

On a class of projectively flat (α, β) -metrics

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Abstract. In this paper, we find a sufficient and necessary condition for an important class of (α, β) -metrics in the form $F = \alpha\phi(\beta/\alpha)$ to be locally projectively flat, where $\phi = \phi(s)$ is a positive C^∞ function satisfying certain conditions, characterized by a polynomial or a power series of s , α is a Riemannian metric and β is a 1-form.

1. Introduction

Hilbert's Fourth Problem in the regular case requires to study and characterize Finsler metrics $F = F(x, y)$ on an open domain $\mathbf{U} \subset R^n$ whose geodesics are straight lines [4]. Finsler metrics on $\mathbf{U}$ with this property are called *projectively flat* metrics. In [3], G. HAMEL first found a simple system of partial differential equations that characterizes projectively flat Finsler metrics on an open domain $\mathbf{U} \subset R^n$. That is $F = F(x, y)$ on $\mathbf{U}$ is projectively flat if and only if the following PDE's hold:

$$F_{x^m y^i} y^m = F_{x^i}. \quad (1)$$

It is one of the important problems in Finsler geometry to characterize projectively flat metrics. According to Beltrami's Theorem, a Riemannian metric is projectively flat if and only if it is of constant sectional curvature [7], [8]. Further, it is known that a Randers metric $F = \alpha + \beta$ is projectively flat if and only if α is projectively flat and β is closed [1], where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a 1-form with $b := \|\beta_x\| < 1$ for $x \in M$.

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In this paper, we are going to consider a class of Finsler metrics on a manifold M which are expressed in the following form:

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\phi = \phi(s)$ is a C^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(0) = 1, \quad \phi(s) > 0, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad (2)$$

where s and b are arbitrary numbers with $|s| \leq b < b_0$. It is known that $F = \alpha\phi(s)$, $s = \beta/\alpha$ is a Finsler metric if and only if the condition (2) holds. Finsler metrics in the above form are called (α, β) -metrics. The class of (α, β) -metrics contains all Riemannian metrics ($\phi = 1$) and all Randers metrics ($\phi = 1 + s$). In the past several years, various curvatures in Finsler geometry have been studied and their geometric meaning is better understood. This is partially due to the study of (α, β) -metrics. Thus, this motivates people to study (α, β) -metrics more deeply.

From now on, we are going to consider a special class of (α, β) -metrics $F = \alpha\phi(s)$, where $\phi = \phi(s)$ is a function satisfying (2) and

$$\phi - s\phi' = (p + rs^2)\phi''(s), \quad (3)$$

where p, r are constants. Recently, Z. SHEN has proved the following

Theorem 1.1. ([11]) *Assume that $\phi = \phi(s)$ satisfies (2) and (3). Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β) -metric on a manifold M . If*

$$b_{i|j} = 2\tau\{(p + b^2)a_{ij} + (r - 1)b_ib_j\} \quad (4)$$

and the spray coefficients G_α^i of α are of the form:

$$G_\alpha^i = \xi y^i - \tau\alpha^2 b^i, \quad (5)$$

where $b := \sqrt{a^{ij}b_ib_j}$, $b_{i|j}$ denote the coefficients of the covariant derivative of β with respect to α , $\tau = \tau(x)$ is a scalar function and $\xi = \xi_i(x)y^i$ is a 1-form on M , then F is locally projectively flat.

Unfortunately, we are not sure that the conditions (4) and (5) are necessary. A key step is to prove that β is closed. However, some progress has been made for certain types of functions ϕ recently. In [12], Z. SHEN and G. C. YILDIRIM first showed that $F = (\alpha + \beta)^2/\alpha$ is projectively flat if and only if the conditions

(4) and (5) hold with $p = 1/2$ and $r = -1/2$. Then they proved that the (α, β) -metrics $F = \alpha + \varepsilon\beta + k\beta^2/\alpha$ are projectively flat if and only if the conditions (4) and (5) hold with $p = 1/(2k)$ and $r = -1/2$, where ε and $k \neq 0$ are constants. Further, Y. SHEN and L. ZHAO have shown that $\phi = 1 + \varepsilon s + 2ks^2 - \frac{k^2}{3}s^4$ satisfies (2) and (3) with $p = 1/(4k)$ and $r = -1/4$, and the Finsler metrics in the form $F = \alpha\phi(\beta/\alpha)$ are projectively flat if and only if (4) and (5) hold [6]. In this paper we will show that, for a larger class of Finsler metrics $F = \alpha\phi(s)$ satisfying (2) and (3), they are projectively flat if and only if (4) and (5) hold.

We will firstly show that, if $p = 0$ then the solutions of (3) will not satisfy (2). Hence, we always assume that $p \neq 0$ in this paper. In this case, the solutions of (3) are analytic near the origin and the power series of the solutions are of the form

$$\phi(s) = C_0 + C_1s + C_2s^2 + C_4s^4 + \cdots + C_{2n}s^{2n} + \cdots, \quad (6)$$

where C_0 and C_1 are arbitrary constants and the coefficients of ϕ satisfy

$$C_{2n+2} = \left(-\frac{1}{p}\right) \frac{(2n-1)(2nr+1)}{(2n+2)(2n+1)} C_{2n}.$$

By a simple analysis, we can see that $\phi(s)$ satisfies (2) and (3) if and only if $C_0 = 1$ and $b \in (0, \sqrt{|p|})$ is sufficiently small. In particular, if we take $r = -\frac{1}{2k}$ and k is a positive integer, then the $\phi(s)$ in (6) are polynomials of the following form

$$\phi(s) = C_0 + C_1s + C_2s^2 + C_4s^4 + \cdots + C_{2k}s^{2k}. \quad (7)$$

Furthermore, if we take $k = 1$ and then $k = 2$, then the (α, β) -metrics $F = \alpha\phi(s)$, $s = \beta/\alpha$, given by (7) have the form

$$F = \alpha + C_1\beta + \frac{1}{2p} \frac{\beta^2}{\alpha}, \quad (8)$$

and

$$F = \alpha + C_1\beta + \frac{1}{2p} \frac{\beta^2}{\alpha} - \frac{1}{48p^2} \frac{\beta^4}{\alpha^3} \quad (9)$$

respectively. These are just the (α, β) -metrics discussed in [12] and [6] respectively.

By using the form (6) of ϕ , we have the following main

Theorem 1.2. *Assume that $\phi = \phi(s)$ is a function in the form (6) satisfying (2) and (3). If $C_1 \neq 0$ and $r \neq 1$ or $C_1 = 0$ but $r = -1/(2k)$, where k is any positive integer, then the (α, β) -metric $F = \alpha\phi(\beta/\alpha)$ is projectively flat if and only if (4) and (5) hold.*

According to the discussion above, Theorem 1.2 generalizes the results in [12] and [6].

2. The analysis of the solutions

In this section, we will study the second order linear ODE (3) by the power series method. If $p = 0$, then equation (3) has explicit solutions as follows:

$$\phi(s) = \begin{cases} Cs & \text{if } r = 0 \\ Cs + \tilde{C}s \ln s & \text{if } r = -1 \\ Cs + \tilde{C}s^{-\frac{1}{r}} & \text{if } r \neq 0, -1. \end{cases}$$

In this case, $\phi(0) \neq 1$. Now, the resulting $\phi(s)$ does not satisfy (2) and the $F = \alpha\phi(\beta/\alpha)$ defined by these $\phi(s)$ are not Finsler metrics. Therefore, we always assume that $p \neq 0$.

By a theorem on the power series method of ODE, we know that the solutions of (3) are analytic near the origin. Let the power series expressions of $\phi(s)$ be

$$\phi(s) = \sum_{n=0}^{\infty} C_n s^n.$$

The first and the second order derivatives of $\phi(s)$ are

$$\phi'(s) = \sum_{n=0}^{\infty} (n+1)C_{n+1}s^n, \quad \phi''(s) = \sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2}s^n.$$

Plugging them into (3) yields

$$(2pC_2 - C_0) + 6pC_3s + \sum_{n=0}^{\infty} \{p(n+4)(n+3)C_{n+4} + (n+1)[r(n+2)+1]C_{n+2}\}s^{n+2} = 0. \quad (10)$$

From (10) we know that the coefficients of $\phi(s)$ must satisfy

$$C_{2n+1} = 0, \quad n \geq 1, \quad (11)$$

$$C_{2n+2} = \left(-\frac{1}{p}\right) \frac{(2n-1)(2nr+1)}{(2n+2)(2n+1)} C_{2n}, \quad n \geq 0. \quad (12)$$

Hence the power series expression of $\phi(s)$ is

$$\phi(s) = C_0 + C_1s + C_2s^2 + C_4s^4 + \cdots + C_{2n}s^{2n} + \cdots. \quad (13)$$

We assert that the $F = \alpha\phi(\beta/\alpha)$ defined by $\phi(s)$ in the form (13) are Finsler metrics if and only if $C_0 = 1$ and $b \in (0, \sqrt{|p|})$ is sufficiently small. Because of $\phi(0) = C_0$ and (2), we know that $C_0 = 1$. Then $\phi(s) > 0$ in a sufficiently small neighborhood of the origin. Note that

$$[\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s)]_{s=0} = \frac{p + b^2}{p}.$$

Thus, if $b \in (0, \sqrt{|p|})$ is sufficiently small and $|s| \leq b$, then

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0.$$

This proves our assertion. In this case,

$$C_2 = \frac{1}{2p} \neq 0. \quad (14)$$

3. (α, β) -metrics

In this section, for a function $\phi = \phi(s)$ satisfying (2), we will recall some well-known properties of an (α, β) -metric $F = \alpha\phi(\beta/\alpha)$. Let $b_{i|j}dx^i \otimes dx^j$ denote covariant derivatives of β with respect to α . Let

$$\begin{aligned} r_{ij} &:= \frac{1}{2}(b_{i|j} + b_{j|i}), & s_{ij} &:= \frac{1}{2}(b_{i|j} - b_{j|i}), \\ s^i_j &:= a^{il}s_{lj}, & s_j &:= b^i s_{ij} \end{aligned}$$

and

$$r_{00} := r_{ij}y^i y^j, \quad s_0 := s_i y^i, \quad s^i_0 := s^i_j y^j, \quad s_{l0} := s_{lj} y^j.$$

Clearly, β is closed if and only if $s_{ij} = 0$.

The geodesic coefficients G^i of $F = \alpha\phi(\beta/\alpha)$ are given by

$$G^i = G^i_\alpha + \alpha Q s^i_0 + H(-2\alpha Q s_0 + r_{00}) \left\{ \chi \frac{y^i}{\alpha} + b^i \right\}, \quad (15)$$

where

$$\begin{aligned} \chi &:= \frac{(\phi - s\phi')\phi'}{\phi\phi''} - s, \\ Q &:= \frac{\phi'}{\phi - s\phi'}, \\ H &:= \frac{\phi''}{2[(\phi - s\phi') + (b^2 - s^2)\phi'']}. \end{aligned}$$

The formula (15) is given in [2], [10], [11]. Further, by (1) and (15) we have the following

Lemma 3.1 ([12]). *An (α, β) -metric $F = \alpha\phi(\beta/\alpha)$ is projectively flat on an open domain $\mathbf{U} \subset R^n$ if and only if*

$$(a_{ml}\alpha^2 - y_my_l)G_\alpha^m + \alpha^3 Qs_{l0} + H(-2\alpha Qs_0 + r_{00})(b_l\alpha^2 - \beta y_l) = 0, \quad (16)$$

where $y_l := a_{il}y^i$.

4. Proof of Theorem 1.2

In this section, we are going to prove Theorem 1.2 using Lemma 3.1.

From (3) and (15) we have

$$Q = \frac{\phi'}{(p + rs^2)\phi''}, \quad H = \frac{1}{2[(p + b^2) + (r - 1)s^2]}. \quad (17)$$

If (4) holds, then β is closed and $r_{00} = 2\tau\{(p + b^2)\alpha^2 + (r - 1)\beta^2\}$. Hence (15) reduces to

$$G^i = G_\alpha^i + \tau \left\{ \chi \frac{y^i}{\alpha} + b^i \right\} \alpha^2.$$

In addition, if (5) holds, we may obtain

$$G^i = (\xi + \tau\chi\alpha)y^i.$$

Therefore F is projectively flat.

Conversely, assume that $\phi = \phi(s)$ is a function satisfying (2) and (3) and the (α, β) -metric $F = \alpha\phi(\beta/\alpha)$ is projectively flat.

If β is closed, i.e. $s_{ij} = 0$, (16) reduces to

$$(a_{ml}\alpha^2 - y_my_l)G_\alpha^m + \frac{\alpha^2 r_{00}}{2[(p + b^2)\alpha^2 + (r - 1)\beta^2]}(b_l\alpha^2 - \beta y_l) = 0,$$

that is,

$$2(a_{ml}\alpha^2 - y_my_l)[(p + b^2)\alpha^2 + (r - 1)\beta^2]G_\alpha^m = -\alpha^2 r_{00}(b_l\alpha^2 - \beta y_l). \quad (18)$$

Contracting (18) with b^l , we get

$$2[(p + b^2)\alpha^2 + (r - 1)\beta^2](b_m\alpha^2 - y_m\beta)G_\alpha^m = -\alpha^2(b^2\alpha^2 - \beta^2)r_{00}.$$

Note that the polynomial $(p+b^2)\alpha^2 + (r-1)\beta^2$ is not divisible by α^2 and $b^2\alpha^2 - \beta^2$ as $r \neq 1$. Thus $(b_m\alpha^2 - y_m\beta)G_\alpha^m$ is divisible by $\alpha^2(b^2\alpha^2 - \beta^2)$. Therefore, there is a function $\tau = \tau(x)$ such that

$$r_{00} = 2\tau\{(p+b^2)\alpha^2 + (r-1)\beta^2\}. \quad (19)$$

Note that (19) is equivalent to (4) since $s_{ij} = 0$. Now the formula (15) for G^i can be simplified to

$$G^i = G_\alpha^i + \tau\chi\alpha y^i + \tau\alpha^2 b^i. \quad (20)$$

On the other hand, it is well-known that F is projectively flat if and only if there is a scalar function $P(x, y)$ satisfying $P(x, \lambda y) = \lambda P(x, y)$ for $\lambda > 0$, such that $G^i = Py^i$. Thus, by (20), we have

$$G_\alpha^i = (P - \tau\chi\alpha)y^i - \tau\alpha^2 b^i.$$

Because both of G_α^i and $\tau\alpha^2 b^i$ are quadratic forms of (y^i) , we assert that $\xi := P - \tau\chi\alpha y^i$ must be a 1-form and we get (5). Thus the key step of the proof is to prove that β is closed.

It is easy to see by (17) that (16) can be rewritten as

$$\begin{aligned} & 2[(p+b^2) + (r-1)s^2](p+rs^2)(a_{ml}\alpha^2 - y_my_l)G_\alpha^m\phi'' \\ & + 2[(p+b^2) + (r-1)s^2]\alpha^3 s_{l0}\phi' - 2\alpha s_0(b_l\alpha^2 - \beta y_l)\phi' \\ & + r_{00}(p+rs^2)(b_l\alpha^2 - \beta y_l)\phi'' = 0. \end{aligned} \quad (21)$$

Contracting (21) with b^l yields

$$\begin{aligned} & 2[(p+b^2) + (r-1)s^2](p+rs^2)(b_m\alpha^2 - y_m\beta)G_\alpha^m\phi'' \\ & + 2[(p+b^2) + (r-1)s^2]\alpha^3 s_0\phi' - 2\alpha s_0(b^2\alpha^2 - \beta^2)\phi' \\ & + r_{00}(p+rs^2)(b^2\alpha^2 - \beta^2)\phi'' = 0. \end{aligned} \quad (22)$$

Then we have

$$\begin{aligned} & \{2[(p+b^2) + (r-1)s^2](b_m\alpha^2 - y_m\beta)G_\alpha^m + r_{00}(b^2\alpha^2 - \beta^2)\}(p+rs^2)\phi'' \\ & + 2\alpha s_0(p\alpha^2 + r\beta^2)\phi' = 0. \end{aligned} \quad (23)$$

Case 1: $C_1 \neq 0$ and $r \neq 1$. In this case, we have

$$\begin{aligned} \phi(s) &= 1 + C_1 s + C_2 s^2 + C_4 s^4 + \cdots + C_{2n} s^{2n} + \cdots, \\ \phi'(s) &= C_1 + 2C_2 s + 4C_4 s^3 + \cdots + 2nC_{2n} s^{2n-1} + \cdots, \\ \phi''(s) &= 2C_2 + 4 \cdot 3C_4 s^2 + \cdots + 2n \cdot (2n-1)C_{2n} s^{2n-2} + \cdots \end{aligned}$$

Let

$$\psi(s) = 2C_2 + 4C_4s^2 + \cdots + 2nC_{2n}s^{2n-2} + \cdots.$$

Then $\phi'(s) = C_1 + s\psi(s)$ and (23) becomes

$$\begin{aligned} & \{2[(p+b^2) + (r-1)s^2](p+rs^2)(b_m\alpha^2 - y_m\beta)G_\alpha^m \\ & + r_{00}(p+rs^2)(b^2\alpha^2 - \beta^2)\}\phi'' + 2\beta s_0(p\alpha^2 + r\beta^2)\psi \\ & + 2\alpha s_0(p\alpha^2 + r\beta^2)C_1 = 0. \end{aligned} \quad (24)$$

Note that, when we change y into $-y$ in (24), only $2\alpha s_0(p\alpha^2 + r\beta^2)C_1$ changes its sign. Hence

$$2\alpha s_0(p\alpha^2 + r\beta^2)C_1 = 0. \quad (25)$$

Because of $p \neq 0$, we assert that $p\alpha^2 + r\beta^2 \neq 0$. Thus

$$s_0 = 0.$$

Substituting it back into (21), by a similar discussion and by paying attention to $\phi'(-s) = C_1 - s\psi(s)$, we get $C_1s_{l0} = 0$. By assumption, we obtain

$$s_{l0} = 0.$$

That is, β is closed.

Case 2: $C_1 = 0$ but $r = -\frac{1}{2k}$, where k is a positive integer.

When $k = 1$, the metrics are just those of the form (8). In [12], Z. SHEN and G. C. YIDIRIM have proved that such metrics are projectively flat if and only if (4) and (5) hold. So, in the following, we will always assume that $k \geq 2$. In this case, we have

$$\begin{aligned} \phi(s) &= 1 + C_2s^2 + C_4s^4 + \cdots + C_{2k}s^{2k}, \\ \phi'(s) &= 2C_2s + 4C_4s^3 + \cdots + 2kC_{2k}s^{2k-1}, \\ \phi''(s) &= 2C_2 + 4 \cdot 3C_4s^2 + \cdots + 2k \cdot (2k-1)C_{2k}s^{2k-2}. \end{aligned}$$

Let

$$\psi(s) = 2C_2 + 4C_4s^2 + \cdots + 2kC_{2k}s^{2k-2}.$$

Then $\phi'(s) = s\psi(s)$ and (23) can be rewritten as

$$\begin{aligned} & \{2[(p+b^2) + (r-1)s^2](b_m\alpha^2 - y_m\beta)G_\alpha^m + r_{00}(b^2\alpha^2 - \beta^2)\}(p+rs^2)\phi'' \\ & + 2\beta s_0(p\alpha^2 + r\beta^2)\psi = 0. \end{aligned} \quad (26)$$

Multiplying (26) by α^{2k+2} yields

$$\begin{aligned} \{2[(p+b^2)\alpha^2 + (r-1)\beta^2](b_m\alpha^2 - y_m\beta)G_\alpha^m + r_{00}\alpha^2(b^2\alpha^2 - \beta^2)\}\eta \\ + 2\alpha^4\beta s_0\theta = 0, \end{aligned} \quad (27)$$

where

$$\begin{aligned} \eta &= 2C_2\alpha^{2k-2} + 4 \cdot 3C_4\alpha^{2k-4}\beta^2 + \cdots + 2k \cdot (2k-1)C_{2k}\beta^{2k-2}, \\ \theta &= 2C_2\alpha^{2k-2} + 4C_4\alpha^{2k-4}\beta^2 + \cdots + 2kC_{2k}\beta^{2k-2}. \end{aligned}$$

When $k \geq 2$, it is clear that θ and η are relatively prime polynomials of (y^i) . Hence $\alpha^4\beta s_0$ must be divisible by η . Because both of α^2 and β are irreducible polynomials of (y^i) , $\alpha^4\beta$ and η are obviously relatively prime polynomials of (y^i) . Then s_0 must be divisible by η , which implies that

$$s_0 = 0.$$

Substituting this back into (21) yields that

$$\begin{aligned} \{2[(p+b^2) + (r-1)s^2](a_{ml}\alpha^2 - y_my_l)G_\alpha^m + r_{00}(b_l\alpha^2 - \beta y_l)\}(p+rs^2)\phi'' \\ + 2[(p+b^2) + (r-1)s^2]\alpha^3 s_{l0}\phi' = 0, \end{aligned}$$

that is,

$$\begin{aligned} \{2[(p+b^2) + (r-1)s^2](a_{ml}\alpha^2 - y_my_l)G_\alpha^m + r_{00}(b_l\alpha^2 - \beta y_l)\}(p+rs^2)\phi'' \\ + 2[(p+b^2) + (r-1)s^2]\alpha^2\beta s_{l0}\psi = 0. \end{aligned} \quad (28)$$

Multiplying (28) by α^{2k+2} yields

$$\begin{aligned} \{2[(p+b^2)\alpha^2 + (r-1)\beta^2](a_{ml}\alpha^2 - y_my_l)G_\alpha^m \\ + r_{00}\alpha^2(b_l\alpha^2 - \beta y_l)\}(p\alpha^2 + r\beta^2)\eta \\ + 2[(p+b^2)\alpha^2 + (r-1)\beta^2]\alpha^4\beta s_{l0}\theta = 0. \end{aligned} \quad (29)$$

Hence $[(p+b^2)\alpha^2 + (r-1)\beta^2]s_{l0}$ must be divisible by η .

If $k = 2$, we know from (12) and (14) that

$$\begin{aligned} \eta &= \frac{1}{p^2} \left(p\alpha^2 - \frac{1}{12}\beta^2 \right), \\ (p+b^2)\alpha^2 + (r-1)\beta^2 &= 15 \left[\left(\frac{p+b^2}{15} \right) \alpha^2 - \frac{1}{12}\beta^2 \right]. \end{aligned}$$

It is clear that $(p+b^2)\alpha^2+(r-1)\beta^2$ and η are relatively prime polynomials of (y^i) . Hence s_{l0} must be divisible by η . This yields that $s_{l0} = 0$.

If $k \geq 3$, the degrees of $[(p+b^2)\alpha^2+(r-1)\beta^2]s_{l0}$ and η show that $s_{l0} = 0$.

Thus we may conclude that β is closed when $k \geq 2$. This completes the proof of Theorem 1.2.

From Theorem 1.2 and (12), (14) and the discussion in section 2, we have the following

Corollary 4.1. *Assume that*

$$\phi(s) = 1 + C_1 s + \frac{1}{2p} s^2 + \cdots + C_{2n} s^{2n} + \cdots$$

and $b \in (0, \sqrt{|p|})$ is sufficiently small, where

$$C_{2n+2} = (-1)^n \frac{(2n-1)!!}{(2n+2)!p^{n+1}} \prod_{i=1}^n (2ir+1), \quad n \geq 1.$$

If $C_1 \neq 0$ and $r \neq 1$ or $C_1 = 0$ but $r = -1/(2k)$ where k is any positive integer, then the (α, β) -metric $F = \alpha\phi(\beta/\alpha)$ is projectively flat if and only if (4) and (5) hold.

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