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On homomorphisms of an abelian group into the group of invertible formal power series

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Abstract. We study solutions of the translation equation in rings of formal power series $\mathbb{K}[\![X]\!]$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ (the so called one-parameter groups or flows), and even, more generally, homomorphisms Θ from an abelian group $(G, +)$ into the group $(\Gamma, \circ)$ of invertible power series in $\mathbb{K}[\![X]\!]$. This problem can equivalently be formulated as the question of finding homomorphisms Φ from $(G, +)$ into the differential group $L^1_\infty = (Z_\infty, \cdot)$ describing the chain rules of higher order of C^∞-functions with fixed point 0.

We prove the general form of the homomorphisms $\Theta : G \to \Gamma$, $\Theta(t) = \sum_{k=1}^{\infty} c_k(t) X^k$ and $\Phi : G \to Z_\infty$, $\Phi = (f_n)_{n\geq 1}$, for which c_1 and f_1 take infinitely many values (Theorems 5 and 6). These representations use sequences $(P_n)_{n\geq 2}$ of universal polynomials in c_1, and $(v_n)_{n\geq 2}$ of universal polynomials in f_1, and some sequences of parameters, which determine the individual homomorphism. We describe the connection between these forms of the homomorphisms. These results are deduced from the special case $|f_1| \neq 1$ (Theorem 3) and the case when c_1 is a regular function (Theorem 4).

1. Introduction

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be the field of real or complex numbers. The aim of this paper is to find a general form of homomorphisms $\Theta : G \to \Gamma$, $\Theta(t) = \sum_{k=1}^{\infty} c_k(t) X^k$, from an abelian group $(G, +)$ into the group $(\Gamma, \circ)$ of invertible formal power series, under the condition that c_1 takes infinitely many values (the case $c_1 = 1$ has been considered in our previous paper [6]). Since the group $(\Gamma, \circ)$ and the differential group $L^1_\infty = (Z_\infty, \cdot)$ are in a natural way isomorphic (see [6, Theorem 2]), the above mentioned problem is the same as the problem to find the

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general form of the homomorphisms $\Phi = (f_n)_{n\geq 1} : G \to Z_\infty$ with the condition that f_1 takes infinitely many values. Then c_1 and f_1 are homomorphisms from $(G,+)$ to $(\mathbb{K}\setminus\{0\},\cdot)$ (see (8) and (15)), or in the terminology of the theory of functional equations, generalized exponential functions on G. Our approach is similar to that one used in [6] to give the explicit form of homomorphisms Φ from an abelian group $(G,+)$ to L^1_∞ in the case when $f_1 = 1$. But it differs from the approach in [6] in some important details, and hence the results are also different.

In Section 4 (Theorem 3) we show that in each homomorphism $\Phi = (f_n)_{n\geq 1} : G \to Z_\infty$ with $|f_1| \neq 1$, the component functions f_n can necessarily be represented by universal polynomials in f_1 and a sequence $(p_n)_{n\geq 2}$ of parameters in $\mathbb{K}$. Using this special case $|f_1| \neq 1$ we will show the same representation of the components f_n of a homomorphism $\Phi : G \to Z_\infty$, if we only assume that the exponential function f_1 takes infinitely many values (Theorem 5). Next, in Section 5 (Theorem 4, Corollary 2), we construct the general regular solution $(F(t,X))_{t\in\mathbb{K}}$, $F(t,X) = \sum_{k=1}^\infty c_k(t)X^k$ (the coefficient functions c_k are C^∞, if $\mathbb{K} = \mathbb{R}$, and entire, if $\mathbb{K} = \mathbb{C}$) of the translation equation in Γ, under the hypothesis that c_1 is a nontrivial exponential function. We give a representation of the regular coefficient functions c_k by means of a sequence of universal polynomials in the arbitrary regular nontrivial exponential function $c_1(t) = e^{\lambda_1 t}$ ($\lambda_1 \neq 0$) and an arbitrary sequence $(\lambda_k)_{k\geq 2}$ of parameters. Using the results of Sections 4 and 5, we show in Section 6 (Theorem 5) that the form of homomorphisms $\Phi = (f_n)_{n\geq 1} : G \to Z_\infty$, where f_1 takes infinitely many values, is also *sufficient*. The same is also true for solutions of translation equation (cf. Theorem 6). In Corollary 4 we give the general form of homomorphisms $\Phi_s = (f_n)_{1\leq n\leq s} : G \to Z_s$ for $1 \leq s < \infty$. This form is the same as for $s = \infty$, there are no new arbitrary functions involved, in contrary to the case $f_1{=}1$ (see [7, Theorem 1]). Therefore, in a Section 7 we prove (Theorem 7) that each homomorphism $\Phi_s : G \to Z_s$ where f_1 takes infinitely many values can be extended to a homomorphism from G to Z_{s+l} for every l with $1 \leq l \leq \infty$.

The problem for homomorphisms Φ with f_1 taking only finitely many values will be treated in a separate paper.

2. The group of invertible formal power series

Let $\mathbb{K}[\![X]\!]$ denote the ring of all formal power series $f(X) = \sum_{k=0}^\infty c_k X^k$ with $c_k \in \mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. Let $\Gamma = \{f(X) \in \mathbb{K}[\![X]\!] : \operatorname{ord} f(X) = 1\}$. Then the

set Γ with the substitution $\circ$ as a binary operation is a group. Finally, $\Gamma_1 = \left\{f(X) = \sum_{k=1}^{\infty} c_k X^k \in \Gamma : c_1 = 1\right\}$ is a subgroup of Γ.

Definition 1. By a *one-parameter group of formal power series* we understand every homomorphism of a group $(G, +)$ into $(\Gamma, \circ)$, i.e. each function $\Theta_G : G \to \Gamma$ which satisfies the equation

$$\Theta_G(s+t) = \Theta_G(s) \circ \Theta_G(t) \quad \text{for } s, t \in G. \tag{1}$$

Remark 1. If Θ_G is a one-parameter group of formal power series, then for an arbitrary group $(H, +)$ and for each homomorphism $a : H \to G$, also $\Theta_H := \Theta_G \circ a$ is a one-parameter group of formal power series.

Let $F(t, X) = \Theta_G(t)(X) \in \Gamma$. In the case when Θ_G is one-parameter group of formal power series we will also say that the family $(F(t, X))_{t \in G}$ forms a one-parameter group of formal power series. From (1) we then obtain, as an equivalent formulation, the so called translation equation

$$F(s+t, X) = F(s, F(t, X)) \quad \text{for } s, t \in G. \tag{2}$$

In the following we use the standard notation $\frac{\partial F(t,X)}{\partial X} := \sum_{k=1}^{\infty} k c_k(t) X^{k-1}$, and $\frac{\partial F(t,X)}{\partial t} := \sum_{k=1}^{\infty} c_k'(t) X^k$ in the case when $G = \mathbb{K}$ and the coefficient functions are differentiable. For $G = \mathbb{K}$ the following theorem describes the general regular solution of the translation equation (2) in the ring of formal power series, which means that the coefficient functions are analytic when $\mathbb{K} = \mathbb{C}$, or C^∞, if $\mathbb{K} = \mathbb{R}$.

Theorem 1 (Cf. [14])**.**

(i) *If a family $(F(t, X))_{t \in \mathbb{K}}$ is a regular one-parameter group of formal power series, then there exists a formal power series $H(X) \in \mathbb{K}[\![X]\!]$ such that*

$$\begin{cases} \dfrac{\partial F(t,X)}{\partial t} = H(F(t,X)), & \text{for } t \in \mathbb{K}, \\ F(0, X) = X. \end{cases} \tag{3}$$

(ii) *For each formal power series $H(X) \in \mathbb{K}[\![X]\!]$ with $\operatorname{ord} H \geq 1$ the family $(F(t, X))_{t \in \mathbb{K}}$ defined by* (3) *is a regular one-parameter group of formal power series.*

(iii) *The series H is uniquely determined by $(F(t, X))_{t \in \mathbb{K}}$. It is given by the formula $H(X) := \frac{\partial F(t,X)}{\partial t}|_{t=0}$, in particular,* $\operatorname{ord} H \geq 1$.

Remark 2. Condition (iii) establishes a $1-1$-correspondence between regular one-parameter groups and formal series H with $\operatorname{ord} H \geq 1$.

3. The one-dimensional differential group

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let k, l be positive integers. By $|k, l|$ we denote the set of all integers n such that $k \leq n \leq l$. Similarly, by $|k, \infty|$ we mean the set of all integers n with $n \geq k$. Finally, $\mathbb{N}$ stands for the set of all positive integers. We assume that $\sum_{t\in\emptyset} a_t = 0$ and $\prod_{t\in\emptyset} a_t = 1$.

Definition 2. Let $s \in \mathbb{N}$. By *the group* L^1_s we mean the set

$$Z_s = \{\overline{x}_s := (x_1, \dots, x_s) \in \mathbb{K}^s : x_1 \neq 0\}$$

equipped with the operation

$$\overline{x}_s \cdot \overline{y}_s = \overline{z}_s$$

defined by

$$z_n = \sum_{k=1}^{n} x_k \sum_{\overline{u}_n \in U_{n,k}} A_{\overline{u}_n} \prod_{j=1}^{n} y_j^{u_j} \quad \text{for} \quad n \in |1, s|, \tag{4}$$

where

$$U_{n,k} := \left\{\overline{u}_n := (u_1, \dots, u_n) \in |0, k|^n : \sum_{j=1}^{n} u_j = k \wedge \sum_{j=1}^{n} j u_j = n\right\},$$

$$A_{\overline{u}_n} := \frac{n!}{\prod\limits_{j=1}^{n} (u_j!(j!)^{u_j})}.$$

Definition 3. By *the group* L^1_∞ we understand the set

$$Z_\infty = \{\overline{x}_\infty := (x_1, x_2, \dots) \in \mathbb{K}^{\mathbb{N}} : x_1 \neq 0\}$$

endowed with the operation

$$\overline{x}_\infty \cdot \overline{y}_\infty = \overline{z}_\infty,$$

given by

$$z_n = \sum_{k=1}^{n} x_k \sum_{\overline{u}_n \in U_{n,k}} A_{\overline{u}_n} \prod_{j=1}^{n} y_j^{u_j} \quad \text{for } n \in \mathbb{N}, \tag{5}$$

where $U_{n,k}$ and $A_{\overline{u}_n}$ are given above.

In the case $\mathbb{K} = \mathbb{R}$ the formulas (4) and (5) describe the higher chain rules for C^s- and C^∞-functions, respectively, with fixed point 0 (cf. [2], [9]). We call the group L^1_s one-dimensional s-th differential group. As examples of (5) we have

$$z_1 = x_1 y_1, \quad z_2 = x_1 y_2 + x_2 y_1^2, \quad z_3 = x_1 y_3 + 3x_2 y_1 y_2 + x_3 y_1^3. \tag{6}$$

In what follows we will need some property of the group operation in L^1_∞ (and in Γ) related to formula (5).

Lemma 1.

(i) *For every $n \geq 2$, $k \in |2, n-1|$ and $\overline{u}_n \in U_{n,k}$ we have (cf. [3])*

1) *there exists $j \in |2, n-k+1|$ such that $u_j \geq 1$;*

2) *for every $j \in |n-k+2, n|$ we have $u_j = 0$.*

(ii) *For every $n \geq 2$ we have (cf. [5])*

$$z_n = x_1 y_n + \sum_{k=2}^{n-1} x_k \sum_{\overline{u}_n \in U_{n,k}} A_{\overline{u}_n} \prod_{j=1}^{n-k+1} y_j^{u_j} + x_n y_1^n.$$

As we have mentioned earlier, the groups Γ and L^1_∞ are isomorphic, which is stated in the following theorem.

Theorem 2 (cf. Theorem 2 in [6])**.** *The groups L^1_∞ and $(\Gamma, \circ)$ are isomorphic. A natural isomorphism from L^1_∞ to $(\Gamma, \circ)$ is given by $\Psi : Z_\infty \to \Gamma$,*

$$\Psi(x_1, x_2, \dots)(X) = \sum_{k=1}^{\infty} \frac{x_k}{k!} X^k. \tag{7}$$

4. Homomorphisms into the group L^1_∞

Throughout this section we assume that $(G, +)$ is an abelian group which admits generalized exponential functions $f : G \to (\mathbb{K} \setminus \{0\})$, i.e. functions f satisfying

$$f(x+y) = f(x)f(y) \quad \text{for } x, y \in G,$$

which are not characters, i.e. such that there exists an $x_0 \in G$ with $|f(x_0)| \neq 1$.

It is known (cf. [1]) that if $f : G \to (\mathbb{K} \setminus \{0\})$ is a generalized exponential function, then $f(x) = e^{a(x)}$ with an additive function $a : G \to \mathbb{R}$ in the real case, and $f(x) = e^{a(x)+ib(x)}$ with an additive function $a : G \to \mathbb{R}$ and a function $b : G \to \mathbb{R}$ additive modulo 2π in the complex case. This means that the assumption that G admits generalized exponential functions which are not characters is equivalent to the one that G admits nonzero homomorphisms into the additive group $(\mathbb{R}, +)$.

Let s be a positive integer or $s = \infty$. Consider a function $\Phi_s : G \to Z_s$,

$$\Phi_s = (f_j)_{j \in |1,s|},$$

where $f_1 : G \to \mathbb{K} \setminus \{0\}$, $f_n : G \to \mathbb{K}$ for $n \in |2, s|$.

The function Φ_s is a homomorphism if and only if

$$\Phi_s(x+y) = \Phi_s(x) \cdot \Phi_s(y) \quad \text{for } x, y \in G,$$

i.e.

$$(f_j(x+y))_{j\in|1,s|} = (f_j(x))_{j\in|1,s|} \cdot (f_j(y))_{j\in|1,s|} \quad \text{for } x, y \in G.$$

Then we obtain (see (6) and Lemma 1) that Φ_s is a homomorphism if and only if the functions f_n solve the system of functional equations

$$\begin{cases} f_1(x+y) = f_1(x)f_1(y), \\ f_2(x+y) = f_1(x)f_2(y) + f_2(x)f_1(y)^2, \\ f_n(x+y) = f_1(x)f_n(y) \\ \quad + \sum_{k=2}^{n-1} f_k(x) \sum_{\overline{u}_n \in U_{n,k}} A_{\overline{u}_n} \prod_{j=1}^{n-1} f_j(y)^{u_j} + f_n(x)f_1(y)^n, \quad n \in |3,s|, \end{cases} \tag{8}$$

for $x, y \in G$.

As one can see, $f_1 : G \to \mathbb{K}\setminus\{0\}$ is then an exponential function. Here we will investigate in detail the form of the solution of the system (8) in the case where f_1 is a general exponential function such that $|f_1(x_0)| \neq 1$ for at least one $x_0 \in G$. We will write $|f_1|$ for the function $|f_1| : G \to (\mathbb{K} \setminus \{0\})$ with $|f_1|(x) = |f_1(x)|$ for $x \in G$. The case when we only assume that f_1 takes infinitely many values (so that f_1 may be also a character) will be treated in Section 6. Let us only mention here that the results on the general form of the solution of translation equation (2) are also valid, if we assume that a generalized exponential function f takes infinitely many values. We will prove

Theorem 3. *Assume that the abelian group $(G,+)$ admits nonzero additive functions. Then there exists a sequence $(v_n)_{n\in|2,s|}$ of universal polynomials such that for each solution $(f_n)_{n\in|1,s|}$ of the system of equations (8) (that is for each homomorphism $\Phi_s = (f_j)_{j\in|1,s|}$ from G to L_s^1) with a generalized exponential function f_1 which is not a character, there is a sequence of constants $(p_n)_{n\in|2,s|}$ such that*

(i) $v_2(X) = 0$;

(ii) $v_n \in \mathbb{Z}[X; p_2, \ldots, p_{n-1}]$ *for* $n \in |2,s|$;

(iii) $\deg_X v_n(X; p_2, \ldots, p_{n-1}) \leq n-2$;

(iv) *for every* $n \in |2,s|$

$$f_n = (f_1^2 - f_1)\left(p_n \sum_{l=0}^{n-2} f_1^l + v_n(f_1; p_2, \ldots, p_{n-1})\right).$$

The polynomials v_n are given for $n \in |2,s|$ by the recurrent formula

$$\begin{cases} v_2(X) = 0,\ V_2(X;p_2) = p_2, \\ v_n(X;p_2,\dots,p_{n-1}) = \displaystyle\sum_{k=2}^{n-1} \sum_{\overline{u}_n \in U_{n,k}} A_{\overline{u}_n} \\ \qquad V_k(0;p_2,\dots,p_k)(X^2-X)^{k-1-u_1} X^{u_1} \displaystyle\prod_{j=2}^{n-k+1} V_j(X;p_2,\dots,p_j)^{u_j}, \\ V_n(X;p_2,\dots,p_n) = p_n \displaystyle\sum_{l=0}^{n-2} X^l + v_n(X;p_2,\dots,p_{n-1}), \end{cases} \tag{9}$$

where V_n are polynomials in X and in $p_2,\dots,p_n$.

Proof. The proof is by induction on n. Consider the second equation of the system (8) that is

$$f_2(x+y) = f_1(x)\, f_2(y) + f_1(y)^2 f_2(x) \quad \text{for } x,y \in G. \tag{10}$$

From the symmetry of the left hand side of (10) we get

$$f_1(x)\, f_2(y) + f_1(y)^2 f_2(x) = f_1(y)\, f_2(x) + f_1(x)^2 f_2(y)$$

and hence $f_2(x)(f_1(y)^2 - f_1(y)) = f_2(y)(f_1(x)^2 - f_1(x))$. Take y_0 such that $f_1(y_0)^2 - f_1(y_0) \neq 0$. Such a y_0 exists since every generalized exponential function f_1, $|f_1| \neq 1$, takes infinitely many values. Then $f_2(x) = p_2(f_1(x)^2 - f_1(x))$ with $p_2 := \frac{f_2(y_0)}{f_1(y_0)^2 - f_1(y_0)}$.

Denote $v_2(f_1) = 0$ and assume that for some $n \in |3,s|$ there exist polynomials $v_j \in \mathbb{Z}[X;p_2,\dots,p_{j-1}]$ with $\deg_X v_j(X;p_2,\dots,p_{j-1}) \leq j-2$ for $j \in |2,n-1|$, such that for a solution $(f_n)_{n\in|2,s|}$ of the system (8) there exist constants $p_2,\dots,p_{n-1} \in \mathbb{K}$ such that

$$f_j = (f_1^2 - f_1) V_j(f_1;p_2,\dots,p_j) \quad \text{for } j \in |2,n-1|, \tag{11}$$

where $V_j(f_1;p_2,\dots,p_j) = p_j \sum_{l=0}^{j-2} f_1^l + v_j(f_1;p_2,\dots,p_{j-1})$. Consider the n-th equation of the system (8). From the symmetry of the left hand side of this equation and by Lemma 1(ii) we obtain

$$\begin{aligned} &f_1(x)\, f_n(y) + \sum_{k=2}^{n-1} f_k(x) \sum_{\overline{u}_n \in U_{n,k}} A_{\overline{u}_n} \prod_{j=1}^{n-k+1} f_j(y)^{u_j} + f_1(y)^n f_n(x) \\ &\qquad = f_1(y)\, f_n(x) + \sum_{k=2}^{n-1} f_k(y) \sum_{\overline{u}_n \in U_{n,k}} A_{\overline{u}_n} \prod_{j=1}^{n-k+1} f_j(x)^{u_j} + f_1(x)^n f_n(x), \end{aligned}$$

and hence

$$f_n(x)(f_1(y)^n - f_1(y)) = f_n(y)(f_1(x)^n - f_1(x)) + \sum_{k=2}^{n-1} \sum_{\overline{u}_n \in U_{n,k}} A_{\overline{u}_n} \Big[f_k(y) \prod_{j=1}^{n-k+1} f_j(x)^{u_j} - f_k(x) \prod_{j=1}^{n-k+1} f_j(y)^{u_j} \Big]. \quad (12)$$

It is known (see Lemma 1(i)) that for $k \in |2, n-1|$ and $\overline{u}_n \in U_{n,k}$ there exists $j \in |2, n-k+1|$ such that $u_j \geq 1$. Then $k - 1 - u_1 \geq 0$. Note that for each $k \in |2, n-1|$ and for every $\overline{u}_n \in U_{n,k}$ we have

$$\begin{aligned} \prod_{j=2}^{n-k+1} f_j^{u_j} &= f_1^{u_1} \prod_{j=2}^{n-k+1} \big((f_1^2 - f_1) V_j(f_1; p_2, \ldots, p_j)\big)^{u_j} \\ &= f_1^{u_1} \prod_{j=2}^{n-k+1} (f_1^2 - f_1)^{u_j} \prod_{j=2}^{n-k+1} V_j(f_1; p_2, \ldots, p_j)^{u_j} \\ &= f_1^{u_1} (f_1^2 - f_1)^{k-u_1} \prod_{j=2}^{n-k+1} V_j(f_1; p_2, \ldots, p_j)^{u_j}, \end{aligned}$$

since (cf. Lemma 1(i)) $\sum_{j=2}^{n-k+1} u_j = \sum_{j=2}^{n} u_j = k - u_1$. Thus we can write

$$\prod_{j=2}^{n-k+1} f_j^{u_j} = (f_1^2 - f_1)\Big((f_1^2 - f_1)^{k-1-u_1} f_1^{u_1} \prod_{j=2}^{n-k+1} V_j(f_1; p_2, \ldots, p_j)^{u_j} \Big).$$

From (12), using the induction hypothesis (11) one can deduce (with the abbreviation $\overline{V}_j(f_1) = V_j(f_1; p_2, \ldots, p_j)$)

$$\begin{aligned} f_n(x) \sum_{l=0}^{n-2} f_1(y)^l &= (f_1(x)^2 - f_1(x)) \Bigg[\frac{f_n(y)}{f_1(y)^2 - f_1(y)} \sum_{l=0}^{n-2} f_1(x)^l \\ &+ \sum_{k=2}^{n-1} \sum_{\overline{u}_n \in U_{n,k}} A_{\overline{u}_n} \Big[\overline{V}_k(f_1(y))(f_1(x)^2 - f_1(x))^{k-1-u_1} f_1(x)^{u_1} \prod_{j=2}^{n-k+1} \overline{V}_j(f_1(x))^{u_j} \\ &- \overline{V}_k(f_1(x))(f_1(y)^2 - f_1(y))^{k-1-u_1} f_1(y)^{u_1} \prod_{j=2}^{n-k+1} \overline{V}_j(f_1(y))^{u_j} \Big] \Bigg]. \end{aligned}$$

Fix $x \in G$ such that $f_1(x)^n - f_1(x) \neq 0$. Then clearly $\sum_{l=0}^{n-2} f_1(x)^l \neq 0$. Since f_1 is a generalized exponential function with $|f_1| \neq 1$, we find a sequence

$(y_m)_{m\in\mathbb{N}}$ of elements of G such that $\lim_{m\to\infty} f_1(y_m) = 0$. This can be seen as follows. From $|f_1| \neq 1$ we deduce the existence of $y_1 \in G$ with $|f_1|(y_1) \neq 1$. If $|f_1|(y_1) < 1$, then for $y_m = my_1$ we get $f_1(y_m) = f(y_1)^m$, hence $\lim_{m\to\infty} f_1(y_m) = \lim_{m\to\infty} f_1(y_1)^m = 0$. If $|f_1|(y_1) > 1$, then $f_1(-y_1) = f_1(y_1)^{-1}$, and so the first case with $y_1' = -y_1$ occurs.

Put $y = y_m$ and let m tend to ∞. Then on the left hand side we have $\lim_{m\to\infty} f_n(x)\sum_{l=0}^{n-2} f_1(y_m)^l = f_n(x)$. On the right hand side we see immediately that the limit $\lim_{m\to\infty} \overline{V}_k(f_1(y_m))$ exists since $\overline{V}_k$ is a polynomial in f_1. Moreover, we have either $u_1 \geq 1$ and $k-1-u_1 \geq 0$ or $u_1 \geq 0$ and $k-1-u_1 \geq 1$, so

$$\lim_{m\to\infty} (f_1(y_m)^2 - f_1(y_m))^{k-1-u_1} f_1(y_m)^{u_1} \prod_{j=2}^{n-k+1} \overline{V}_j(f_1(y_m))^{u_j} = 0.$$

Since $\sum_{l=0}^{n-2} f_1(x)^l \neq 0$, the limit $\lim_{m\to\infty} \frac{f_n(y_m)}{f_1(y_m)^2 - f_1(y_m)} =: p_n$ exists.

Fix $x \in G$ arbitrarily and put $y = y_m$. Then letting $m \to \infty$ we get

$$\begin{aligned} f_n(x) &= \big(f_1(x)^2 - f_1(x)\big)\Bigg[p_n \sum_{l=0}^{n-2} f_1(x)^l \\ &\quad + \sum_{k=2}^{n-1} \sum_{\overline{u}_n \in U_{n,k}} A_{\overline{u}_n} \overline{V}_k(0)(f_1(x)^2 - f_1(x))^{k-1-u_1} f_1(x)^{u_1} \prod_{j=2}^{n-k+1} \overline{V}_j(f_1(x))^{u_j}\Bigg] \\ &= \big(f_1(x)^2 - f_1(x)\big)\Bigg[p_n \sum_{l=0}^{n-2} f_1(x)^l + v_n(f_1(x); p_2, \ldots, p_{n-1})\Bigg] \end{aligned}$$

and the recurrent formula (9) for universal polynomials v_n. Finally, on account of Lemma 1 (ii) we get

$$\begin{aligned} &\deg_X \Bigg\{(X^2 - X)^{k-1-u_1} X^{u_1} \prod_{j=2}^{n-k+1} \overline{V}_j(X)^{u_j}\Bigg\} \\ &\qquad \leq 2(k-1-u_1) + u_1 + \sum_{j=2}^{n-k+1} (j-2)u_j \\ &\qquad = 2k - 2 - u_1 + \sum_{j=2}^{n} (j-2)u_j = 2k - 2 - u_1 + \sum_{j=2}^{n} ju_j - 2\sum_{j=2}^{n} u_j \\ &\qquad = 2k - 2 - u_1 + (n - u_1) - 2(k - u_1) = n - 2, \end{aligned}$$

so $\deg_X v_n(X; p_2, \ldots, p_{n-1}) \leq n-2$. □

Remark 3. Note that the form of the polynomials v_n in Theorem 3 does not depend on s, that is if $s_1 < s_2$ and $(v_n)_{n\in|2,s_1|}$, $(\tilde{v}_n)_{n\in|2,s_2|}$ are sequences of polynomials existing in virtue of Theorem 3 corresponding to s_1 and s_2, respectively, then $v_n = \tilde{v}_n$ for $n \in |2, s_1|$.

5. Solution of the translation equation in rings of formal power series

From Theorem 2, using (7), we obtain that if

$$\sum_{k=1}^{\infty} a_k \left(\sum_{l=1}^{\infty} b_l X^l \right)^k = \sum_{n=1}^{\infty} d_n X^n,$$

then

$$d_n = \sum_{k=1}^{n} a_k \sum_{\overline{u}_n \in U_{n,k}} B_{\overline{u}_n} \prod_{j=1}^{n} b_j^{u_j} \quad \text{for } n \in \mathbb{N}, \tag{13}$$

where

$$B_{\overline{u}_n} := \frac{k!}{\prod_{j=1}^{n} u_j!}.$$

We mention as examples of (13)

$$e_1 = c_1 d_1, \qquad e_2 = c_1 d_2 + c_2 d_1^2, \qquad e_3 = c_1 d_3 + 2c_2 d_1 d_2 + c_3 d_1^3. \tag{14}$$

As a Corollary from Theorem 2 and Lemma 1 we obtain

Corollary 1. *For every $n \geq 2$*

$$e_n = c_1 d_n + \sum_{k=2}^{n-1} c_k \sum_{\overline{u}_n \in U_{n,k}} B_{\overline{u}_n} \prod_{j=1}^{n-k+1} d_j^{u_j} + c_n d_1^n.$$

Let $F(t, X) = \sum_{k=1}^{\infty} c_k(t) X^k$, where $c_1 : G \to \mathbb{K} \setminus \{0\}$, $c_k : G \to \mathbb{K}$ for $k \geq 2$. Then we get from (2)

$$\sum_{k=1}^{\infty} c_k(s+t) X^k = \sum_{l=1}^{\infty} c_l(t) \left(\sum_{j=1}^{\infty} c_j(t) X^j \right)^l$$

for $s,t\in\mathbb{K}$, which in virtue of (13), (14) and Corollary 1 is the same, by comparing coefficients, as the infinite system of functional equations

$$\begin{cases} c_1(s+t)=c_1(s)c_1(t)\\ c_2(s+t)=c_1(s)c_2(t)+c_2(s)c_1(t)^2\\ c_n(s+t)=c_1(s)c_n(t)+\sum\limits_{k=2}^{n-1}c_k(s)\sum\limits_{\overline{u}_n\in U_{n,k}}B_{\overline{u}_n}\prod\limits_{j=1}^{n-1}c_j(t)^{u_j}\\ \qquad\qquad +c_n(s)c_1(t)^n,\quad n\in|2,\infty| \end{cases} \tag{15}$$

for $s,t\in G$. We are interested here in the case when c_1 *is a nontrivial exponential function*.

Now, let us consider the case $G=\mathbb{K}$. Assume that $F(t,X)=\sum_{k=1}^{\infty}c_k(t)X^k$ is a regular one-parameter group of formal power series (which means that the sequence of regular functions $(c_n)_{n\in\mathbb{N}}$ satisfies the system of equations (15)). Then $c_1'(0)\neq 0$ and, by Theorem 1(i) and (iii), there exists a power series

$$H(X)=\sum_{k=1}^{\infty}c_k'(0)X^k=:\lambda_1X+\sum_{k=2}^{\infty}((k-1)\lambda_1\lambda_k)X^k,$$

(here $\lambda_k:=\frac{c_k'(0)}{\lambda_1(k-1)}$) the so called infinitesimal generator of the one-parameter group, such that

$$\sum_{k=1}^{\infty}c_k'(t)X^k=\lambda_1\sum_{l=1}^{\infty}c_l(t)X^l+\sum_{k=2}^{\infty}((k-1)\lambda_1\lambda_k)\left(\sum_{l=1}^{\infty}c_l(t)X^l\right)^k,$$

and $\sum_{k=1}^{\infty}c_k(0)X^k=X$, i.e.

$$\begin{cases} c_1'(t)=\lambda_1c_1(t),\quad c_1(0)=1,\\ c_2'(t)=\lambda_1c_2(t)+\lambda_1\lambda_2c_1(t)^2,\quad c_2(0)=0,\\ c_n'(t)=\lambda_1c_n(t)+\sum\limits_{k=2}^{n-1}((k-1)\lambda_1\lambda_k)\sum\limits_{\overline{u}_n\in U_{n,k}}B_{\overline{u}_n}\prod\limits_{j=1}^{n-k+1}c_j(t)^{u_j}\\ \qquad +(n-1)\lambda_1\lambda_nc_1(t)^n,\ \ c_n(0)=0,\ n\in|3,\infty|. \end{cases} \tag{16}$$

Then $c_1(t)=e^{\lambda_1t}$, and $c_2(t)=\lambda_2e^{\lambda_1t}(e^{\lambda_1t}-1)=c_1(t)(\lambda_2c_1(t)-\lambda_2)$. Finally, let us assume that $P_2(X)=0$ and for some $n\in\mathbb{N}$, $n\geq 3$ there are polynomials $P_j\in\mathbb{Q}[X;\lambda_2,\dots,\lambda_{j-1}]$ such that for every $j\in|2,n-1|$

$$c_j=c_1\big(\lambda_j(c_1^{j-1}-1)+P_j(c_1;\lambda_2,\dots,\lambda_{j-1}\big)=c_1R_j(c_1;\lambda_2,\dots,\lambda_j),$$

$$P_j(0;\lambda_2,\dots,\lambda_{j-1})=0,\qquad \deg_X P_j(X;\lambda_2,\dots,\lambda_{j-1})\leq j-1$$

holds. Then, from the n-th equation of the system (16), one can derive

$$c_n(t) = c_1(t)\Bigg(\lambda_n(c_1(t)^{n-1}-1) + \sum_{k=2}^{n-1}((k-1)\lambda_1\lambda_k)\sum_{\overline{u}_n\in U_{n,k}} B_{\overline{u}_n} \\ \times \int_0^t \Bigg((c_1(s))^{k-1}\prod_{j=2}^{n-k+1}(R_j(c_1(s);\lambda_2,\dots,\lambda_j))^{u_j}\Bigg)ds\Bigg). \tag{17}$$

From the equality (17) one can deduce, by explicitly calculating the involved integrals, the existence of a polynomial P_n such that

$$c_n(t) = e^{\lambda_1 t}\left(\lambda_n\big(e^{(n-1)\lambda_1 t}-1\big) + P_n(e^{\lambda_1 t};\lambda_2,\dots,\lambda_{n-1})\right). \tag{18}$$

We have thus proved that there is a sequence of polynomials $(P_n)_{n\geq 2}$ such that for every regular solution of (15) there exists a sequence of constants $(\lambda_n)_{n\in\mathbb{N}}$ such that $c_1(t) = e^{\lambda_1 t}$, $\lambda_1 \neq 0$ and (18) holds for every $n \geq 2$. The polynomials P_n are recurrently defined by the formula

$$\begin{cases} P_2(X) = 0; \quad R_2(X;\lambda_2) = \lambda_2 X - \lambda_2 \\ P_n(X;\lambda_2,\dots,\lambda_{n-1}) = \displaystyle\sum_{k=2}^{n-1}((k-1)\lambda_k \sum_{\overline{u}_n\in U_{n,k}} B_{\overline{u}_n} \\ \qquad\qquad \displaystyle\int_1^X s^{k-2}\prod_{j=2}^{n-k+1}(R_j(s;\lambda_2,\dots,\lambda_j))^{u_j}\,ds. \\ R_n(X;\lambda_2,\dots,\lambda_n) = \lambda_n(X^{n-1}-1) + P_n(X;\lambda_2,\dots,\lambda_{n-1}). \end{cases} \tag{19}$$

Moreover $P_n(0,\lambda_2,\dots,\lambda_{n-1}) = 0$ and

$$\deg_X P_n(X;\lambda_2,\dots,\lambda_{n-1}) = \max_{\substack{k\in|2,n-1|,\\ \overline{u}_n\in U_{n,k}}} \deg_X \int_0^X s^{k-2}\prod_{j=2}^{n-k+1} R_j(s;\lambda_2,\dots,\lambda_j)^{u_j}\,ds$$

$$\leq 1 + \left(k-2+\sum_{k=2}^{n-k+1}(j-1)u_j\right) = k-1+\sum_{k=2}^{n} ju_j - \sum_{k=2}^{n} u_j$$

$$= k-1+\sum_{k=1}^{n} ju_j - \sum_{k=1}^{n} u_j = k-1+n-k = n-1.$$

Now, fix arbitrarily a sequence of numbers $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_1 \neq 0$. Then, by Theorem 1(ii), the family $(F(t,X))_{t\in\mathbb{K}}$ given by $F(t,X) = \sum_{n=1}^{\infty} c_n(t)X^n$ with functions c_n defined by (17) is a one-parameter group of formal power series, which means that the functions $(c_n)_{n\in\mathbb{N}}$ satisfy the system of equations (15).

We have proved

Theorem 4. *There exists a sequence of polynomials $(P_n)_{n\geq 2}$ defined by* (19) *such that for every regular solution $F(t,X)=\sum_{k=1}^{\infty}c_k(t)X^k$ of the translation equation* (2) *with a nontrivial exponential function c_1 there exists a sequence of constants $(\lambda_n)_{n\in\mathbb{N}}$, $\lambda_1\neq 0$ such that*

$$\begin{aligned} &c_1(t)=e^{\lambda_1 t},\\ &c_n(t)=\lambda_n\left(e^{n\lambda_1 t}-e^{\lambda_1 t}\right)+e^{\lambda_1 t}P_n\left(e^{\lambda_1 t};\lambda_2,\ldots,\lambda_{n-1}\right),\quad n\geq 2. \end{aligned} \tag{20}$$

Conversely, for every sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_1\neq 0$, the family $(F(t,X))_{t\in\mathbb{K}}=\left(\sum_{k=1}^{\infty}c_k(t)X^k\right)_{t\in\mathbb{K}}$ defined by (20) *is a solution of the translation equation* (2).

As a Corollary from Theorem 4 and Remark 1 we obtain

Corollary 2. *If a sequence $(\tilde{c}_n)_{n\in\mathbb{N}}$ is a regular solution of the system* (15) *(i.e. represents a regular solution $F(t,X)=\sum_{k=1}^{\infty}c_k(t)X^k$ of the translation equation), then the sequence $(\tilde{c}_n\circ a)_{n\in\mathbb{N}}$ with an arbitrary homomorphism $a:G\to\mathbb{K}$ (by $\mathbb{K}$ we mean the additive group of the field $\mathbb{K}$), is a solution of this system. This means that for each homomorphism $a:G\to\mathbb{K}$ and for each sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_1=1$ the sequence of functions $(c_n)_{n\in\mathbb{N}}$ defined by $c_1(t)=e^{a(t)}$ and by*

$$c_n(t)=\lambda_n\left(e^{na(t)}-e^{a(t)}\right)+e^{a(t)}P_n\left(e^{a(t)};\lambda_2,\ldots,\lambda_{n-1}\right),\quad n\geq 2, \tag{21}$$

where P_n are polynomials from Theorem 4, satisfies the system of equations (15).

We will show in this paper that (21) is, in fact, the general solution of the system (15) in the case when $\mathbb{K}=\mathbb{R}$ and $c_1(t)=e^{a(t)}$ takes infinitely many values. We will also show that if $\mathbb{K}=\mathbb{C}$ and c_1 takes infinitely many values, then the general solution of (15) has the form

$$c_n(t)=\lambda_n\left(c_1(t)^n-c_1(t)\right)+c_1(t)P_n\left(c_1(t);\lambda_2,\ldots,\lambda_{n-1}\right),\quad n\geq 2,$$

i.e. $e^{a(t)}$ in (21) has to be replaced by $c_1(t)$.

6. Connection between homomorphisms into the group L^1_∞ and one-parameter groups of formal power series

From now on, if we do not make explicitly another assumption, whenever we will consider the systems of equations (8), *we will assume that $s=\infty$*. As we have mentioned, a one-parameter group of formal power series $(F(t,X))_{t\in G}$

with $F(t,X) = \sum_{k=1}^{\infty} c_k(t)X^k$ is a homomorphism $\Theta_G : G \to \Gamma$ such that $\Theta_G(t)(X) = F(t,X)$. Then the functions c_n satisfy the infinite system of functional equations (15). On the other side, a function $\Phi : G \to Z_\infty$, $\Phi = (f_1, f_2, \dots)$ is a homomorphism into L^1_∞ if and only if the functions f_n satisfy the infinite system of functional equations (8). As we have seen, the groups $(\Gamma, \circ)$ and L^1_∞ are isomorphic (see Theorem 2). Consequently, systems (8) and (15) are equivalent in the following sense.

Proposition 1. *The sequence of functions $(c_k)_{k\in\mathbb{N}}$ is a solution of the system of equations* (15) *if and only if $f_k = k!c_k$, where the sequence of functions $(f_k)_{k\in\mathbb{N}}$ is a solution of the system of equations* (8).

As a simple consequence of Proposition 1 and Theorem 4 we obtain

Corollary 3. *Let $G = \mathbb{K}$. There exists a sequence of polynomials $(P_n)_{n\geq 2}$ defined by* (19) *such that for every regular solution $(f_n)_{n\in\mathbb{N}}$ of the system of functional equations*(8) *with a nontrivial exponential function f_1 there exists a sequence of constants $(\lambda_n)_{n\in\mathbb{N}}$, $\lambda_1 \neq 0$ such that $f_1(x) = e^{\lambda_1 x}$ and for $n \geq 2$*

$$f_n(x) = n!\left(\lambda_n\left(e^{n\lambda_1 x} - e^{\lambda_1 x}\right) + e^{\lambda_1 x}P_n\left(e^{\lambda_1 x}; \lambda_2, \dots, \lambda_{n-1}\right)\right). \tag{22}$$

Conversely, for each sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_1 \neq 0$ the sequence $(f_n)_{n\in\mathbb{N}}$ such that $f_1(x) = e^{\lambda_1 x}$ and for every $n \geq 2$ the functions f_n are defined by (22) *is a (regular) solution of the system* (8).

Consider now a sequence of functions $(\overline{f}_n)_{n\in\mathbb{N}}$, $\overline{f}_n : \mathbb{K} \to \mathbb{K}$ with $\overline{f}_1(x) = e^x$. By Theorem 3 the following holds true: there exists a sequence of polynomials $(v_n)_{n\geq 2}$ such that if the sequence $(\overline{f}_n)_{n\in\mathbb{N}}$ with $\overline{f}_1(x) = e^x$ satisfies the system (8) with $G = \mathbb{K}$ ($\overline{f}_1$ is an exponential function!), then there exists a sequence of scalars $(p_n)_{n\geq 2}$ such that for every $n \geq 2$

$$\overline{f}_n = (\overline{f}_1^2 - \overline{f}_1)\left(p_n \sum_{l=0}^{n-2} \overline{f}_1^l + v_n(\overline{f}_1; p_2, \dots, p_{n-1})\right). \tag{23}$$

So, if $(\overline{f}_n)_{n\in\mathbb{N}}$ with $\overline{f}_1(x) = e^x$ is a solution of system of equations (8), then it is a regular solution of this system. First we will show that for every sequence $(p_n)_{n\geq 2}$ this is really the solution of (8). The proof of this fact is divided into two steps.

1. Choose arbitrarily a sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_1 = 1$. Then the sequence of functions $(\widehat{f}_n)_{n\in\mathbb{N}}$ such that $\widehat{f}_1(x) = e^x$ and $\widehat{f}_n$ are defined by (22) with $\lambda_1 = 1$, satisfies the system of equations (8), that is, according to what has been said, there exists a sequence $(p_n)_{n\geq 2}$ such that $\widehat{f}_n$ for $n \geq 2$ is given by (23). Putting

$x = 1$ we get

$$n!(\lambda_n(e^n - e) + eP_n(e; \lambda_2, \ldots, \lambda_{n-1})) = (e^2 - e)\Big(p_n \sum_{l=0}^{n-2} e^l + v_n(e; p_2, \ldots, p_{n-1})\Big)$$

or equivalently

$$n!(\lambda_n(e^n - e) + eP_n(e; \lambda_2, \ldots, \lambda_{n-1})) = p_n(e^n - e) + (e^2 - e)v_n(e; p_2, \ldots, p_{n-1}).$$

Then for $n \geq 2$ we have

$$p_n = n!\left(\lambda_n + \frac{e}{e^n - e}P_n(e; \lambda_2, \ldots, \lambda_{n-1})\right) - \frac{e^2 - e}{e^n - e}v_n(e; p_2, \ldots, p_{n-1}). \quad (24)$$

Thus the sequence $(p_n)_{n\geq 2}$ is uniquely determined by the sequence $(\lambda_n)_{n\geq 2}$, and moreover we find, by induction on n,

$$\begin{cases} p_2 = 2\lambda_2, \\ p_n = n!\lambda_n + S_n(\lambda_2, \ldots, \lambda_{n-1}), \end{cases} \quad (25)$$

with universal polynomials $(S_n)_{n\geq 3}$. By the same argument we get

$$\begin{cases} \lambda_2 = \dfrac{1}{2}p_2, \\ \lambda_n = \dfrac{p_n}{n!} + T_n(p_2, \ldots, p_{n-1}), \end{cases} \quad (26)$$

with universal polynomials $(T_n)_{n\geq 3}$. Moreover, from (24) it follows that the mappings $S : (\lambda_n)_{n\geq 2} \mapsto (p_n)_{n\geq 2}$ and $T : (p_n)_{n\geq 2} \mapsto (\lambda_n)_{n\geq 2}$ defined by (25) and (26) respectively, are one-to-one and onto and S represents the inverse mapping to T.

2. Now take a sequence of functions $(\overline{f}_n)_{n\in\mathbb{N}}$ such that $\overline{f}_1(x) = e^x$,

$$\overline{f}_n(x) = (e^{2x} - e^x)\Big(p_n \sum_{l=0}^{n-2} e^{lx} + v_n(e^x; p_2, \ldots, p_{n-1})\Big), \quad n \geq 2,$$

with arbitrarily fixed sequence $(p_n)_{n\geq 2}$ and define a sequence $(\lambda_n^*)_{n\geq 2}$ by (26). Then the sequence of functions $(f_n)_{n\in\mathbb{N}}$ given by $f_1(x) = e^x$,

$$f_n(x) = n!e^x\big(\lambda_n^*(e^{(n-1)x} - 1) + P_n(e^x; \lambda_2^*, \ldots, \lambda_{n-1}^*)\big), \quad n \geq 2,$$

is a solution of the system of equations (8), that is there exists a sequence of scalars $(p_n^*)_{n\geq 2}$ with such that for every $n \geq 2$

$$f_n(x) = (e^{2x} - e^x)\Big(p_n^* \sum_{l=0}^{n-2} e^{lx} + v_n(e^x; p_2^*, \ldots, p_{n-1}^*)\Big).$$

Clearly

$$p_n^* = n!\lambda_n^* + S_n(\lambda_2^*, \dots, \lambda_{n-1}^*). \tag{27}$$

Since $(\lambda_n^*)_{n\geq 2} = T\left((p_n)_{n\geq 2}\right)$, and S is the inverse mapping of T, also

$$p_n = n!\lambda_n^* + S_n\left(\lambda_2^*, \dots, \lambda_{n-1}^*\right) \tag{28}$$

Consequently, since S is one-to-one, we get $p_n = p_n^*$ from (27) and (28) for $n \geq 2$. Then $f_n = \overline{f}_n$, which means that $\overline{f}_n$ is a solution of the system of equations (8). We have thus proved

Proposition 2. *Let $G = \mathbb{K}$. There exists a sequence of polynomials $(v_n)_{n\geq 2}$ (the one from Theorem 3) such that the general solution $(f_n)_{n\in\mathbb{N}}$ with $f_1(x) = e^x$ of the system of equations* (8) *is given by*

$$f_n(x) = (e^{2x} - e^x)\left(p_n \sum_{l=0}^{n-2} e^{lx} + v_n(e^x; p_2, \dots, p_{n-1})\right) \quad \text{for } n \geq 2, \tag{29}$$

where $(p_n)_{n\geq 2}$ is an arbitrary sequence of scalars.

Using the above result, we prove useful polynomial identities for the sequences of polynomials (V_n), $V_n(X; p_2, \dots, p_n) = p_n \sum_{l=0}^{n-2} X^l + v_n(X; p_2, \dots, p_{n-1})$.

Proposition 3. *Let $(V_n)_{n\geq 1}$ be given by* (9)*. Then for every $n \geq 2$ we have*

$$\begin{aligned}((XY)^2 - XY)V_n(XY; p_2, \dots, p_n) &= X(Y^2 - Y)V_n(Y; p_2, \dots, p_n)\\ &+ \sum_{k=2}^{n-1}(X^2 - X)V_k(X; p_2, \dots, p_k) \sum_{\overline{u}_n \in U_{n,k}} A_{\overline{u}_n} Y^{u_1} \prod_{j=2}^{n-k+1} ((Y^2 - Y)V_j(Y; p_2, \dots, p_j))^{u_j}\\ &+ Y^n(X^2 - X)V_n\left(X; p_2, \dots, p_n\right).\end{aligned} \tag{30}$$

Proof. From Proposition 2 it follows that there exists a sequence of polynomials $(v_n)_{n\geq 2}$, such that a sequence of functions (f_n) such that $f_1(x) = e^x$ and f_n are given by (29) satisfies the system of equations (8) with arbitrary constants p_n. This means that for each $n \geq 2$ and every $x, y \in \mathbb{K}$ we have (cf. (9))

$$\begin{aligned}((e^{x+y})^2 - e^{x+y})V_n(e^{x+y}; p_2, \dots, p_n) &= e^x((e^y)^2 - e^y)V_n(e^y; p_2, \dots, p_n)\\ &+ \sum_{k=2}^{n-1}((e^x)^2 - e^x)V_k(e^x; p_2, \dots, p_k) \sum_{\overline{u}_n \in U_{n,k}} A_{\overline{u}_n}(e^y)^{u_1}\\ &\times \prod_{j=2}^{n-k+1} (((e^y)^2 - e^y)V_j(e^y; p_2, \dots, p_j))^{u_j} + (e^y)^n((e^x)^2 - e^x)V_n(e^x; p_2, \dots, p_n)\end{aligned}$$

which implies the polynomial identity (30). □

We present now a complete answer to our problem to find the general form of homomorphisms from the abelian group $(G,+)$ to the differential group L^1_∞.

Theorem 5. *Let $(G,+)$ be an abelian group which admits generalized exponential functions mapping G into $\mathbb{K}\setminus\{0\}$ having infinitely many values. There exists a sequence of polynomials $(v_n)_{n\geq 2}$ (the one from Theorem 3) such that the general solution $(f_n)_{n\in\mathbb{N}}$ of the system of equations (8) (that is the homomorphism $\Phi_\infty = (f_j)_{j\in|1,\infty|}$ from G to L^1_∞) with a generalized exponential function f_1 taking infinitely many values is given by*

$$f_n = (f_1^2 - f_1)\bigg(p_n \sum_{l=0}^{n-2} f_1^l + v_n(f_1; p_2, \ldots, p_{n-1})\bigg) \quad \text{for } n \geq 2, \tag{31}$$

where f_1 is an arbitrary nontrivial exponential function and $(p_n)_{n\geq 2}$ is an arbitrary sequence of scalars.

Proof. In the first step we will show that if f_1 is a generalized exponential function (not necessarily taking infinitely many values) then the sequence $(f_n)_{n\in\mathbb{N}}$ of functions with

$$f_n = (f_1^2 - f_1)V_n(f_1, p_2, \ldots, p_n) \quad \text{for } n \geq 2,$$

where $(V_n)_{n\geq 2}$ is the sequence of universal polynomials from Theorem 3 and $(p_n)_{n\geq 2}$ is a sequence of constants, is a solution of the system of equations (8).

Indeed, for each $n \geq 2$ and for every $x, y \in G$, using (30) we obtain

$$\begin{aligned}
f_n(x+y) &= (f_1(x+y)^2 - f_1(x+y))V_n(f_1(x+y); p_2, \ldots, p_n)\\
&= \big((f_1(x)f_1(y))^2 - f_1(x)f_1(y)\big)V_n(f_1(x)f_1(y); p_2, \ldots, p_n)\\
&= f_1(x)(f_1(y)^2 - f_1(y))V_n(f_1(y); p_2, \ldots, p_n)\\
&\quad + \sum_{k=2}^{n-1}(f_1(x)^2 - f_1(x))V_k(f_1(x); p_2, \ldots, p_k)\\
&\quad \times \sum_{\overline{u}_n \in U_{n,k}} A_{\overline{u}_n} f_1(y)^{u_1} \prod_{j=2}^{n-k+1} (f_1(y)^2 - f_1(y))V_j(f_1(y); p_2, \ldots, p_j)^{u_j}\\
&\quad + f_1(y)^n(f_1(x)^2 - f_1(x))V_n(f_1(x); p_2, \ldots, p_n)\\
&= f_1(x)f_n(y) + \sum_{k=2}^{n-1} f_k(x) \sum_{\overline{u}_n \in U_{n,k}} A_{\overline{u}_n} \prod_{j=1}^{n-1} f_j(y)^{u_j} + f_n(x)f_1(y)^n
\end{aligned}$$

which means that every equation of the system (8) is satisfied with an arbitrary generalized exponential function f_1 and an arbitrary sequence of constants $(p_n)_{n\geq 2}$.

Now assume that f_1 takes infinitely many values. Consider once more the second equation of the system (8) that is

$$f_2(x+y) = f_1(x)\, f_2(y) + f_1(y)^2 f_2(x) \quad \text{for } x, y \in G.$$

As we have seen in the proof of Theorem 3, $f_2(x) = p_2(f_1(x)^2 - f_1(x))$ with $p_2 := \frac{f_2(y_0)}{f_1(y_0)^2 - f_1(y_0)}$. Thus $f_2 = (f_1^2 - f_1)(p_2 \cdot 1 + 0)$.

Now, assume that for some $n \in |3, s|$ and for a solution $(f_n)_{n\in|2,s|}$ of the system (8) there exist constants $p_2, \ldots, p_{n-1} \in \mathbb{K}$ such that

$$f_j = (f_1^2 - f_1)\Big(p_j \sum_{l=0}^{j-2} f_1^l + v_j(f_1; p_2, \ldots, p_{j-1})\Big), \quad j \in |2, n-1|, \tag{32}$$

where the polynomials v_n for $v \geq 2$ are given by (9). Consider the n-th equation of the system (8), that is

$$\begin{aligned} f_n(x+y) &= f_1(x) f_n(y) \\ &\quad + \sum_{k=2}^{n-1} f_k(x) \sum_{\overline{u}_n \in U_{n,k}} A_{\overline{u}_n} \prod_{j=1}^{n-1} f_j(y)^{u_j} + f_n(x) f_1(y)^n \quad \text{for } x, y \in G. \end{aligned} \tag{33}$$

We know from the first step of the proof that every function

$$\hat{f}_n = (f_1^2 - f_1)\Big(q_n \sum_{l=0}^{n-2} f_1^l + v_n(f_1; p_2, \ldots, p_{n-1})\Big),$$

with arbitrary q_n satisfies the equation (33). Therefore the function f_n' given by

$$f_n' = (f_1^2 - f_1) v_n(f_1; p_2, \ldots, p_{n-1})$$

is a particular solution of (33). Furthermore, let f_n, f_n' be solutions of (33). Note that the sum $\sum_{k=2}^{n-1} f_k(x) \sum_{\overline{u}_n \in U_{n,k}} A_{\overline{u}_n} \prod_{j=1}^{n-1} f_j(y)^{u_j}$ contains only functions f_j with $j \in |1, n-1|$. Then the function $F := f_n - f_n'$ satisfies the equation

$$F(x+y) = f_1(x)\, F(y) + f_1(y)^n F(x) \quad \text{for } x, y \in G.$$

This functional equation for F is similar to (10) for f_2, which is the special case with $n = 2$. By the same argumentation as used for (10), we get $F(x) = p_n(f_1(x)^n - f_1(x))$ with some $p_n \in \mathbb{K}$. Thus two arbitrary solutions f_n, f_n' of the equation (33) differ by such a function F with some $p_n \in \mathbb{K}$. This means

that an arbitrary solution of (33) is the sum of a particular solution of (33) and a function F with some $p_n \in \mathbb{K}$. Finally, every solution of the equation (33) must be of the form

$$\begin{aligned} f_n(x) &= f_n'(x) + F(x) = (f_1^2 - f_1)v_n(f_1; p_2, \ldots, p_{n-1}) + p_n(f_1(x)^n - f_1(x)) \\ &= (f_1(x)^2 - f_1(x))\left(p_n \sum_{l=0}^{n-2} f_1(x)^l + v_n(f_1(x); p_2, \ldots, p_{n-1})\right) \end{aligned}$$

We have thus proved inductively that every solution $(f_n)_{n\in|1,s|}$ of the system of equations (8) with a generalized exponential function f_1 taking infinitely many values is given by (32) with some sequence of constants $(p_n)_{n\in|2,s|}$. By the first step of the proof we know, that this is, in fact, the general solution of (8). □

Remark 4. Let us note once more that for an arbitrary abelian group $(G, +)$, a sequence of functions $(f_n)_{n\in\mathbb{N}}$ with an arbitrary exponential function f_1 and with f_n given by (31) satisfies the system of equations (8). However, we do not know whether there exists another solution of this system when $f_1 \neq 1$ and takes only finitely many values.

Now let us consider the case where s is a finite number. From Theorem 3 it follows that there exists a sequence of polynomials $(v_n)_{n\in|2,s|}$ not depending on s (cf. Remark 1) such that if a sequence of functions $(f_n)_{n\in|2,s|}$ with a generalized exponential function f_1 with $|f_1| \neq 1$ is a solution of the system of equations (8) for a finite s, then there exists a sequence of constants $(p_n)_{n\in|2,s|}$ such that

$$f_n = (f_1^2 - f_1)\left(p_n \sum_{l=0}^{n-2} f_1^l + v_n(f_1; p_2, \ldots, p_{n-1})\right) \tag{34}$$

holds for $n \in |2, s|$. Conversely, consider the infinite sequence of polynomials $(v_n)_{n\in\mathbb{N}}$ given by (9). Then the sequence of functions $(f_n)_{n\in\mathbb{N}}$ with arbitrary exponential function f_1, arbitrary sequence $(p_n)_{n\geq 2}$ and such that f_n are given by (34) for $n \in |2, \infty|$, satisfies the system (8) for $s = \infty$. Thus the sequence $(f_n)_{n\in|2,s|}$ with arbitrary exponential function f_1 and such that is given by (34) for $n \in |2, s|$, satisfies the system (8) for a finite s with arbitrary sequence $(p_n)_{n\in|2,s|}$. We have thus proved

Corollary 4. *Let $(G, +)$ be an abelian group which admits a generalized exponential function from G into $\mathbb{K} \setminus \{0\}$ with infinite image. There exists a sequence of polynomials $(v_n)_{n\in|2,s|}$ (the one from Theorem 3) such that the general*

solution $(f_n)_{n\in|2,s|}$ *of the system of equations* (8) *with a generalized exponential function* f_1 *which takes infinitely many values, is given by*

$$f_n = (f_1^2 - f_1)\Big(p_n \sum_{l=0}^{n-2} f_1^l + v_n(f_1; p_2, \dots, p_{n-1})\Big) \quad \text{for } n \in |2, s|,$$

where f_1 *is an arbitrary nontrivial exponential function and* $(p_n)_{n\in|2,s|}$ *is an arbitrary sequence of scalars.*

At the end we prove that the general solution of the system of equations (15) can be written (as we have in advance mentioned) in the form

$$c_n(t) = \lambda_n \left(c_1(t)^n - c_1(t)\right) + c_1(t) P_n(c_1(t); \lambda_2, \dots, \lambda_{n-1}), \quad n \geq 2 \tag{35}$$

with arbitrary exponential function c_1 and arbitrary sequence $(\lambda_n)_{n\geq 2}$, where P_n are the polynomials from Theorem 4.

Theorem 6. *Let* $(G, +)$ *be an abelian group which admits a generalized exponential function from* G *into* $\mathbb{K} \setminus \{0\}$ *with infinite image. There exists a sequence of polynomials* $(P_n)_{n\geq 2}$ *defined by* (19) *such that for every solution* $(c_n)_{n\in\mathbb{N}}$ *of the system of functional equations* (15) *(that is for each solution* $F(t, X) = \sum_{k=1}^{\infty} c_k(t) X^k$ *of the translation equation* (2)*) with a generalized exponential function* c_1 *taking infinitely many values, there exist a sequence of constants* $(\lambda_n)_{n\geq 2}$ *such that* (35) *holds. Conversely, for each exponential function* c_1 *and for each sequence* $(\lambda_n)_{n\geq 2}$ *the sequence* $(c_n)_{n\in\mathbb{N}}$ *defined by* (35) *is a solution of the system* (15).

Proof. Assume that a sequence $(c_n)_{n\in\mathbb{N}}$ is a solution of the system of equations (15) with a generalized exponential function c_1 which takes infinitely many values. From Proposition 1 we know, that the sequence $(f_n)_{n\in\mathbb{N}} := (n! c_n)_{n\in\mathbb{N}}$ is a solution of the system of equations (8). Then $f_1 = c_1$ is a generalized exponential function which takes infinitely many values and from Theorem 5 it follows

$$f_n = (f_1^2 - f_1)\Big(p_n \sum_{l=0}^{n-2} f_1^l + v_n(f_1; p_2, \dots, p_{n-1})\Big) \quad \text{for } n \geq 2$$

with some constants $p_2, \dots, p_n$. Consider a sequence $(\overline{f}_n)_{n\in\mathbb{N}}$ given by

$$\overline{f}_1(t) = e^t, \quad t \in \mathbb{K},$$

$$\overline{f}_n(t) = (e^{2t} - e^t)\Big(p_n \sum_{l=0}^{n-2} e^{lt} + v_n(e^t; p_2, \dots, p_{n-1})\Big), \quad t \in \mathbb{K},\ n \geq 2.$$

Then clearly the sequence $(\overline{f}_n)_{n\geq p+2}$ is a regular solution of the system of equations (8) with $G=\mathbb{K}$. If we denote $f_n = W_n(f_1; p_2, \ldots, p_n)$ with suitable polynomials $(W_n)_{n\geq 2}$, then also $\overline{f}_n(t) = W_n\left(e^t; p_2, \ldots, p_n\right)$.

Since $(\overline{f}_n)_{n\geq p+2}$ is a regular solution of (8), it follows from Proposition 1 that the sequence $(\overline{c}_n)_{n\geq p+2} := \left(\frac{\overline{f}_n}{n!}\right)_{n\geq p+2}$ is a regular solution of the system of equations (15) with $G=\mathbb{K}$. We have clearly $\overline{c}_1(t) = e^t$ for $t\in\mathbb{K}$. Using Theorem 4 we find constants $\lambda_2, \ldots, \lambda_n$ such that

$$\overline{c}_n(t) = \lambda_n\left(e^{nt} - e^t\right) + e^t P_n\left(e^t; \lambda_2, \ldots, \lambda_{n-1}\right), \quad n\geq 2.$$

Similarly as above we can write $\overline{c}_n(t) = L\left(e^t; \lambda_2, \ldots, \lambda_n\right)$. Thus we have the polynomial relation

$$n!L(X; \lambda_2, \ldots, \lambda_n) = W_n(X; p_2, \ldots, p_n)$$

with suitable constants $\lambda_2, \ldots, \lambda_n, p_2, \ldots, p_2$. Consequently, for $n\geq 2$,

$$\begin{aligned} c_n(t) &= \frac{f_n(t)}{n!} = \frac{W_n(f_1(t); p_2 \ldots, p_n)}{n!} = L(f_1(t); \lambda_2, \ldots, \lambda_n) \\ &= L(c_1(t); \lambda_2, \ldots, \lambda_n) = \lambda_n(c_1(t)^n - c_1(t)) + c_1(t)P_n(c_1(t); \lambda_2, \ldots, \lambda_{n-1}), \end{aligned}$$

which finishes the proof. □

7. Note on extensibility of homomorphisms

One of the authors posed the following question (see [13], p. 309): "When does a homomorphism Φ_s of $(\mathbb{R}, +)$ into L_s^r (the group of truncated formal power series transformations in r variables) have an extension $\overline{\Phi}_s$ from $(\mathbb{R}, +)$ into L_{s+1}^r?" For $r=1$ the term "extensibility of homomorphisms" one should understand as follows. Given a homomorphism $\Phi_s = (f_j)_{j\in|1,s|}$ of the group $(\mathbb{R}, +)$ into L_s^1, does there exist a function f_{s+1} such that $\overline{\Phi}_s = (f_j)_{j\in|1,s+1|}$ is a homomorphism from $(\mathbb{R}, +)$ into L_{s+1}^1? If such a function exists, we call $\overline{\Phi}_s$ an extension of Φ_s, and the homomorphism Φ_s extensible.

Instead of the problem of extensibility of homomorphisms which we formulated in the introduction, we may consider more generally the notion of l-extensibility, where l is a positive integer or $l=\infty$.

Definition 4. Let $(G, +)$ ba a group and let s be a positive integer. We call a homomorphism $\Phi_s = (f_n)_{n\in|1,s|}$ of the group $(G, +)$ into L_s^1 l-extensible, if there are functions f_n with $n\in|s+1, s+l|$ such that the function $\overline{\Phi}_s := (f_n)_{n\in|1,s+l|}$ is a homomorphism from $(G, +)$ into L_{s+l}^1.

Theorem 7. *Let $(G,+)$ be an abelian group which admits a generalized exponential function from G into $\mathbb{K}\setminus\{0\}$ taking infinitely many values and let l be a natural number or $l=\infty$. Every homomorphism $\Phi_s=(f_n)_{n\in|1,s|}$ with a generalized exponential function f_1 taking infinitely many values is l-extensible.*

Proof. Let us fix a homomorphism $\Phi_s=(f_n)_{n\in|1,s|}$ with a nontrivial exponential function f_1, which takes infinitely many values. From Theorem 5 it follows that there exists a sequence of polynomials $(v_n)_{n\in|2,s|}$, independent on s, such that there exists a sequence of constants $(p_n)_{n\in|2,s|}$ with

$$f_n=(f_1^2-f_1)\bigg(p_n\sum_{k=0}^{n-2}f_1^k+v_n(f_1;p_2,\dots,p_{n-1})\bigg) \tag{36}$$

for $n\in|2,s|$. Consider the sequence of polynomials $(v_n)_{n\in|2,s+l|}$ given by (9). Then the sequence of functions $(f_n)_{n\in\mathbb{N}}$ with arbitrary exponential function f_1 and such that f_n is given by (36) for $n\in|2,s+l|$, satisfies the system (8) for $s+l$ with an arbitrary sequence of constants $(p_n)_{n\in\mathbb{N}}$. Thus $\overline{\Phi}_s=(f_n)_{n\in|1,s+l|}$ is an l-extension of Φ_s. □

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