

On the diophantine equation $x^2 + 2^\alpha 3^\beta 5^\gamma 7^\delta = y^n$

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Abstract. Let $S = \{p_1, \dots, p_s\}$ be a set of distinct primes and denote by $\mathbf{S}$ the set of non-zero integers composed only of primes from S . Further, denote by Q the product of the primes from S . Let $f \in \mathbb{Z}[X]$ be a monic quadratic polynomial with negative discriminant D_f contained in $\mathbf{S}$. Consider equation $f(x) = y^n$ (2) in integer unknowns x, y, n with $n \geq 3$ prime and $y > 1$. It follows from a general result of [13] that in (2) n can be bounded from above by an effectively computable constant depending only on Q . This bound is, however, large and is not given explicitly. Using some results of BUGEAUD and SHOREY [8] we derive, apart from certain exceptions, a good and completely explicit upper bound for n in (2) (see Theorems 1 and 2). Further, combining our Theorem 2 with some deep results of COHN [12] and DE WEGER [25] we give all non-exceptional (see Section 1) solutions of equation $x^2 + 2^\alpha 3^\beta 5^\gamma 7^\delta = y^n$ (6), where $x, y, n, \alpha, \beta, \gamma, \delta$ are unknown non-negative integers with $x \geq 1$, $\gcd(x, y) = 1$ and $n \geq 3$ (cf. Theorem 3). When, in (6), $\alpha \geq 1$ is also assumed then our Theorem 3 is a generalization of a result of LUCA [19]. In this case all the solutions of equation (6) are listed.

1. Introduction

There are many results concerning the generalized Ramanujan–Nagell equation

$$x^2 + D = \mu y^n, \tag{1}$$

where $D > 0$ is a given integer, $\mu \in \{1, 4\}$ and x, y, n are positive integer unknowns with $n \geq 3$ and $\gcd(x, y) = 1$. First consider the case $\mu = 1$. Then the first result was due to V. A. LEBESQUE [15] who proved that there are no

Mathematics Subject Classification: 11D41, 11D61.

Key words and phrases: exponential diophantine equations, primitive divisors.

solutions for $D = 1$. LJUNGGREN [16] solved (1) for $D = 2$, and NAGELL [22], [23] solved it for $D = 3, 4$ and 5 . In his elegant paper [11], COHN gave a fine summary of work on equation (1). Further, he developed a method by which he found all solutions of the above equation for 77 positive values of $D \leq 100$. For $D = 74$ and $D = 86$, equation (1) was solved by MIGNOTTE and DE WEGER [20]. By using the theory of Galois representations and modular forms BENNETT and SKINNER [5] solved (1) for $D = 55$ and $D = 95$. On combining the theory of linear forms in logarithms with Bennett and Skinner's method and with several additional ideas, BUGEAUD, MIGNOTTE and SIKSEK [7] gave all the solutions of (1) for the remaining 19 values of $D \leq 100$. BUGEAUD and SHOREY [8] used a beautiful result of BILU, HANROT and VOUTIER [6] to solve completely several equations of type (1) both for $\mu = 1$ and for $\mu = 4$ when D is an odd positive square-free integer, $n \geq 3$ is an odd prime not dividing the class number of the field $\mathbb{Q}(\sqrt{-D})$ and $D \not\equiv 7 \pmod{8}$ if $\mu = 1$ (see Corollaries 3, 5 and 7 of [8]).

Let $S = \{p_1, \dots, p_s\}$ denote a set of distinct primes and $\mathbf{S}$ the set of non-zero integers composed only of primes from S . Denote by P and Q the greatest and the product of the primes of S , respectively. In recent years, equation (1) has been considered also in the more general case when D is no longer fixed but $D \in \mathbf{S}$ with $D > 0$. It follows from Theorem 2 of [24] that in (1) n can be bounded from above by an effectively computable constant depending only on f, P and s . In [13] an effective upper bound was derived for n which depends only on Q . By using the powerful method of BILU, HANROT and VOUTIER [6] equation (1) can be completely solved for $\mu = 1$ and some special sets of primes S . Namely, if in (1) $D \in \mathbf{S}$ with $S = \{2\}$ then all solutions of (1) were given by COHN [10] and ARIF and MURIEFAH [1] and [3]. For $S = \{3\}$, equation (1) was solved completely by ARIF and MURIEFAH [2] and LUCA [18]. When $S = \{q\}$, where $q \geq 5$ is an odd prime with $q \not\equiv 7 \pmod{8}$, ARIF and MURIEFAH [4] determined all solutions of the equation $x^2 + q^{2k+1} = y^n$, where $\gcd(n, 3h_0) = 1$ and $n \geq 3$. Here h_0 denotes the class number of the field $\mathbb{Q}(\sqrt{-q})$. For $S = \{2, 3\}$, LUCA [19] gave the complete solution of (1).

To formulate our results we introduce some notation. Let $f(x) = x^2 + Ax + B$ where $A, B \in \mathbb{Z}$ and denote by D_f the discriminant of f . Set

$$\Delta = \begin{cases} -\frac{D_f}{4} & \text{if } D_f \text{ is even,} \\ -D_f & \text{if } D_f \text{ is odd.} \end{cases}$$

Suppose that $\Delta \in \mathbf{S}$ and $\Delta > 0$. Let c and d be non-zero integers such that $\Delta = dc^2$ and $d > 0$ denotes the square-free part of Δ . Further, for any $k \in \mathbb{Z}$ and rational prime p denote by $\text{ord}_p(k)$ the greatest power of p to which p divides k .

Consider the equation

$$f(x) = y^n \quad (2)$$

in integer unknowns x, y, n with $n \geq 3$ prime and $y > 1$. We say that a solution (x, y, n) of (2) is *exceptional* if

$$\text{ord}_2(D_f) = 2, \ y \text{ is even and } d \equiv 7 \pmod{8}.$$

Write h for the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Further, denote by $h(-4\Delta)$ the number of classes of positive binary quadratic forms with discriminant -4Δ (for the definition see Section 2).

Theorem 1. *If (x, y, n) is a non-exceptional solution of (2) with $x \neq -\frac{A}{2}$ and $\gcd(y, \Delta) = 1$ then, except for the infinite families of equations*

$$x^2 + Ax + B = y^n$$

where $(A, B, x, y, \Delta, n) \in \{(A, (A^2 + 7)/4, (11 - A)/2, 2, 7, 5), (A, (A^2 + 7)/4, (181 - A)/2, 2, 7, 13), (A, (A^2 + 11)/4, (31 - A)/2, 3, 11, 5), (A, (A^2 + 19)/4, (559 - A)/2, 5, 19, 7)\}$, where A is odd and $(A, B, x, y, \Delta, n) \in \{(A, (A^2 + 76)/4, (44868 - A)/2, 55, 19, 5), (A, (A^2 + 1364)/4, (5519292 - A)/2, 377, 341, 5)\}$, where A is even, we have

$$n = 3 \quad \text{or} \quad n \mid h(-4\Delta).$$

Further, in the latter case

$$n \leq \max\{3, P\} \quad \text{if} \quad n \nmid h$$

and

$$n < \frac{4}{\pi} \sqrt{Q} \log(2e\sqrt{Q}) \quad \text{if} \quad n \mid h.$$

We note that the assumption $x \neq -\frac{A}{2}$ is necessary. Otherwise using (2) and supposing that D_f is even we get $y^n = \Delta$, whence by $\Delta \in \mathbf{S}$ we see that n cannot be bounded.

Equation (2) can be reduced to an equation of the type

$$X^2 + \Delta = \mu Y^n, \quad (3)$$

where $\mu \in \{1, 4\}$,

$$\gcd(X, Y) = \gcd(Y, \Delta) = 1 \quad (4)$$

and

$$\begin{aligned} \mu &= 1 \text{ if } D_f \text{ is even, } \mu = 4 \text{ if } D_f \text{ is odd,} \\ \Delta &\in \mathbf{S}, \Delta > 0, X \geq 1, Y > 1, n \geq 3 \text{ prime.} \end{aligned} \quad (5)$$

We shall deduce Theorem 1 from the following Theorem 2. We say that a solution (X, Y, n) of (3) is *exceptional* if

$$\mu = 1, \text{ord}_2(D_f) = 2, Y \text{ is even and } d \equiv 7 \pmod{8}.$$

Theorem 2. *If (X, Y, n) is a non-exceptional solution of equation (3) satisfying (4) and (5) then, except for $(\mu, Y, \Delta, n) \in \{(4, 2, 7, 5), (4, 2, 7, 13), (4, 3, 11, 5), (4, 5, 19, 7), (1, 55, 19, 5), (1, 377, 341, 5)\}$, we have*

$$n = 3 \quad \text{or} \quad n \mid h(-4\Delta).$$

Further, in the latter case

$$n \leq \max\{3, P\} \quad \text{if } n \nmid h$$

and

$$n < \frac{4}{\pi} \sqrt{Q} \log(2e\sqrt{Q}) \quad \text{if } n \mid h.$$

This should be compared with Corollaries 5 and 7 of BUGEAUD and SHOREY [8], where equations of type (3) were considered with square-free $\Delta > 0$. In Corollary 5 they showed that the equation $x^2 + 4\Delta = y^n$ has no solution with $n \geq 5$. Here Δ is square-free and n is an odd prime not dividing the class number of the field $\mathbb{Q}(\sqrt{-\Delta})$. Further, in Corollary 7 of [8] the authors considered the equation (3), where $\mu \in \{1, 4\}$, Δ is an odd positive square-free integer and $n \geq 3$ is an odd prime not dividing the class number of the field $\mathbb{Q}(\sqrt{-\Delta})$. Under these assumptions they solved completely equation (3) in the case when

$$\mu = 1, \Delta \equiv 1 \pmod{4}, n \geq 3 \quad \text{or} \quad \mu = 4, \Delta \equiv 7 \pmod{8}, n \geq 3 \quad \text{or}$$

$$\mu = 4, \Delta \equiv 3 \pmod{8}, n \geq 5.$$

In contrast with [8], in our Theorem 2 it is not assumed that Δ is square-free. Using the approach of [8] we give completely explicit upper bounds for n in (3) depending only on P and Q . This allows us to solve completely equation (3) in the case when $S = \{2, 3, 5, 7\}$ and $\mu = 1$. Combining Theorem 2 with some results of COHN [12] and DE WEGER [25], we give all non-exceptional solutions of the equation

$$x^2 + 2^\alpha 3^\beta 5^\gamma 7^\delta = y^n \quad (6)$$

where $x, y, n, \alpha, \beta, \gamma, \delta$ are unknown non-negative integers with $x \geq 1, y \geq 2, \gcd(x, y) = 1$ and $n \geq 3$. We recall that in this special case a solution is called *exceptional* if $\alpha = 0, y$ is even and $3^\beta 5^\gamma 7^\delta$ is either of the form $7c^2$ or of the form $15c^2$. We note that if our equation (6) is of the form $x^2 + 7c^2 = y^n$ or $x^2 + 15c^2 = y^n$ and (x, y, n) is an exceptional solution of (6), then we cannot use the parametrization for (x, y) provided by Lemma 2 (see e.g. [11]). Hence we consider only the non-exceptional solutions of (6). We note that using another approach BUGEAUD, MIGNOTTE and SIKSEK [7] solved the equations $x^2 + 7c^2 = y^n$ and $x^2 + 15c^2 = y^n$ when $1 \leq 7c^2 < 15c^2 \leq 100$.

Theorem 3. *All non-exceptional solutions of equation (6) are listed in the table occurring in Section 4.*

If in (6) $\alpha \geq 1$ is assumed then, by $\gcd(x, y) = 1, y$ is odd. Hence the solutions (x, y, n) of (6) are always non-exceptional. Thus in this case we can list all the solutions of equation (6).

Corollary. *All solutions of (6) with $\alpha \geq 1$ are listed in the table in Section 4.*

We note that the solutions of equation (6) with $\alpha \geq 1$ are those which are not marked with an asterisk in the table. Further, in this case our Theorem 3 is a generalization of a result of LUCA [19] mentioned above.

2. Auxiliary results

We keep the notations of the preceding section. For a non-zero integer m denote by $\omega(m)$ the number of distinct prime factors of m . By definition, for $a, b, c \in \mathbb{Z}$, the discriminant of the binary quadratic form $aX^2 + 2bXY + cY^2$ is $4b^2 - 4ac$, thus -4Δ is the discriminant of the form $X^2 + \Delta Y^2$. We say that a binary quadratic form is positive if $a > 0$. The set of positive binary quadratic forms of discriminant -4Δ is partitioned into a finite number of equivalence classes which we denote by $h(-4\Delta)$.

The next lemma is a special case of Lemma 1 of [8] (see also LE [14]).

Lemma 1. *Consider equation*

$$X_1^2 + \Delta Y_1^2 = \mu Y^{Z_1} \quad (7)$$

in integer unknowns X_1, Y_1, Z_1 with $Z_1 > 0$ and $\gcd(X_1, Y_1) = 1$. Then the solutions of the above equation can be put into at most $2^{\omega(Y)-1}$ classes. Further, in each class there is a unique solution (X_1, Y_1, Z_1) such that $X_1 > 0, Y_1 > 0$ and

Z_1 is minimal among the solutions of the class. This minimal solution satisfies $Z_1 \mid h(-4\Delta)$, where $h(-4\Delta)$ is the number of classes of positive binary forms of discriminant -4Δ .

PROOF. See [8]. □

Lemma 2. Suppose that equation (3) has a solution under the assumptions (4) and (5) with $\mu = 1$. Denote by $d > 0$ the square-free part of $\Delta = dc^2$. If $d \not\equiv 7 \pmod{8}$ or $d \equiv 7 \pmod{8}$ and Y is odd then one of the following cases holds:

- (a) there exist $a_1, b_1 \in \mathbb{Z}$ with $b_1 \mid c$, $b_1 \neq \pm c$ such that $Y = a_1^2 + b_1^2 d$ and $\pm X + c\sqrt{-d} = (a_1 + b_1\sqrt{-d})^n$;
- (b) $n \mid h$, where h denotes the class number of the field $\mathbb{Q}(\sqrt{-d})$;
- (c) $d \equiv 3 \pmod{8}$, $n = 3$ and there exist odd integers A_1, B_1 with $B_1 \mid c$ such that $Y = \frac{1}{4}(A_1^2 + B_1^2 d)$, $\pm X + c\sqrt{-d} = \frac{1}{8}(A_1 + B_1\sqrt{-d})^3$;
- (d) $(n, \Delta, X) = (3, 3u^2 \pm 8, u^3 \pm 3u)$ or $(n, \Delta, X) = (3, 3u^2 \pm 1, 8u^3 \pm 3u)$, where $u \in \mathbb{Z}$;
- (e) $(n, \Delta, X) = (5, 19, 22434)$ or $(n, \Delta, X) = (5, 341, 2759646)$.

PROOF. If $d \not\equiv 7 \pmod{8}$ then the lemma is a reformulation of a theorem of COHN [12]. So, it remains the case when in (3) $d \equiv 7 \pmod{8}$ and Y is odd. In this case we may apply a result of LJUNGGREN [17] (pp. 593–594) to conclude that if in equation (3) $n \nmid h$ then there exist $a_1, b_1 \in \mathbb{Z}$ such that

$$\pm X + c\sqrt{-d} = \left(\frac{a_1 + b_1\sqrt{-d}}{2} \right)^n, \quad a_1 \equiv b_1 \pmod{2}. \quad (8)$$

If in (8) a_1 and b_1 are both odd then since $d \equiv 7 \pmod{8}$, we get

$$a_1^2 + db_1^2 \equiv 0 \pmod{8},$$

whence, by

$$Y = \frac{a_1^2 + db_1^2}{4},$$

it follows that Y is even, a contradiction. So a_1 and b_1 are both even and the lemma is proved. □

The next lemma provides an upper bound for the class number of an imaginary quadratic field.

Lemma 3. Let $\mathcal{D} > 0$ be a square-free integer, and denote by h the class number of the field $\mathbb{K} = \mathbb{Q}(\sqrt{-\mathcal{D}})$. Then

$$h < \frac{4}{\pi} \sqrt{\mathcal{D}} (\log 2e\sqrt{\mathcal{D}}).$$

PROOF. Denote by $h(-4\mathcal{D})$ the class number of the unique quadratic order in $\mathbb{K}$ with discriminant $-4\mathcal{D}$. Then $h(-4\mathcal{D})$ is the number of classes of positive quadratic forms of discriminant $-4\mathcal{D}$ (see e.g. COHEN [9], Definition 5.2.7). Further, we have

$$h(-4\mathcal{D}) < \frac{4}{\pi} \sqrt{\mathcal{D}} (\log 2e\sqrt{\mathcal{D}})$$

(cf. e.g. Proposition 1 of [8]). Since $h \mid h(-4\mathcal{D})$ (see e.g. [21]), the assertion follows. $\square$

Lemma 4. *Denote by $h(-4\Delta)$ the number of classes of positive binary forms of discriminant -4Δ . Then, for $d \equiv 3 \pmod{4}$,*

$$h(-4\Delta) = h(-4c^2d) = h2c \prod_{p|2c} \left(1 - \frac{(-d/p)}{p}\right) \frac{1}{u},$$

where $u = 3$, if $d = 3$ and $u = 1$ otherwise; for $d \equiv 1, 2 \pmod{4}$,

$$h(-4\Delta) = h(-4c^2d) = hc \prod_{p|c} \left(1 - \frac{(-4d/p)}{p}\right) \frac{1}{u},$$

where $u = 2$, if $d = 1$ and $u = 1$ otherwise. Here $(\frac{\cdot}{p})$ denotes the Kronecker symbol.

PROOF. See MOLLIN [21]. $\square$

The next lemma is a deep result of DE WEGER [25]. It will be utilized in the proof of Theorem 3.

Lemma 5. *Let $S = \{2, 3, 5, 7\}$. Consider the equation $U + V = W^2$ in unknowns U, V, W , where U, V or $-V \in \mathbf{S} \cap \mathbb{Z}_{>0}$, $W \in \mathbb{Z}_{>0}$. Suppose that $U \geq V$ and that $\gcd(U, V)$ is square-free. Then the above equation has exactly 388 solutions which are given explicitly in [25].*

PROOF. This is Theorem 7.2 of [25]. $\square$

3. Proofs of theorems

PROOF OF THEOREM 2. Consider equation (3) satisfying (4) and (5). We follow the approach of [8] and we introduce two infinite sets. Denote by F_k the Fibonacci sequence defined by $F_0 = 0, F_1 = 1$ and satisfying $F_k = F_{k-1} + F_{k-2}$

for all $k \geq 2$ and by L_k the Lucas sequence defined by $L_0 = 2$, $L_1 = 1$ and satisfying $L_k = L_{k-1} + L_{k-2}$ for all $k \geq 2$. Then

$$F := \{(F_{k+\varepsilon}, L_{k-\varepsilon}, F_k) \mid k \geq 2, \varepsilon \in \{\pm 1\}\},$$

and

$$H := \{(1, \Delta, Y) \mid \text{there exist } r, s \in \mathbb{Z}_{>0} \text{ such that} \\ s^2 + \Delta = \mu Y^r \text{ and } 3s^2 - \Delta = \mp \mu\}.$$

If (X, Y, n) is a non-exceptional solution of (3) then it corresponds to a solution $(X_1, Y_1, Z_1) = (X, 1, n)$ of (7). Since by Lemma 1 the solutions of (7) can be put into at most $2^{\omega(Y)-1}$ classes we have to distinguish two cases. Firstly, if $(X, 1, n)$ is the minimal solution in the class then by Lemma 1 we have $n \mid h(-4\Delta)$. Secondly, if $(X, 1, n)$ is not the minimal solution then there exist at least two solutions of (7) in the class. By using Theorem 2 of [8] and noting that n is an odd prime we see that in this case either $(\mu, Y, \Delta, n) \in \{(4, 2, 7, 3), (4, 7, 3, 3), (4, 2, 7, 5), (4, 2, 7, 13), (4, 3, 11, 5), (4, 5, 19, 7), (1, 55, 19, 5), (1, 377, 341, 5)\}$ or we have

$$n \in \{1, 5\} \quad \text{and} \quad (1, \Delta, Y) \in F$$

or

$$n \in \{r, 3r\} \quad \text{and} \quad (1, \Delta, Y) \in H, \quad \text{with } r \in \mathbb{Z}_{>0}.$$

Since n is an odd prime we obtain that

$$n = 5 \quad \text{and} \quad (1, \Delta, Y) \in F \quad \text{or} \quad n = 3 \quad \text{and} \quad (1, \Delta, Y) \in H.$$

If $n = 5$ and $(1, \Delta, Y) \in F$ then by the definition of the set F we get

$$F_{k-2} = 1, \quad L_{k+1} = \Delta, \quad F_k = Y$$

or

$$F_{k+2} = 1, \quad L_{k-1} = \Delta, \quad F_k = Y.$$

We see that $F_{k+2} = 1$ cannot hold since in this case it follows that $k+2 \in \{1, 2\}$ and hence $k = 0$. Thus $F_0 = Y = 0$ follows which contradicts the assumption $Y > 1$. If $F_{k-2} = 1$ we get $k-2 \in \{1, 2\}$, whence $k \in \{3, 4\}$ which implies by $F_k = Y$ that

$$(Y, \Delta) \in \{(2, 7), (3, 11)\}.$$

Hence using (3) we get

$$X^2 + 7 = \mu \cdot 2^5 \quad \text{and} \quad X^2 + 11 = \mu \cdot 3^5.$$

We see that if $\mu = 4$ the above equations have solutions which are already listed (i.e. $(\mu, Y, \Delta, n) \in \{(4, 2, 7, 5), (4, 3, 11, 5)\}$). If $\mu = 1$ then $X^2 + 11 = 3^5$ is impossible, while the equation $X^2 + 7 = 2^5$ leads to an exceptional solution of (3), which contradicts the assumption that (X, Y, n) is non-exceptional. Hence we obtain that

$$n \mid h(-4\Delta) \quad \text{or} \quad n = 3,$$

according as $(X, 1, n)$ is the minimal solution in the class or not. We recall that n is an odd prime and $\Delta = dc^2 \in \mathbf{S}$. Thus if $n \mid h(-4\Delta)$ but $n \nmid h$ then by Lemma 4 we obtain that n cannot exceed the greatest prime lying in $S = \{p_1, \dots, p_s\}$. Hence

$$n \leq \max\{3, P\}.$$

If $n \nmid h$ then since d is the square-free part of Δ we have

$$d \leq Q = p_1 \cdots p_s.$$

Hence using Lemma 3 the assertion follows. $\square$

PROOF OF THEOREM 1. Put $f(x) = x^2 + Ax + B$, where $A, B \in \mathbb{Z}$. One can easily see that equation (2) leads to the equation of type (3)

$$X^2 + \Delta = \mu Y^n, \tag{9}$$

where

$$(X, \Delta, \mu, Y) = \begin{cases} \left(x + \frac{A}{2}, -\frac{D_f}{4}, 1, y\right) & \text{if } D_f \text{ is even,} \\ (2x + A, -D_f, 4, y) & \text{if } D_f \text{ is odd.} \end{cases}$$

According to the definition of Δ and the assumption $x \neq -\frac{A}{2}$, we may suppose that in equation (9)

$$\Delta \in \mathbf{S}, \quad \Delta > 0, \quad X \geq 1, \quad n \geq 3 \text{ prime.} \tag{10}$$

Since, by assumption, $\gcd(Y, \Delta) = 1$ we can apply Theorem 2 to equation (9) and we get Theorem 1. $\square$

PROOF OF THEOREM 3. There is no loss of generality by supposing that in (6) $n = 4$ or n is an odd prime. Keeping the notations of the preceding sections we have $dc^2 = \Delta = 2^\alpha 3^\beta 5^\gamma 7^\delta$, where $d \in \mathcal{H}$ with $\mathcal{H} = \{1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210\}$. Assume that n is an odd prime. Since (x, y, n) is a non-exceptional solution of (6) and for every $d \in \mathcal{H}$ the class number h of the imaginary

quadratic field $\mathbb{Q}(\sqrt{-d})$ is 1 or a power of 2 by Theorem 2 we get $n \leq 7$. Hence (6) can have a solution only if $n \in \{3, 4, 5, 7\}$.

The case $n \in \{5, 7\}$. We recall that $dc^2 = \Delta = 2^\alpha 3^\beta 5^\gamma 7^\delta$, where $d \in \mathcal{H}$ with $\mathcal{H} = \{1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210\}$.

Consider equation (6) with $n = 5$. Assume first that $\alpha \geq 0, \beta \geq 0, \gamma \geq 6, \delta \geq 0$. By Lemma 2 we get

$$\pm x + c\sqrt{-d} = (a + b\sqrt{-d})^5 \quad (11)$$

where $a, b \in \mathbb{Z}, b \mid c, y = a^2 + db^2$. Hence, by comparing the imaginary parts of (11) we obtain

$$5a^4b - 10a^2b^3d + b^5d^2 = c. \quad (12)$$

Since $\gamma \geq 6$ we have $\text{ord}_5(c) \geq 3$.

Case 1. $1 \leq \text{ord}_5(b) \leq \text{ord}_5(c) - 2$

Since $b \mid c$ and $\text{ord}_5(b) \leq \text{ord}_5(c) - 2$, we see that $\frac{c}{5b} \in \mathbb{Z}$ and $\text{ord}_5(\frac{c}{5b}) \geq 1$. Using (12) we get

$$a^4 - 2a^2b^2d + \frac{b^4}{5}d^2 = \frac{c}{5b}. \quad (13)$$

Since $\text{ord}_5(\frac{c}{5b}) \geq 1$ and $\text{ord}_5(b) \geq 1$, we obtain by (13) that $5 \mid a^4$, whence by $a \mid x$ we get $5 \mid x$. Thus using equation (6) and the assumption $\gamma \geq 6$, we obtain that $5 \mid y$ which is impossible since x and y are relatively prime.

Case 2. $\text{ord}_5(b) = \text{ord}_5(c)$.

In this case we have $\text{ord}_5(b) \geq 3$ since $\text{ord}_5(c) \geq 3$. By (12) it follows that

$$5a^4 - 10a^2b^2d + b^4d^2 = \frac{c}{b}. \quad (14)$$

Hence the left-hand side of (14) is divisible by 5 but the right-hand side is not, a contradiction.

Case 3. $\text{ord}_5(b) = 0$

By $\text{ord}_5(b) = 0$ we have $\text{ord}_5(c) = \text{ord}_5(\frac{c}{b}) \geq 3$. Thus using (14) we see that $5 \mid b^4d^2$ follows, whence by $\text{ord}_5(b) = 0$ we get $5 \mid d$. Hence from (14) we infer that

$$a^4 - 2a^2b^2d + \frac{b^4d^2}{5} = \frac{c}{5b}. \quad (15)$$

Clearly $\frac{b^4d^2}{5}$ and $\frac{c}{5b}$ are integers and $\text{ord}_5(\frac{c}{5b}) \geq 2$. Thus by (15) it follows that $5 \mid a^4$, whence by $a \mid x$ we get $5 \mid x$. Thus using equation (6) and the assumption $\gamma \geq 6$, we obtain that $5 \mid y$ which contradicts $\gcd(x, y) = 1$.

Case 4. $\text{ord}_5(b) = \text{ord}_5(c) - 1$

By assumption we see that in (15) $\frac{b^4 d^2}{5}$ and $\frac{c}{5b}$ are integers and $\text{ord}_5(\frac{c}{5b})=0$. If now $\text{ord}_2(\frac{c}{5b}) \geq 1$ then clearly $\alpha \geq 1$ and by (15) we get

$$a^4 \equiv \frac{b^4 d^2}{5} \pmod{2}. \quad (16)$$

We may suppose that a is odd since otherwise we obtain by (6), $a \mid x$ and $\alpha \geq 1$ that $2 \leq \gcd(x, y)$ contradicting the assumption $\gcd(x, y) = 1$. Thus by (16) we see that b and d are odd integers, whence it follows that $2 \mid y = a^2 + db^2$. Using equation (6) and $\alpha \geq 1$ we get $2 \mid x$ which cannot hold since x and y are relatively prime integers.

Suppose now that $\text{ord}_2(\frac{c}{5b}) = 0$ and $\text{ord}_3(\frac{c}{5b}) \geq 1$. Then obviously $\beta \geq 1$. We may assume that $3 \nmid d$ and $3 \nmid b$ since otherwise we get by (15) that $3 \mid a^4$ whence $3 \mid x$. Thus by (6) and $\beta \geq 1$ we see that $3 \mid y$ which leads to a contradiction.

By $\Delta = dc^2, y = a^2 + db^2$ and (6) we have

$$x^2 + dc^2 = (a^2 + db^2)^5 \quad (17)$$

Clearly $3 \nmid x$ since otherwise we obtain a contradiction by (6), $\beta \geq 1$ and $\gcd(x, y) = 1$. Thus by $3 \nmid x$ and $\beta \geq 1$ we have

$$x^2 + dc^2 \equiv 1 \pmod{3}. \quad (18)$$

If $3 \mid a$, then by $a \mid x, \beta \geq 1$ and (6) we obtain a contradiction. Hence

$$(a^2 + db^2)^5 \equiv (1 + d)^5 \pmod{3}. \quad (19)$$

Since for every $d \in \mathcal{H}$ with $3 \nmid d$ we have $1 + d \equiv -1, 0 \pmod{3}$ we see by (19) that

$$(a^2 + db^2)^5 \equiv -1, 0 \pmod{3}. \quad (20)$$

Combining (17), (18) and (20) we get a contradiction.

If $\text{ord}_2(\frac{c}{5b}) = 0$ and $\text{ord}_3(\frac{c}{5b}) = 0$ then we have by (15) that

$$a^4 - 2a^2 b^2 d + \frac{b^4 d^2}{5} = \pm 7^{\delta'}, \quad (21)$$

for some non-negative integer δ' . If $\delta' \geq 1$ and $7 \mid d$ then by (21) we infer that $7 \mid a^4$, whence by $a \mid x$ we have $7 \mid x$ and $7 \mid y = a^2 + db^2$. This cannot hold by $\gcd(x, y) = 1$.

If $\delta' \geq 1$ and $7 \nmid d$ then (21) is impossible mod 7 for every $d \in \mathcal{H}$.

If $\delta' = 0$ then (21) is a Thue equation. By solving (21) for every $d \in \mathcal{H}$ we obtain the solution $(\Delta, y, n) = (2 \cdot 5^3, 11, 5)$. It remains the case when in (6) $n = 5$ and $\gamma \in \{0, 1, 2, 3, 4, 5\}$. If $\gamma \in \{0, 3, 4, 5\}$ then we may apply the argument used in the *Case 4* to the equation

$$5a^4 - 10a^2b^2d + b^4d^2 = \frac{c}{b} \quad \text{and} \quad a^4 - 2a^2b^2d + \frac{b^4d^2}{5} = \frac{c}{5b},$$

respectively. Thus in this case there are no other solutions with $n = 5$.

If $\gamma \in \{1, 2\}$ we see that $\text{ord}_5(c) = 0$ and $\text{ord}_5(d) = 1$, or $\text{ord}_5(c) = 1$ and $\text{ord}_5(d) = 0$. This leads to a contradiction in view of (12).

In the case when in (6) $n = 7$ we can work as above and we conclude that there are no solutions to (6) with $n = 7$.

The case $n \in \{3, 4\}$.

First consider equation (6) with $n = 4$. Then factorizing in (6) we get

$$(y^2 + x)(y^2 - x) = \Delta. \quad (22)$$

Hence $y^2 + x \in \mathbf{S}$ and $y^2 - x \in \mathbf{S}$, where $S = \{2, 3, 5, 7\}$. Thus we have the following equations

$$\begin{cases} \frac{y^2 + x}{2} + \frac{y^2 - x}{2} = y^2, & \text{if } x \equiv y \pmod{2}, \\ 2(y^2 + x) + 2(y^2 - x) = (2y)^2, & \text{if } x \not\equiv y \pmod{2}. \end{cases} \quad (23)$$

Since x and y are relatively prime we have $\gcd\left(\frac{y^2+x}{2}, \frac{y^2-x}{2}\right) = 1$ and $\gcd(2(y^2 + x), 2(y^2 - x)) = 2$. Further, since $x > 0$ we get that $\frac{y^2+x}{2} \geq \frac{y^2-x}{2}$ and $2(y^2 + x) \geq 2(y^2 - x)$ always holds. Thus we see that equations (23) satisfies the conditions of Lemma 5 with

$$(U, V, W) = \left(\frac{y^2 + x}{2}, \frac{y^2 - x}{2}, y\right) \quad \text{and}$$

$$(U, V, W) = (2(y^2 + x), 2(y^2 - x), 2y).$$

By applying Lemma 5 to (23) we obtain all non-exceptional solutions of (6) with $n = 4$ (see the table).

Next suppose that in (6) $n = 3$. By Lemma 2 we see that (6) can have a solution with $n = 3$ only in the following cases:

- (I) $3a_1^2b_1 - b_1^3d = c$, where $a_1, b_1 \in \mathbb{Z}$ and $b_1 \mid c$,
- (II) $3A_1^2B_1 - B_1^3d = 8c$, where A_1, B_1 are odd integers and $B_1 \mid c$,
- (III) $\Delta = 3u^2 \pm 8$ and $x = u^3 \pm 3u$, where $u \in \mathbb{Z}$,
- (IV) $\Delta = 3u^2 \pm 1$ and $x = 8u^2 \pm 3u$, where $u \in \mathbb{Z}$.

Each of the above cases leads to an equation of the form

$$U + V = W^2, \quad U, V \in \mathbf{S}$$

with the following choices of the triple (U, V, W) .

Case (I) In this case we have the equation

$$3a_1^2b_1 - b_1^3d = c,$$

where $a_1, b_1 \in \mathbb{Z}$ and $b_1 \mid c$. We distinguish three subcases according to $3 \mid d$ or $3 \nmid db_1$ or $3 \nmid d$ and $3 \mid b_1$.

If $3 \mid d$ then obviously $3 \mid c$ and hence we get from the above equation

$$a_1^2 = \frac{c}{3b_1} + \frac{d}{3}b_1^2.$$

We see that $\gcd\left(\frac{c}{3b_1}, \frac{d}{3}b_1^2\right)$ is square-free since $\frac{c}{3b_1}$ and b_1^2 are relatively prime and $\frac{d}{3}$ is square-free. The last two subcases can be reduced in a

similar way. Thus we have to solve for every $d \in \mathcal{H}$ the following equations:

$$(U, V, W) = \begin{cases} \left(\frac{d}{3}b_1^2, \frac{c}{3b_1}, a_1\right), & \text{if } 3 \mid d \text{ and } \frac{d}{3}b_1^2 \geq \frac{c}{3b_1}, \\ \left(\frac{c}{3b_1}, \frac{d}{3}b_1^2, a_1\right), & \text{if } 3 \mid d \text{ and } \frac{c}{3b_1} > \frac{d}{3}b_1^2, \\ \left(3db_1^2, \frac{3c}{b_1}, 3a_1\right), & \text{if } 3 \nmid d, 3 \nmid b_1 \text{ and } 3db_1^2 \geq \frac{3c}{b_1}, \\ \left(\frac{3c}{b_1}, 3db_1^2, 3a_1\right), & \text{if } 3 \nmid d, 3 \nmid b_1 \text{ and } \frac{3c}{b_1} > 3db_1^2, \\ \left(3db_1'^2, \frac{c}{9b_1'}, a_1\right), & \\ b_1 = 3b_1', & \text{if } 3 \nmid d, 3 \mid b_1, \text{ and } 3db_1'^2 \geq \frac{c}{9b_1'}, \\ \left(\frac{c}{9b_1'}, 3db_1'^2, a_1\right), & \\ b_1 = 3b_1', & \text{if } 3 \nmid d, 3 \mid b_1, \text{ and } \frac{c}{9b_1'} > 3db_1'^2. \end{cases} \quad (24)$$

Case (II) In this case we deal only with those values of $d \in \mathcal{H}$ for which $d \equiv 3 \pmod{8}$. Thus $d \in \{3, 35\}$ and we get the following equations:

$$(U, V, W) = \begin{cases} \left(B_1^2, \frac{8c}{3B_1}, A_1\right), & \text{if } d = 3 \text{ and } B_1^2 \geq \frac{8c}{3B_1}, \\ \left(\frac{8c}{3B_1}, B_1^2, A_1\right), & \text{if } d = 3 \text{ and } \frac{8c}{3B_1} > B_1^2, \\ \left(105B_1^2, \frac{24c}{B_1}, 3A_1\right), & \text{if } d = 35, 3 \nmid B_1 \text{ and } 105B_1^2 \geq \frac{24c}{B_1}, \\ \left(\frac{24c}{B_1}, 105B_1^2, 3A_1\right), & \text{if } d = 35, 3 \nmid B_1 \text{ and } \frac{3c}{B_1} > 105B_1^2, \\ \left(105B_1'^2, \frac{8c}{9B_1'}, A_1\right), & \\ \quad B_1 = 3B_1', & \text{if } d = 35, 3 \mid b_1, \text{ and } 105B_1'^2 \geq \frac{8c}{9B_1'}, \\ \left(\frac{8c}{9B_1'}, 105B_1'^2, A_1\right), & \\ \quad B_1 = 3B_1', & \text{if } d = 35, 3 \mid b_1, \text{ and } \frac{8c}{9B_1'} > 105B_1'^2. \end{cases} \quad (25)$$

Case (III)

$$(U, V, W) = \begin{cases} (3\Delta, \pm 24, 3u), & \text{if } \alpha \in \{0, 1\}, \\ \left(\frac{3\Delta}{4}, \pm 6, \frac{3u}{2}\right), & \text{if } \alpha \geq 2. \end{cases} \quad (26)$$

Case (IV)

$$(U, V, W) = (3\Delta, \pm 3, 3u). \quad (27)$$

One can easily see that each of the above equations satisfies the conditions of Lemma 5. By applying Lemma 5 to equations (24)–(27) we get all non-exceptional solutions to (6) with $n = 3$ (see the table). $\square$

4. Non-exceptional solutions of equation (6)

The solutions of (6) with $\alpha = 0$ are marked with an asterisk.

Table

Δ	y	n	Δ	y	n	Δ	y	n
2^2	5	3	$3^7 7^2 *$	67	3	$2^8 3^7 5^4 7^2$	50401	3
$3^4 *$	13	3	$3^5 5^2 7^2 *$	79	3	$3^{13} 5^2 7^4 *$	59539	3
$7^2 *$	65	3	$2^6 3^7 7^2$	193	3	$3^{13} 5^6 7^2 *$	60799	3
$2^4 3^4$	193	3	$2^4 3^7$	73	3	$2^2 3^{13} 5^4 7^6$	93349	3
$3^6 7^2 *$	585	3	$3^7 5^2 *$	91	3	$2^8 3^5 5^6 7^6$	129649	3

Δ	y	n	Δ	y	n	Δ	y	n
$3^6 5^2 *$	2701	3	$3^7 5^2 7^2 *$	151	3	$2^{10} 3^7 5^8 7^4$	362401	3
$2^6 3^4 7^2$	37633	3	$2^4 3^5 5^4 7^4$	1801	3	$2^{18} 3^5 5^2 7^6$	1053721	3
2	3	3	$3^5 5^2 7^2 *$	211	3	$2^8 3^5 5^4 7^8$	5762401	3
$2 \cdot 5^4$	11	3	$2^6 3^5 5^2$	241	3	$3^{17} 5^8 7^2 *$	19136251	3
$2^3 5^2$	9	3	$2^6 3^7 5^4$	481	3	$5 \cdot 7^2 *$	9	3
$2 \cdot 3^4 5^2$	19	3	$2^6 3^5 5^2 7^2$	361	3	$2^2 5 \cdot 7^2$	29	3
$2^5 5^2$	41	3	$3^7 5^4 7^2 *$	499	3	$3^4 \cdot 5 *$	61	3
$2^7 5^6$	129	3	$2^2 3^5 5^2 7^2$	421	3	$3^4 5 \cdot 7^4 *$	109	3
$2^3 3^4$	97	3	$2^2 3^9 5^2 7^2$	589	3	$2^2 5 \cdot 7^6$	141	3
$2^3 3^4 5^4$	121	3	$2^2 3^5 5^4 7^2$	541	3	$2^2 5^3 7^2$	669	3
$2 \cdot 3^6 5^2$	211	3	$2^4 3^5 5^4$	601	3	$2^4 3^4 5 \cdot 7^4$	1009	3
$2^5 3^4 5^4$	409	3	$3^9 5^4 *$	679	3	$2^6 3^4 \cdot 5$	3841	3
$2^9 5^2$	681	3	$2^2 3^5 5^4 7^2$	709	3	$5^5 7^6 *$	4281	3
$2^3 3^6 5^8$	1489	3	$2^8 3^3 5^4 7^2$	849	3	$2^4 3^6 5 \cdot 7^4$	8689	3
$2 \cdot 3^4 5^6 7^2$	1051	3	$2^2 3^9 7^2$	757	3	$3^6 5^3 7^8 *$	15901	3
$2 \cdot 3^4 5^2 7^2$	1171	3	$2^8 3^9 5^2$	889	3	$2^8 3^4 5 \cdot 7^8$	17761	3
$2 \cdot 3^8 5^6$	1819	3	$2^4 3^9 5^2 7^2$	1009	3	$3^8 5 \cdot 7^2 *$	238141	3
$2^9 3^4 5^{12}$	21769	3	$2^8 3^5 5^2 7^2$	1129	3	$2 \cdot 3^3$	7	3
$2^9 3^6 5^2$	6129	3	$2^2 3^7 5^2 7^2$	1261	3	$2^3 3^3 7^2$	25	3
$2^5 3^4 5^2$	9601	3	$2^4 3^7 5^2 7^4$	2041	3	$2 \cdot 3^3 5^2 7^4$	151	3
$2^7 3^6 5^4$	13849	3	$2^2 3^7 5^2 7^4$	2221	3	$2 \cdot 3^5 7^2$	79	3
$2^5 3^4 5^8 7^2$	19441	3	$2^6 3^5 7^4$	2353	3	$2^5 3^3 7^2$	121	3
$2^3 3^6 5^4 7^2$	42361	3	$3^3 5^4 7^4 *$	2451	3	$2 \cdot 3^3 5^2$	199	3
$2 \cdot 3^6 5^{14} 7^4$	440491	3	$2^8 3^1 17^4$	4993	3	$2 \cdot 3^5 7^6$	415	3
$2^7 3^4 5^2 7^4$	92198401	3	$3^9 5^2 7^4 *$	2671	3	$2^3 3^5 7^4$	337	3
$3^5 *$	7	3	$2^{10} 3^5 5^2 7^4$	3361	3	$2^7 3^3 7^2$	505	3
$3^5 5^2 *$	19	3	$2^4 3^{11} 5^4 7^2$	5161	3	$2 \cdot 3^7 7^2$	655	3
$2^2 3^5$	13	3	$2^{10} 3^7 7^2$	4033	3	$2^5 3^5$	1153	3
$2^2 3^5 7^2$	37	3	$2^6 3^{11} 5^2$	6481	3	$2^9 3^3 7^6$	1705	3
$2^2 3^7 5^2$	61	3	$3^9 5^6 7^4 *$	12979	3	$2^3 3^5 5^2 7^4$	7249	3
$2^4 3^5 5^2$	49	3	$3^7 5^6 7^2 *$	15751	3	$2^3 3^3 5^2 7^2$	39201	3
$3^5 5^2 *$	31	3	$2^{12} 3^5 5^6$	16009	3	$2^7 3^7 7^8$	43873	3
$2^4 3^5 5^2 7^2$	169	3	$2^{12} 3^9 5^2 7^2$	17329	3	$2 \cdot 3^{11} 7^{10}$	69295	3
$3^5 7^2 *$	43	3	$2^{10} 3^9 5^6 7^2$	27721	3	$2^{15} 3^3 7^2$	131065	3
$2^2 3^5 5^2 7^2$	109	3	$2^{14} 3^{11} 5^4 7^2$	51361	3	$5^2 7 *$	11	3
$2^2 5^4 7$	29	3	$2^2 3^9 5^4 7^3$	70189	3	$2^{10} 3^4 7^2$	65	4
$2^2 7$	37	3	$2^4 3^{23} 5^2 7^3$	607849	3	$2^{10} 3^2 5^2 7^2$	113	4
$3^4 5^2 7 *$	79	3	$2^{10} 3^{19} 5^2 7$	723361	3	$2^{12} 3^4 5^4 7^2$	337	4
$5^{10} 7^3 *$	1499	3	$3^{31} 5^4 7$	4800469	3	$2^{10} 3^2 5^4 7^4$	1201	4
$3^6 5^6 7 *$	631	3	$2 \cdot 3^7 5$	31	3	2^5	3	4
$2^4 3^4 5^4 7$	1369	3	$2^3 3^7 5$	169	3	$2^7 7^2$	9	4
$2^4 3^4 5^8 7$	1969	3	$2^3 3^{11} 5$	241	3	$2^5 3^2 7^2$	11	4

Δ	y	n	Δ	y	n	Δ	y	n
$3^4 5^2 7^3 *$	4111	3	$2^5 3^7 5$	649	3	$2^7 3^2$	17	4
$2^6 3^4 5^4 7$	5401	3	$2 \cdot 3^{11} 5^3$	919	3	$2^5 3^2 5^2 7^2$	43	4
$2^{10} 5^2 7^3$	9569	3	$2^3 3^5 5$	1441	3	$2^5 5^2 7^4$	51	4
$2^4 3^6 7$	12097	3	$2^7 3^{15} 5$	3289	3	$2^9 5^2 7^2$	57	4
$2^2 3^8 5^6 7^3$	26341	3	$2 \cdot 3^7 5^5$	24991	3	$2^9 3^4 7^4$	113	4
$2^2 5^8 7^5$	89429	3	$2^5 3^{11} 5 \cdot 7^2$	31441	3	$2^5 5^2 7^2$	99	4
$2^4 5^{14} 7^3$	182441	3	$2^3 3^{23} 5 \cdot 7^2$	66889	3	$2^5 3$	5	4
$2^8 3^6 5^{12} 7$	209161	3	$2^3 3^{15} 5^3 7^2$	196729	3	$2^6 3$	7	4
$2^{12} 3^4 5^4 7^3$	16859161	3	$2^9 3^7 5^3$	256009	3	$2^4 3^5 5^2 7^2$	49	4
$2 \cdot 5 \cdot 7^2$	11	3	$2^4 3^4 5 \cdot 7$	6721	3	$2^8 37^2$	97	4
$2 \cdot 5^3 7^2$	331	3	$2 \cdot 3^5 5^6 7$	379	3	$2^4 3^3 5^4 7^2$	133	4
$2^3 3^4 5$	481	3	$2 \cdot 3^5 5^2 7$	499	3	$2^4 5$	3	4
$2^3 3^4 5 \cdot 7^4$	529	3	$2^5 3^3 5^4 7^3$	721	3	$2^4 3^2 5$	7	4
$2^5 3^4 5 \cdot 7^4$	1969	3	$2^3 3^5 5^4 7$	2041	3	$2^6 5$	9	4
$2^3 3^6 5 \cdot 7^8$	6721	3	$2^7 3^3 5^8 7$	4209	3	$5^3 7^2 *$	21	4
$2^7 3^4 5^3 7^{12}$	309649	3	$2^3 3^5 5^{12} 7$	17641	3	$2^4 3^2 5 \cdot 7^2$	47	4
$2^9 3^6 5 \cdot 7^4$	276529	3	$2 \cdot 3^9 5^2 7$	40819	3	$2^8 3^4 5$	161	4
$2^3 3^8 5^3 7^4$	972049	3	$2^{11} 3^3 5^4 7^3$	57169	3	$2^5 3 \cdot 5^2$	7	4
$2^3 3^4 7$	673	3	$2 \cdot 3^3 5^6 7^5$	134331	3	$2^7 3 \cdot 5^2$	11	4
$2 \cdot 3^{10} 7^3$	122479	3	$2 \cdot 3^7 5^{10} 7^3$	219139	3	$2^5 3^3 5^4$	29	4
$2^3 3^{10} 5^4 7$	306180001	3	$2^7 3^5 5^2 7$	806401	3	$2^7 3 \cdot 5^2$	49	4
$3^3 5 *$	19	3	$2^3 3^4 5 \cdot 7$	3361	3	$2^9 3^3 5^2$	59	4
$2^6 3^7 5$	103681	3	$2^4 3^5 5 \cdot 7$	20161	3	$2^7 3 \cdot 5^4 7^2$	73	4
$3^5 7 *$	25	3	$2^3 3^7 5 \cdot 7$	1129	3	$2^5 3^3 5^2 7^2$	103	4
$3^7 7 *$	37	3	$2^3 3^{11} 5 \cdot 7$	1201	3	$2^{11} 3 \cdot 5^6$	131	4
$2^2 3^9 7$	85	3	$2^5 3^{15} 5 \cdot 7$	5209	3	$2^3 3^7 7^2$	175	4
$2^2 3^5 7$	109	3	$2^3 3^{23} 5^3 7$	87049	3	$2^9 3 \cdot 5^2 7^4$	4801	4
$3^5 7 *$	253	3	$2^7 3^7 5 \cdot 7$	17929	3	$7 *$	2	4
$2^4 3^{15} 7$	1177	3	$2^3 3^5 5^3 7$	252001	3	$2^3 7$	3	4
$2^4 3^7 7^3$	385	3	$2^6 3^2$	5	4	$5^2 7 *$	4	4
$2^4 3^7 7$	457	3	$7^2 *$	5	4	$2^6 3^2 7$	11	4
$3^5 5^2 7^3 *$	721	3	$2^6 3^2 5^2 7^2$	29	4	$5^4 7^3 *$	22	4
$3^{11} 5^2 7 *$	781	3	$2^6 3^2 5^2$	13	4	$2^8 3^4 7$	23	4
$2^6 3^{11} 7$	1873	3	$2^8 3^2 5^2$	17	4	$2^6 3^6 7$	29	4
$2^2 3^5 7^3$	5485	3	$2^8 3^2 7^2$	25	4	$2^6 3^2 5^2 7$	53	4
$2^8 3^{11} 7^3$	6601	3	$2^6 3^2 5^2 7^2$	37	4	$2^{10} 3^2 7$	127	4
$2^4 3^7 5^2 7$	11209	3	$2^6 3^4 5^2 7^2$	53	4	$2^6 3^{10} 5^2 7^3$	443	4
$2^8 3^5 7$	64513	3	$2^8 3^4 5^2$	41	4	$2^{12} 3^8 5^2 7$	431	4
$2^5 3^2 5$	7	4	$3^3 5 *$	4	4	$2^{11} 3^3 5 \cdot 7^2$	263	4

Δ	y	n	Δ	y	n	Δ	y	n
$2^5 3^4 5$	11	4	$3 \cdot 5^3 *$	8	4	$2^9 3 \cdot 5^3 7^6$	407	4
$2^9 3^6 5$	37	4	$2^6 3 \cdot 5 \cdot 7^2$	17	4	$2^5 3^5 5^3 7^2$	493	4
$2^7 3^2 5$	13	4	$2^8 3 \cdot 5$	31	4	$2^6 3^2 5 \cdot 7$	71	4
$2^5 3^2 5$	19	4	$2^6 3^3 5 \cdot 7^2$	47	4	$2^5 3 \cdot 7$	13	4
$2^7 3^4 5 \cdot 7^2$	89	4	$3^7 5^3 *$	34	4	$2^5 3^7 5^4 7$	8749	4
$2^5 3^6 5^3 7^2$	223	4	$2^4 3 \cdot 7$	5	4	$2^5 3^2 5 \cdot 7$	17	4
$2^5 3^{10} 5 \cdot 7^2$	247	4	$2^4 3 \cdot 5^2 7$	11	4	$2^5 3^4 5 \cdot 7$	19	4
$2^{11} 3^2 5^3$	253	4	$2^4 3^3 5^2 7$	17	4	$2^7 3^6 5 \cdot 7$	43	4
$2^3 3^2 7$	5	4	$2^6 3 \cdot 5^4 7$	31	4	$2^9 3^2 5 \cdot 7$	67	4
$2^5 5^2 7$	9	4	$2^6 3 \cdot 5^2 7$	19	4	$2^5 3^{10} 5^3 7$	257	4
$2^3 3^4 7$	13	4	$2^4 3 \cdot 5^2 7$	23	4	$2^5 3^2 5^3 7$	251	4
$2^7 7$	15	4	$2^8 3 \cdot 5^2 7$	37	4	$2^6 3 \cdot 5 \cdot 7$	11	4
$2^9 5^4 7$	39	4	$2^6 3^3 7$	55	4	$2^8 3 \cdot 5 \cdot 7$	13	4
$2^3 3^2 7^3$	19	4	$2^{10} 3^3 5^6 7$	721	4	$2^{10} 3 \cdot 5 \cdot 7$	19	4
$2^5 3^2 5^2 7$	23	4	$2^4 3^5 5^4 7^7$	293	4	$2^6 3^3 5 \cdot 7$	31	4
$2^3 3^8 7^5$	173	4	$2^{14} 3 \cdot 5^2 7^3$	2053	4	$2^6 3 \cdot 5 \cdot 7$	41	4
$2^5 3^2 5^6 7$	127	4	$2^5 3 \cdot 5 7^2$	13	4	$2^{14} 3^3 5 \cdot 7$	71	4
$2^7 3^4 5^4 7$	137	4	$2^5 3 \cdot 5$	11	4	$2^{12} 3^3 5^3 \cdot 7$	157	4
$2^9 3^2 5^2 7$	449	4	$2^7 3 \cdot 5 \cdot 7^2$	23	4	$2^{10} 3 \cdot 5^3 7^3$	359	4
$3 \cdot 5 *$	2	4	$2^5 3^3 5 \cdot 7^4$	59	4	$2^{20} 3 \cdot 5 \cdot 7^3$	517	4
						$2^5 3 \cdot 5 \cdot 7$	29	4
						$2 \cdot 5^3$	11	5

ACKNOWLEDGEMENTS. The author is grateful to KÁLMÁN GYÖRÝ, ÁKOS PINTÉR, LAJOS HAJDU and the referee for their help and numerous valuable remarks.

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(Received May 24, 2005; revised January 9, 2006)