

On a family of connections in Finsler geometry

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Abstract. In this paper, we introduce a new family of linear torsion-free connections for Finsler metrics. This family of connections defines a Riemannian curvature tensor R and a non-Riemannian quantity P . We show that P contains all the non-Riemannian information, namely, $P = 0$ if and only if the Finsler metric is Riemannian. In fact, this family of connections makes a systematical analysis of connections that characterize Riemannian metrics.

1. Introduction

A Finsler space is a manifold M equipped with a family of smoothly varying Minkowski norms; one on each tangent space. Riemannian metrics are examples of Finsler norms that arise from an inner-product. After Einstein's formulation of general relativity, Riemannian geometry became fashionable and one of the connections, namely that due to Christoffel (Levi-Civita), came to the forefront. This connection is both torsion-free and metric-compatible. Likewise, connections in Finsler geometry can be prescribed on π^*TM and its tensor products. Examples of such connections were proposed by J. L. SYNGE (1925), J. H. TAYLOR (1925), L. BERWALD (1928) [9], but most important of all is ELIE CARTAN's connection (1934) [10]. There is also such a connection given by CHERN [11] in 1948. It is torsion-free but not completely compatible with the inner product (on π^*TM) defined by the g_{ij} 's. Incidentally, in the generic Finslerian setting, it is not possible to have a connection on π^*TM which is both torsion-free and

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compatible with the said inner product. The Chern connection, like many other connections, solves the equivalence problem for Finsler structures [7]. Namely, it gives rise to a list of criteria which decide when two such structures differ only by a change of coordinates. For a treatment of this connection using moving frames, see CHERN's article [12]. The Chern connection coincides with the Rund connection, as pointed out by ANASTASIEI [4]. ASANOV [5], MIRON and ANASTASIEI [17], BEJANCU [6], IKEDA [14], KOZMA [25] and TAMÁSSY [23], [24] have worked on connection theory. Recently Z. SHEN [20] has found a new torsion-free connection in Finsler geometry. He proved that $P = 0$ if and only if F is Riemannian.

In this paper we will give a new family of torsion-free linear connections in π^*TM , which are torsion-free and compatible with the Finsler structure in a certain sense, where as torsion-free connections, in our connection we define two curvature tensors R and P . The R -term is the so-called Riemannian curvature tensor which is a natural extension of the usual Riemannian curvature tensor of Riemannian metrics, while the P -term is a purely non-Riemannian quantity. The main result of this paper states that $P = 0$ if and only if the Finsler metric is Riemannian. This is the second torsion-free linear connection with such property ever discovered since SHEN's work [20]. We know there are already several well-known linear connections in Finsler geometry which are introduced from various points of view, in particular the connection constructed by CHERN and BAO [7], that shows its extraordinary usefulness in treating global problems in Finsler geometry. However, the non-Riemannian quantity of our connections as well as the Shen connection seems to capture all non-Riemannian information on the Finsler metric.

2. Preliminaries

Let M be an n -dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM := \cup_{x \in M} T_xM$ the tangent bundle of M . Each element of TM has the form (x, y) , where $x \in M$ and $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM \rightarrow M$ is given by $\pi(x, y) := x$.

The pull-back tangent bundle π^*TM is a vector bundle over TM_0 whose fiber π_v^*TM at $v \in TM_0$ is T_xM , where $\pi(v) = x$. Then

$$\pi^*TM = \{(x, y, v) \mid y \in T_xM_0, v \in T_xM\}.$$

Some authors prefer to define connections in the pull-back tangent bundle π^*TM . From a geometrical point of view the construction of these connections

on π^*TM seems to be simple because here the fibers are n -dimensional (i.e., $\pi^*(TM)_u = T_{\pi(u)}M$, $\forall u \in TM$) thus torsions and curvatures are obtained quickly from the structure equations. When the construction is done on $T(TM)$ many geometrical objects appear twice and one needs to split $T(TM)$ into the vertical and horizontal parts where the later is called horizontal distribution or non-linear connection. Nevertheless we do not need to split π^*TM . Indeed, the connection on $\pi^*(TM)$ is the most natural connection for Physicists.¹ In order to define curvatures, it is more convenient to consider the pull-back tangent bundle than the tangent bundle, because our geometric quantities depend on directions.

For the sake of simplicity, we denote by $\{\partial_i|_v := (v, \frac{\partial}{\partial x^i})|_x\}_{i=1}^n$ the natural basis for π_v^*TM . In Finsler geometry, we study connections and curvatures in (π^*TM, g) , rather than in (TM, F) . The pull-back tangent bundle π^*TM is a very special tangent bundle.

Throughout this paper, we use the *Einstein summation convention* for expressions with indices.²

Finsler structure

A (globally defined) Finsler structure on a manifold M is a function

$$F : TM \rightarrow [0, \infty)$$

with the following properties:

- (i) F is a differentiable function on the manifold TM_0 and F is continuous on the null section of the projection $\pi : TM \rightarrow M$.
- (ii) F is a positive function on TM_0 .
- (iii) F is positively 1-homogeneous on the fibers of the tangent bundle TM .
- (iv) The Hessian of F^2 with elements

$$(g_{ij}) := \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right)$$

is positive definite on TM_0 .

Given a manifold M and a Finsler structure F on M , the pair (M, F) is called a Finsler manifold. F is called Riemannian if $g_{ij}(x, y)$ are independent of $y \neq 0$.

Every Finsler metric on a manifold defines a length structure on oriented piecewise C^∞ curves. Let C be an oriented piecewise C^∞ curve from p to q in

¹For more details on the structure of $\pi^*(TM)$ see [19] and [8].

²That is, if an index appears twice, namely as a subscript as well as a superscript, then that term is assumed to be summed over all values of that index.

a Finsler manifold (M, F) . Let $C : [a, b] \rightarrow M$ be a parameterization of C with $C(a) = p$ and $C(b) = q$. Then the length of C is defined by

$$\mathbf{L}_F(C) := \int_a^b F\left(C(t), \frac{dC(t)}{dt}\right) dt. \quad (*)$$

The homogeneity of F implies that $\mathbf{L}_F(C)$ is independent of the choice of the particular parameterization of C .

The Finsler structure F defines a fundamental tensor $g : \pi^*TM \otimes \pi^*TM \rightarrow [0, \infty)$ by the formula $g(\partial_i|_v, \partial_j|_v) = g_{ij}(x, y)$, where $v = y^i \frac{\partial}{\partial x^i}|_x$. Let

$$g_{ij}(x, y) := FF_{y^i y^j} + F_{y^i} F_{y^j},$$

where $F_{y^i} = \frac{\partial F}{\partial y^i}$. Then (π^*TM, g) becomes a Riemannian vector bundle over TM_0 . Let

$$A_{ijk}(x, y) = \frac{1}{2} F(x, y) \frac{\partial g_{ij}}{\partial y^k}(x, y).$$

Clearly, A_{ijk} is symmetric with respect to i, j, k . The Cartan³ tensor $A : \pi^*TM \otimes \pi^*TM \otimes \pi^*TM \rightarrow R$ is defined by

$$A(\partial_i|_v, \partial_j|_v, \partial_k|_v) = A_{ijk}(x, y),$$

where $v = y^i \frac{\partial}{\partial x^i}|_x$. The homogeneity condition (iii) holds in particular for positive λ . Therefore, by Euler's theorem we see that

$$y^i \frac{\partial g_{ij}}{\partial y^k}(x, y) = y^j \frac{\partial g_{ij}}{\partial y^k}(x, y) = y^k \frac{\partial g_{ij}}{\partial y^k}(x, y) = 0.$$

We recall that the canonical section ℓ is defined by

$$\ell = \ell(x, y) = \frac{y^i}{F(x, y)} \frac{\partial}{\partial x^i} = \frac{y^i}{F} \frac{\partial}{\partial x^i} := \ell^i \frac{\partial}{\partial x^i}.$$

Put $\ell_i := g_{ij} \ell^j = F_{y^i}$. Thus the canonical section ℓ satisfies

$$g(\ell, \ell) = g_{ij} \frac{y^i}{F} \frac{y^j}{F} = 1$$

and

$$\ell^i A_{ijk} = \ell^j A_{ijk} = \ell^k A_{ijk} = 0.$$

Thus $A(X, Y, \ell) = 0$.

³In some literature $C_{ijk} = \frac{A_{ijk}}{F}$ is called Cartan tensor. Riemannian manifolds are characterized by $A \equiv 0$.

3. Existence and uniqueness of a new Finsler connection on π^*TM

In this section we introduce a new Finsler connection which is torsion-free and almost compatible with Finsler metric.

Bundle Maps μ and ρ .

The bundle map $\rho : T(TM_0) \rightarrow \pi^*TM$ is defined by

$$\rho\left(\frac{\partial}{\partial x^i}\right) = \partial_i, \quad \rho\left(\frac{\partial}{\partial y^i}\right) = 0. \quad (1)$$

Put $VTM := \ker \rho = \text{span}\left\{\frac{\partial}{\partial y^i}\right\}_{i=1}^n$. VTM is an n -dimensional subbundle of $T(TM_0)$, whose fiber V_vTM at v is just the tangent space $T_v(T_xM) \subset T_v(TM_0)$. VTM is called the *vertical tangent bundle* of TM_0 .

The bundle map $\mu : T(TM_0) \rightarrow \pi^*TM$ is defined by $\mu\left(\frac{\partial}{\partial y^i}\right) = \partial_i$.

Put $HTM := \text{Ker } \mu$. HTM is called the *horizontal tangent bundle* of TM_0 .

We have the direct decomposition $T(TM_0) = HTM \oplus VTM$. Tangent vectors in HTM are called *horizontal* and vectors in VTM are called *vertical*. We summarize: $\text{Ker } \rho = VTM$, $\text{Ker } \mu = HTM$, ρ restricted to HTM is an isomorphism onto π^*TM , and μ restricted to VTM is the bundle isomorphism onto π^*TM .

Definition 3.1. A tensor $T : \pi^*TM \otimes \pi^*TM \otimes \pi^*TM \rightarrow R$ is called compatible if it has the following properties:

- (i) $T(X, Y, Z)$ is symmetric with respect to X, Y, Z .
- (ii) $T(X, Y, \ell) = 0$.
- (iii) T is homogeneous, i.e., $T_{ijk}(x, ty) = T_{ijk}(x, y)$, $\forall t \in R$, where $T_{ijk}(x, y) = T(\partial_i, \partial_j, \partial_k)$.

Let (M, F) be a Finsler n -manifold. Let g, A and T denote the fundamental tensor, the Cartan tensor and a compatible tensor in π^*TM , respectively.

Definition 3.2. Let D be a Finsler connection on M . Then we say that

- (i) D is *torsion-free*, if

$$\mathbf{T}_D(\hat{X}, \hat{Y}) := D_{\hat{X}}\rho(\hat{Y}) - D_{\hat{Y}}\rho(\hat{X}) - \rho([\hat{X}, \hat{Y}]) = 0, \quad \forall \hat{X}, \hat{Y} \in C^\infty(T(TM_0)). \quad (2)$$

- (ii) D is *almost compatible* with the Finsler structure in the following sense: for all $X, Y \in C^\infty(\pi^*TM)$ and $\hat{Z} \in T_v(TM_0)$,

$$\begin{aligned} (D_{\hat{Z}}g)(X, Y) &:= \hat{Z}g(X, Y) - g(D_{\hat{Z}}X, Y) - g(X, D_{\hat{Z}}Y) \\ &= A(\rho(\hat{Z}), X, Y) - 2T(\rho(\hat{Z}), X, Y) + 2F^{-1}A(\mu(\hat{Z}), X, Y), \end{aligned} \quad (3)$$

where $\rho(\hat{Z}) := (v, \pi_*(\hat{Z}))$, $\mu(\hat{Z}) := D_{\hat{Z}}F\ell$, and T is the given compatible tensor.

Theorem 3.1. *Let (M, F) be a Finsler n -manifold and T an arbitrary compatible tensor in π^*TM . Then there is a unique linear torsion-free connection D in π^*TM , which is almost compatible with the above Finsler structure.⁴*

We define the *Landsberg* tensor $\dot{A} = \pi^*TM \otimes \pi^*TM \otimes \pi^*TM \rightarrow R$ by

$$\dot{A}(X, Y, Z) := \bar{\ell}A(X, Y, Z) - A(D_{\bar{\ell}}X, Y, Z) - A(X, D_{\bar{\ell}}Y, Z) - A(X, Y, D_{\bar{\ell}}Z).$$

It is obvious that

$$\ell^i \dot{A}_{ijk} = \ell^j \dot{A}_{ijk} = \ell^k \dot{A}_{ijk} = 0.$$

Then $\dot{A}(X, Y, \ell) = 0$. It is easy to check that $T = \alpha A + \beta \dot{A}$ is a compatible tensor $\forall \alpha, \beta \in R$.

4. Nonlinear connections and Finsler connections

Let M be a real n -dimensional connected manifold of C^∞ -class and (TM, π, M) its tangent bundle with the zero section removed. Every local chart $(U, \varphi = (x^i))$ on M induces a local chart $(\varphi^{-1}(U), \varphi = (x^i, y^i))$ on TM . The kernel of the linear map $\pi_* : TTM \rightarrow TM$ is called the *vertical distribution* and is denoted by VTM . For every $u \in TM$, $\text{Ker } \pi_{*,u} = V_u TM$ is spanned by $\{\frac{\partial}{\partial y^i}|_u\}$. By a *nonlinear connection* on TM we mean a regular n -dimensional distribution $H : u \in TM \rightarrow H_u TM$ which is supplementary to the vertical distribution i.e.

$$T_u(TM) = H_u TM \oplus V_u TM, \quad \forall u \in TM.$$

A basis for $T_u TM$ adapted to the above direct sum is $(\frac{\delta}{\delta x^i}|_u, \frac{\partial}{\partial y^i}|_u)$, where N_j^i are the coefficients of the nonlinear connection and $\frac{\delta}{\delta x^i}|_u = \frac{\partial}{\partial x^i} - N_j^i(u) \frac{\partial}{\partial y^j}|_u$. The dual basis of $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ is given by $(dx^i, \delta y^i + N_j^i dx^j)$. These are the *Berwald bases*.

Let M be a real n -dimensional C^∞ manifold and $VTM = \cup_{v \in TM} V_v TM$ its vertical vector bundle. Suppose that $HTM = \cup_{v \in TM} H_v TM$ is a nonlinear connection on TM and ∇ a linear connection on VTM ; then the pair (HTM, ∇) is called a *Finsler connection* on the manifold M .

⁴In the sequel we will refer to this connection as “*New connection*”.

PROOF OF THEOREM 3.1. In a standard local coordinate system (x^i, y^i) in TM_0 , we write

$$D_{\frac{\partial}{\partial x^i}} \partial_j = \Gamma_{ij}^k \partial_k, \quad D_{\frac{\partial}{\partial y^i}} \partial_j = F_{ij}^k \partial_k.$$

Clearly, (2) and (3) are equivalent to the following:

$$\Gamma_{ij}^k = \Gamma_{ji}^k \quad (4)$$

$$F_{ij}^k = 0 \quad (5)$$

$$\frac{\partial}{\partial x^k} (g_{ij}) = -\Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{li} + 2A_{ijk} - 2T_{ijk} + 2A_{ijl} \Gamma_{km}^l \ell^m \quad (6)$$

$$\frac{\partial}{\partial y^k} (g_{ij}) = -F_{kj}^l g_{li} + F_{ki}^l g_{jl} + 2C_{ijk} - 2T_{ijk} F_{mk}^l \ell^m + 2A_{ijk} F_{mk}^l \ell^m \quad (7)$$

where $g_{ij} = g_{ij}(x, y)$, $A_{ijk} = A_{ijk}(x, y)$ and $T_{ijk} = T_{ijk}(x, y)$ is the given compatible tensor. Notice that (5) and (7) are just the definition of A_{ijk} . We must compute Γ_{ij}^k from (4) and (6). Then permuting i, j, k in (6), and using (4), one obtains

$$\Gamma_{ij}^k = \gamma_{ij}^k - A_{ij}^k + T_{ij}^k + g^{kl} \{A_{ijm} \Gamma_{lb}^m - A_{jlm} \Gamma_{ib}^m - A_{lim} \Gamma_{jb}^m\} \ell^b, \quad (8)$$

where we have put

$$\gamma_{ij}^k = \frac{1}{2} g^{kl} \left\{ \frac{\partial}{\partial x^i} (g_{jl}) + \frac{\partial}{\partial x^j} (g_{il}) - \frac{\partial}{\partial x^l} (g_{ij}) \right\}$$

and $A_{ij}^k = g^{kl} A_{ijl}$. Multiplying (8) by ℓ^i , yields

$$\Gamma_{ib}^k \ell^b = \gamma_{ib}^k \ell^b - A_{im}^k \Gamma_{lb}^m \ell^l \ell^b. \quad (9)$$

Multiplying (9) by ℓ^i gives

$$\Gamma_{ab}^k \ell^a \ell^b = \gamma_{ab}^k \ell^a \ell^b. \quad (10)$$

By substituting (10) into (9) one obtains

$$\Gamma_{ib}^k \ell^b = \gamma_{ib}^k \ell^b - A_{im}^k \gamma_{ab}^m \ell^a \ell^b. \quad (11)$$

By substituting (11) into (8) one obtains

$$\begin{aligned} \Gamma_{ij}^k &= \gamma_{ij}^k - A_{ij}^k + T_{ij}^k + g^{kl} \{A_{ijm} \gamma_{lb}^m - A_{jlm} \gamma_{ib}^m - A_{lim} \gamma_{jb}^m\} \ell^b \\ &\quad + \{A_{jms}^k A_{is}^m + A_{ims}^k A_{js}^m - A_{sms}^k A_{ij}^m\} \gamma_{ab}^s \ell^b \ell^a. \end{aligned} \quad (12)$$

This proves the uniqueness of D . The set $\{\Gamma_{ij}^k, F_{ij}^k = 0\}$, where $\{\Gamma_{ij}^k\}$ are given by (12), satisfy a linear connection D with properties (2) and (3). $\square$

The bundle map $\mu : T(TM_0) \rightarrow \pi^*TM$ defined in Section 3 can be expressed in the following form:

$$\mu \left(\frac{\partial}{\partial x^i} \right) = N_i^k \partial_k, \quad \mu \left(\frac{\partial}{\partial y^i} \right) = \partial_i, \quad (13)$$

where

$$N_i^k = y^j \Gamma_{ij}^k = y^j \gamma_{ij}^k - \frac{1}{F} g^{ks} A_{sil} \gamma_{ab}^l y^a y^b.$$

The above N_j^i are known in the literature as the *nonlinear connection coefficients* on TM_0 . The Berwald connection is most directly related to the nonlinear connection N_j^i , and is most amenable to the study of path geometry.

Defining $G^i := \gamma_{jk}^i y^j y^k$, one can prove that $\frac{\partial G^i}{\partial y^j} = N_j^i$. Finslerian geodesics are curves in M which obey the equation $\dot{y}^i + G^i = 0$. Thus, if the geodesic equation is once known, the nonlinear connection N_j^i can be computed without having to calculate first the Cartan tensor A_{ijk} and the formal Christoffel symbols γ_{ijk} . The formula (12) in terms of the coefficients N_j^i is given by

$$\Gamma_{ij}^k = \gamma_{ij}^k - A_{ij}^k + T_{ij}^k - g^{kl} \{ N_j^s C_{ski} + N_i^s C_{sjk} - N_k^s C_{sij} \}. \quad (14)$$

It is obvious that

$$\Gamma_{ij}^k \ell^i = \Gamma_{ji}^k \ell^i = \frac{N_j^k}{F}. \quad (15)$$

Let us express the Christoffel coefficients of the Berwald, Chern and Shen connections and of the New connection, by ${}^b\Gamma_{ij}^k$, ${}^c\Gamma_{ij}^k$, ${}^s\Gamma_{ij}^k$ and Γ_{ij}^k respectively; then we see that:

$$\Gamma_{ij}^k := {}^b\Gamma_{ij}^k - A_{ij}^k - \dot{A}_{ij}^k + T_{ij}^k, \quad (16)$$

$$\Gamma_{ij}^k := {}^c\Gamma_{ij}^k - A_{ij}^k + T_{ij}^k, \quad (17)$$

$$\Gamma_{ij}^k := {}^s\Gamma_{ij}^k + T_{ij}^k. \quad (18)$$

It is clear that in a locally Minkowski space, $\Gamma_{ij}^k = -A_{ij}^k + T_{ij}^k$ and $N_j^i = 0$. The reader can consult [24] and [15].

5. Curvatures of the New connection

In this section we study the curvature tensor of the “new Finsler connection” introduced in the above section, which is torsion-free and almost compatible with the Finsler metric. As a torsion-free connection, it defines two curvatures R and P . The R -term is the so-called Riemannian curvature tensor, which is a

natural extension of the usual Riemannian curvature tensor of a Riemannian metric, while the P -term is a purely non-Riemannian quantity. We prove also that the hv -curvature P of this connection vanishes if and only if the Finsler structure is Riemannian. The curvature tensor Ω of D is defined by

$$\Omega(\hat{X}, \hat{Y})Z = D_{\hat{X}}D_{\hat{Y}}Z - D_{\hat{Y}}D_{\hat{X}}Z - D_{[\hat{X}, \hat{Y}]}Z, \quad (19)$$

where $\hat{X}, \hat{Y} \in C^\infty(T(TM_0))$ and $Z \in C^\infty(\pi^*TM)$.

Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame field (with respect to g) for the vector bundle π^*TM , such that $g(e_i, e_n) = 0$, $i = 1, \dots, n-1$ and

$$e_n := \frac{y}{F} = \frac{y^i}{F(x, y)} \frac{\partial}{\partial x^i} = \ell.$$

Let $\{\omega^i\}_{i=1}^n$ be its dual co-frame field. These are local sections of the dual bundle π^*TM . One readily finds that

$$\omega^n := \frac{\partial F}{\partial y^i} dx^i = \ell_i dx^i = \omega,$$

which is the *Hilbert form*. It is obvious that

$$\omega(\ell) = 1.$$

We observe that for a curve $x^i = x^i(t)$ with $y^i = \frac{dx^i}{dt}$, Euler's theorem allows us to rewrite the integral (*) as $\int_a^b \omega^n$.

Put

$$\rho = \omega^i \otimes e_i, \quad De_i = \omega_i^j \otimes e_j, \quad \Omega e_i = 2\Omega_i^j \otimes e_j.$$

$\{\Omega_i^j\}$ and $\{\omega_i^j\}$ are called the *curvature forms* and *connection forms* of D with respect to $\{e_i\}$. We have $\mu := DF\ell = F\{\omega_n^i + d(\log F)\delta_n^i\} \otimes e_i$. Put $\omega^{n+i} := \omega_n^i + d(\log F)\delta^i n$. It is easy to see that $\{\omega^i, \omega^{n+i}\}_{i=1}^n$ is a local basis for $T^*(TM_0)$. By definition

$$\rho = \omega^i \otimes e_i, \quad \mu = F\omega^{n+i} \otimes e_i.$$

Use the above formula for Theorem 3.1, then it will re-express the structure equation of the New connection

$$d\omega^i = \omega^j \wedge \omega_j^i \quad (20)$$

$$dg_{ij} = g_{kj}\omega_i^k + g_{ik}\omega_j^k + 2A_{ijk}\omega^k - 2T_{ijk}\omega^k + 2A_{ijk}\omega^{n+k}. \quad (21)$$

Define $g_{ij.k}$ and $g_{ij|k}$ by

$$dg_{ij} - g_{kj}\omega_i^k - g_{ik}\omega_j^k = g_{ij|k}\omega^k + g_{ij.k}\omega^{n+k}, \quad (22)$$

where $g_{ij.k}$ and $g_{ij|k}$ are the vertical and horizontal covariant derivative respectively of g_{ij} with respect to the New connection. This gives

$$g_{ij|k} = 2(A_{ijk} - T_{ijk}), \quad (23)$$

$$g_{ij.k} = 2A_{ijk}. \quad (24)$$

It can be shown that $\delta_{j|s}^i = 0$ and $\delta_{j.s}^i = 0$, thus $(g^{ij}g_{jk})_{|s} = 0$ and $(g^{ij}g_{jk})_{.s} = 0$. So

$$g_{|s}^{ij} = 2(T_s^{ij} - A_s^{ij}) \quad (25)$$

and

$$g_{.s}^{ij} = -2A_s^{ij}. \quad (26)$$

Moreover, torsion freeness is equivalent to the absence of dy^k in $\{\omega_j^i\}$, namely

$$\omega_j^i = \Gamma_{jk}^i(x, y)dx^k. \quad (27)$$

(19) is equivalent to

$$d\omega_i^j - \omega_i^k \wedge \omega_k^j = \Omega_i^j. \quad (28)$$

Since the Ω_j^i are 2-forms on the manifold TM_0 , they can be expanded as

$$\Omega_i^j = \frac{1}{2}R_i^j{}_{kl}\omega^k \wedge \omega^l + P_i^j{}_{kl}\omega^k \wedge \omega^{n+l} + \frac{1}{2}Q_i^j{}_{kl}\omega^{n+k} \wedge \omega^{n+l}. \quad (29)$$

The objects R , P and Q are the hh -, hv - and vv -curvature tensors respectively of the connection D . Let $\{\bar{e}_i, \dot{e}_i\}_{i=1}^n$ be a local basis for $T(TM_0)$, which is dual to $\{\omega^i, \omega^{n+i}\}_{i=1}^n$, i.e., $\bar{e}_i \in HTM$, $\dot{e}_i \in VTM$ such that $\rho(\bar{e}_i) = e_i$, $\mu(\dot{e}_i) = Fe_i$. We have put $R(\bar{e}_k, \bar{e}_l)e_i = R_i^j{}_{kl}e_j$, $P(\bar{e}_k, \dot{e}_l)e_i = P_i^j{}_{kl}e_j$ and $Q(\dot{e}_k, \dot{e}_l)e_i = Q_i^j{}_{kl}e_j$. Since the New connection is torsion-free we have ([21] and [22]) that

$$Q = 0.$$

The first Bianchi identities for R are

$$R_i^j{}_{kl} + R_k^j{}_{li} + R_l^j{}_{ik} = 0 \quad (30)$$

and

$$P_i^j{}_{kl} = P_k^j{}_{il}. \quad (31)$$

Exterior differentiation of (28) gives the Second Bianchi identities:

$$d\Omega_i^j - \omega_i^k \wedge \Omega_k^j + \omega_k^j \wedge \Omega_i^k = 0. \quad (32)$$

We decompose the covariant derivative of the Cartan tensor on TM

$$dA_{ijk} - A_{ljk}\omega_i^l - A_{ilk}\omega_j^l - A_{ijl}\omega_k^l = A_{ijk|l}\omega^l + A_{ijk.l}\omega^{n+l}, \quad (33)$$

and so for $\dot{A}_{ijk}$ we have

$$d\dot{A}_{ijk} - \dot{A}_{ljk}\omega_i^l - \dot{A}_{ilk}\omega_j^l - \dot{A}_{ijl}\omega_k^l = \dot{A}_{ijk|l}\omega^l + \dot{A}_{ijk.l}\omega^{n+l}. \quad (34)$$

Clearly, in the above relations the tensors $A_{ijk|l}$, $A_{ijk.l}$, $\dot{A}_{ijk|l}$ and $\dot{A}_{ijk.l}$ are symmetric with respect to the indices i, j, k .

Put $\dot{A}_{ijk} = \dot{A}(e_i, e_j, e_k)$, $\dot{A}_{ij}^k = g^{kl}\dot{A}_{ijl}$. The quantity $A_{ijk|n}$ plays a somewhat privileged role in Finsler geometry so much that it deserves perhaps a special notation:

$$A_{ijk|n} = \dot{A}_{ijk}. \quad (35)$$

It follows from (33) and (34) that

$$A_{njk|l} = 0, \quad \text{and} \quad A_{njk.l} = -A_{jkl}. \quad (36)$$

$$\dot{A}_{njk|l} = 0, \quad \text{and} \quad \dot{A}_{njk.l} = -\dot{A}_{jkl}. \quad (37)$$

Theorem 5.1. *Let (M, F) be a Finsler manifold and D be a torsion-free connection defined in theorem (1) with the condition $T_{ijk} := k_1 A_{jk}^{(1)i} + \dots + k_m A_{jk}^{(m)i}$, where $A_{jk}^{(m)} = A_{ijk} \overbrace{|n|n \cdots |n|}^{m \text{ times}}$, $m \in N$. Then F is Riemannian if and only if $P = 0$.*

PROOF. Let (M, F) be a Finsler manifold. Differentiating (21) and using (20), (21), (33), (36) and (37) leads to

$$\begin{aligned} g_{kj}\Omega_i^k + g_{ik}\Omega_j^k &= -2A_{ijk}\Omega_n^k - 2A_{ijk|l}\omega^k \wedge \omega^l + 2A_{ijk.l}\omega^{n+k} \wedge \omega^{n+l} \\ &\quad - 2\{A_{ijk.l} - A_{ijk|l}\}\omega^k \wedge \omega^{n+l} \\ &\quad + k_1(A_{ijk|l}^{(1)}\omega^l + A_{ijk.l}^{(1)}\omega^{n+l}) \wedge \omega^k + \dots \\ &\quad + k_m(A_{ijk|l}^{(m)}\omega^l + A_{ijk.l}^{(m)}\omega^{n+l}) \wedge \omega^k. \end{aligned} \quad (38)$$

Using (29), we get

$$\begin{aligned} R_{ijkl} + R_{jikl} &= 2k_1\{A_{ijl|k}^{(1)} - A_{ijk|l}^{(1)}\} + \dots + 2k_m\{A_{ijl|k}^{(m)} - A_{ijk|l}^{(m)}\} \\ &\quad - 2A_{ijs}R_n{}^s{}_{kl} + 2\{A_{ijk|l} - A_{ijl|k}\}, \end{aligned} \quad (39)$$

$$\begin{aligned} P_{ijkl} + P_{jikl} &= -2\{k_1 A_{ijk.l}^{(1)} + \dots + k_m A_{ijk.l}^{(m)}\} \\ &\quad + 2\{A_{ijk.l} - A_{ijl|k}\} - 2A_{ijs}P_n{}^s{}_{kl}, \end{aligned} \quad (40)$$

$$A_{ijk.l} = A_{ijl.k}. \quad (41)$$

Permuting i, j, k in (40) yields

$$\begin{aligned} P_{ijkl} = & -\{k_1 A_{ijk.l}^{(1)} + \cdots + k_m A_{ijk.l}^{(m)}\} + A_{ijk.l} - (A_{ijl|k} + A_{jkl|i} - A_{kil|j}) \\ & + A_{kis} P_n^s{}_{jl} - A_{jks} P_n^s{}_{il} - A_{ijs} P_n^s{}_{kl} \end{aligned} \quad (42)$$

and

$$P_{n j k l} = \{k_1 A_{j k l}^{(1)} + \cdots + k_m A_{j k l}^{(m)}\} - A_{j k l} - \dot{A}_{j k l}, \quad (43)$$

because of $P_{n j n l} = 0$.

Now if F is Riemannian, then from (42) and (43) we conclude that $P = 0$.

Conversely let $P = 0$. It follows from (43) that

$$\text{By (42) one has} \quad k_1 A_{j k l}^{(1)} + \cdots + k_m A_{j k l}^{(m)} = \dot{A}_{j k l} + A_{j k l}. \quad (44)$$

$$k_1 A_{ijk.l}^{(1)} + \cdots + k_m A_{ijk.l}^{(m)} = A_{ijk.l} + A_{kil|j} - A_{ijl|k} - A_{jkl|i}.$$

Permuting i, j, k in the above identity leads to

$$k_1 A_{ijk.l}^{(1)} + \cdots + k_m A_{ijk.l}^{(m)} = A_{ijk.l} + A_{jkl|i} - A_{kil|j} - A_{ijl|k},$$

and then

$$A_{ijl|k} = A_{jkl|i}.$$

Letting $k = n$, we can conclude

$$\dot{A}_{ijk} = 0. \quad (45)$$

It is obvious that

$$A_{ijk}^{(m)} = 0, \quad \forall m \in N. \quad (46)$$

Therefore we conclude that $A_{ijk} = 0$, and thus F is Riemannian. $\square$

6. Complete Finsler manifolds

Let $\bar{\ell}$ denote the unique vector field in HTM such that $\rho(\bar{\ell}) = \ell$. We call $\bar{\ell}$ the *geodesic field* on TM_0 , because it determines all geodesics and it is called a *spray*.

Let $c : [a, b] \rightarrow (M, F)$ be a unit speed C^∞ curve. The canonical lift of c to TM_0 is defined by $\hat{c} := \frac{dc}{dt} \in TM_0$. It is easy to see that

$$\rho\left(\frac{d\hat{c}}{dt}\right) = \ell_{\hat{c}}.$$

The curve c is called a *geodesic* if its canonical lift $\hat{c}$ satisfies

$$\frac{d\hat{c}}{dt} = \bar{\ell}_{\hat{c}},$$

where $\bar{\ell}$ is the *geodesic field* on TM_0 , i.e., $\ell \in HTM$, $\rho(\bar{\ell}) = \ell$.

Let $I_x M = \{v \in T_x M, F(v) = 1\}$ and $IM = \bigcup_{p \in M} I_x M$. The $I_x M$ is called *indicatrix*, and it is a compact set. We can show that the projection of the integral curve $\varphi(t)$ of $\bar{\ell}$ with $\varphi(0) \in IM$ is a unit speed geodesics c , whose canonical lift is $\hat{c}(t) = \varphi(t)$. A Finsler manifold (M, F) is called *complete* if any unit speed geodesic $c : [a, b] \rightarrow M$ can be extended to a geodesic defined on R . This is equivalent to requiring that the geodesic field $\bar{\ell}$ restricted to IM is complete.

If we put $k_1 = k_3 = \dots = k_m = 0$ and $k_2 = -1$, then we have a connection and we obtain

$$A + \dot{A} + \ddot{A} = 0,$$

where $\ddot{A} := A^{(2)}$ is defined in Theorem 5.1.

Let (M, F) be a Finsler manifold and c a unit speed geodesic in M . A section $X = X(t)$ of $\pi^* TM$ along $\hat{c}$ is said to be parallel if $D_{\frac{d\hat{c}}{dt}} X = 0$. For $v \in TM_0$ we define $\|A\|_v := \sup A(X, Y, Z)$. Then we put $\|A\| = \sup_{v \in IM} \|A\|_v$ where the supremum is taken over all unit vectors of $\pi_v^* TM$.

Theorem 6.1. *Let (M, F) be complete with bounded $\|A\|$. If $k_1 = k_3 = \dots = k_m = 0$ and $k_2 = -1$, then F is Riemannian, whenever*

$$A + \dot{A} + \ddot{A} = 0. \tag{47}$$

PROOF. If F is Riemannian, then (47) is true. Conversely, let the above condition be true. Fix any $X, Y, Z \in \pi^* TM$ at $v \in I_x M$. Let $c : M \rightarrow R$ be the unit speed geodesic with $\frac{dc}{dt}(0) = v$. Let $X(t)$, $Y(t)$ and $Z(t)$ denote the parallel sections along $\hat{c}$ with $X(0) = X$, $Y(0) = Y$, $Z(0) = Z$. Putting $A(t) = A(X(t), Y(t), Z(t))$, $\dot{A}(t) = \dot{A}(X(t), Y(t), Z(t))$, and $\ddot{A}(t) = \ddot{A}(X(t), Y(t), Z(t))$, one has

$$\frac{dA}{dt} = \dot{A} \quad \text{and} \quad \frac{d\dot{A}}{dt} = \ddot{A}. \tag{48}$$

Therefore by (47) and (48) we have

$$\frac{d^2 A}{dt^2} + \frac{dA}{dt} + A = 0. \quad (49)$$

Now

$$A(t) = e^{\frac{-t}{2}} \left(c_1 \cos \frac{\sqrt{3}}{2} t + c_2 \sin \frac{\sqrt{3}}{2} t \right).$$

Using $\|A\| < \infty$ and letting $t \rightarrow -\infty$, we get $c_1 = c_2 = 0$, and $A(0) = A(X, Y, Z) = 0$, which completes the proof. $\square$

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