}\log(4bc), \quad h'(\alpha_2)=\frac{1}{2}\log\alpha_2<\frac{1}{2}\log(4c),$$
$$h'(\alpha_3)\leq\frac{1}{4}\left\{\log(b(c-1))^2+\log(\alpha_3^2)\right\}$$
$$=\frac{1}{2}\log\left\{b(c-1)+\sqrt{bc(b-1)(c-1)}\right\}<\frac{1}{2}\log(2bc),$$

we have

$$\log\Lambda > -18\cdot 4!\cdot 3^4(32\cdot 4)^5\cdot\frac{1}{2}\log(4bc)\cdot\frac{1}{2}\log(4c)\cdot\frac{1}{2}\log(2bc)$$
$$\times\log 24\cdot\log(2n),$$

which together with (20) implies

$$\frac{n-1}{\log(2n)}<4.8\cdot 10^{14}\log(4bc)\cdot\log(4c). \tag{21}$$

Corollary 3.12 and (21) give upper bounds for n:

if $b = 5$, then $c \le c_7 = 974170$ and $n < 6 \cdot 10^{18}$;
if $b = 10$, then $c \le c_7 = 34199105$ and $n < 9 \cdot 10^{18}$;
if $b = 17$, then $c \le c_7 = 482812730$ and $n < 2 \cdot 10^{19}$;
if $b = 26$, then $c \le c_7 = 3947106277$ and $n < 2 \cdot 10^{19}$;
if $b = 37$, then $c \le c_7 = 22480504226$ and $n < 2 \cdot 10^{19}$;
if $b = 50$, then $c \le c_7 = 99106595345$ and $n < 2 \cdot 10^{19}$.

Now, dividing (20) by $\log \alpha_2$ leads to the inequality

$$0 < n\kappa - m + \mu < AB^{-n}, \tag{22}$$

where

$$\kappa := \frac{\log \alpha_1}{\log \alpha_2}, \quad \mu := \frac{\log \alpha_3}{\log \alpha_2}, \quad A := \frac{1.7b}{\log \alpha_2}, \ \ B := \alpha_1.$$

The following is based on the Baker–Davenport lemma ([3, Lemma]).

Lemma 4.2 ([11, Lemma 5 a)]). *Let N be a positive integer. Let p/q be the convergent of the continued fraction expansion of κ such that $q > 6N$. Put $\epsilon := \|\mu q\| - N\,\|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\epsilon > 0$, then the inequality (22) has no solution in the range*

$$\frac{\log(Aq/\epsilon)}{\log B} \le n < N.$$

We apply Lemma 4.2 with N the upper bound for n in each case. In the first step of reduction, we have to examine $36 \cdot 2 = 72$ cases (the doubling comes from the signs "$\pm$" in α_3), of which the second convergent of κ such that $q > 6N$ is needed only in three cases. Thus we obtain a new bound $(n \le) N_1$ with $N_1 \le 6$ in each case. The second step of reduction (with $N = N_1$) requires the second convergent of κ such that $q > 6N_1$ only in four cases. Thus we obtain a new bound $(n \le) N_2$ with $N_2 = 0$ or 1. The former contradicts the assumption. The latter occurs only in five cases. Then the third step of reduction (with $N = 1$) requires the second convergent of κ such that $q > 6$ only in one case and gives $n < 1$, which contradicts the assumption. This completes the proof of Theorem 2.1.

Acknowledgments. We would like to thank Professor Masaki Sudo for telling us some works of Rickert and of Dujella and for sending us the exposition [17] of Rickert's paper [15]. Thanks also go to the referee for careful reading and detailed suggestions.

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(Received April 4, 2005; revised September 15, 2005)