

## Distribution of additive and $q$ -additive functions under some conditions II.

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**Abstract.** Distribution of additive function over the set of integers having fixed number of prime divisors, and the distribution of  $q$ -additive functions over the set of integers for which the value of the sum of divisors function is fixed are investigated.

### §1. Introduction

**1.1. Notation.**  $\mathbb{N}, \mathbb{R}, \mathbb{C}$  as usual denote the set of natural, real and complex numbers, respectively,  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ,  $\mathcal{P}$  be the set of the primes,  $p$  with or without suffixes always denote prime numbers. The letters  $c, c_1, c_2, \dots$  denote constants not necessary the same at every occurrence. Let  $\Phi(y)$  be the Gaussian distribution function,  $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-u^2/2} du$ .

**1.2.  $q$ -additive and  $q$ -multiplicative functions.** Let  $q \geq 2$  be an integer, the  $q$ -ary expansion of  $n \in \mathbb{N}_0$  is defined as

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) q^j, \tag{1.1}$$

where the digits  $\varepsilon_j(n)$  are taken from  $\mathbb{A}_q := \{0, 1, \dots, q-1\}$ . Let  $\mathcal{A}_q$  be the set of  $q$ -additive functions, and  $\overline{\mathcal{M}}_q$  be the set of  $q$ -multiplicative functions of modulus

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1:  $f : \mathbb{N}_0 \rightarrow \mathbb{R}$  belongs to  $\mathcal{A}_q$  if  $f(0) = 0$  and

$$f(n) = \sum_{j=0}^{\infty} f(\varepsilon_j(n)q^j) \quad (n \in \mathbb{N}_0). \quad (1.2)$$

We say that  $g : \mathbb{N}_0 \rightarrow \mathbb{C}$  belongs to  $\overline{\mathcal{M}}_q$ , if  $g(0) = 1$ ,

$$g(n) = \prod_{j=0}^{\infty} g(\varepsilon_j(n)q^j) \quad (n \in \mathbb{N}_0) \quad (1.3)$$

and  $|g(n)| = 1$  ( $n \in \mathbb{N}_0$ ).

Let  $\alpha(n)$ ,  $\beta_h(n)$  be defined as

$$\alpha(n) = \sum_{j=0}^{\infty} \varepsilon_j(n); \quad \beta_h(n) = \sum_{\varepsilon_j(n)=h} 1 \quad (h = 1, \dots, q-1). \quad (1.4)$$

It is clear that  $\alpha, \beta_h \in \mathcal{A}_q$ . H. DELANGE [1] proved that for every  $g \in \overline{\mathcal{M}}_q$  the limit

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} g(n) = M(g) \quad (1.5)$$

exists and  $M(g) \neq 0$ , if

$$m_j := \frac{1}{q} \sum_{c \in \mathbb{A}_q} g(cq^j) \neq 0 \quad (j = 0, 1, 2, \dots) \quad (1.6)$$

and

$$\sum (1 - m_j) \quad (1.7)$$

is convergent. If these conditions hold, then

$$M(g) = \prod_{j=0}^{\infty} m_j. \quad (1.8)$$

Hence he deduced that for  $f \in \mathcal{A}_q$  the values  $f(n)$  possess a limit distribution if and only if both of the series

$$\sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}_q} f(bq^j), \quad (1.9)$$

$$\sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}_q} f^2(bq^j) \quad (1.10)$$

are convergent.

Let  $f \in \mathcal{A}_q$ . Assume that it has the limit distribution

$$F(y) := \lim_{x \rightarrow \infty} \frac{1}{x} \#\{n < x \mid f(n) < y\}. \quad (1.11)$$

Delange proved that  $F(y) = P(\xi < y)$ , where  $\xi$  is the sum of the independent random variables  $\xi_0, \xi_1, \dots$ , where  $P(\xi_j = f(aq^j)) = 1/q$  ( $a \in \mathbb{A}_q$ ). Thus the characteristic function  $\varphi(\tau)$  of  $F(y)$  can be written as

$$\varphi(\tau) = \prod_{j=0}^{\infty} \left\{ \frac{1}{q} \sum_{a=0}^{q-1} e^{i\tau f(aq^j)} \right\}. \quad (1.12)$$

Let  $r_1, r_2, \dots, r_{q-1}$  be nonnegative integers,  $\underline{r} = (r_1, \dots, r_{q-1})$  and  $S_N(\underline{r}) = \{n < q^N \mid \beta_j(n) = r_j, j = 1, \dots, q-1\}$ . Let  $r_0 := N - (r_1 + \dots + r_{q-1})$ .  $S_N(\underline{r})$  is empty if  $r_0 < 0$ . Let  $M(N \mid \underline{r}) := \#S_N(\underline{r})$ .

In [5] we proved the following Theorems A, B, C.

**Theorem A.** *Let  $f \in \mathcal{A}_q$ , and the series (1.9), (1.10) be convergent. Let  $\underline{r}^{(N)} = (r_1^{(N)}, \dots, r_{q-1}^{(N)})$  be such a sequence for which*

$$\left| \frac{qr_j^{(N)}}{N} - 1 \right| < \delta_N \quad (j = 1, \dots, q-1) \quad (1.13)$$

where  $\delta_N \rightarrow 0$  ( $N \rightarrow \infty$ ).

Then

$$\lim_{N \rightarrow \infty} \frac{1}{M(N \mid \underline{r})} \#\{n \in S_N(\underline{r}^{(N)}) \mid f(n) < y\} = F(y), \quad (1.14)$$

where  $F(y) = P(\xi < y)$ .

**Theorem B.** *Let  $g \in \overline{\mathcal{M}}_q$  be such a function for which (1.6) holds and (1.7) is convergent. Let  $\underline{r}^{(N)}$  be a sequence satisfying (1.13). Then*

$$\lim_{N \rightarrow \infty} \frac{1}{M(N \mid \underline{r})} \sum_{n \in S_N(\underline{r}^{(N)})} g(n) = M(g). \quad (1.15)$$

**Theorem C.** *Let  $q = 2$ ,  $f \in \mathcal{A}_2$ ,  $f(2^j) = O(1)$  ( $j \in \mathbb{N}$ ),*

$$\eta_N = \frac{1}{N} \sum_{j=0}^{N-1} f(2^j), \quad B_N^2 := \frac{1}{4} \sum_{j=0}^{N-1} (f(2^j) - \eta_N)^2.$$

Assume that  $B_N \rightarrow \infty$ .

Let  $\rho_N \rightarrow 0$ , and  $k = k^{(N)}$  be such a sequence of integers for which

$$\left| \frac{k}{N} - 1/2 \right| < \rho_N \quad (1.16)$$

holds.

Then

$$\lim_{N \rightarrow \infty} \frac{1}{\binom{N}{k}} \# \left\{ n < 2^N \mid \frac{f(n) - k\eta_N}{B_N} < y, \alpha(n) = k \right\} = \Phi(y), \quad (1.17)$$

the convergence is uniform in  $y$ .

In [6] we continued our work and proved the following Theorems D, E.

Let

$$\eta_{N,k} := \frac{k}{N}, \quad \mathcal{E}_{N,k} = \{n < 2^N \mid \alpha(n) = k\}. \quad (1.18)$$

**Theorem D.** Let  $g \in \overline{\mathcal{M}}_2$  be such a function for which

$$\sum_{j=0}^{\infty} (1 - g(2^j)) \quad (1.19)$$

is convergent. Let

$$M_{\xi} := \prod_{j=0}^{\infty} ((1 - \xi) + g(2^j)\xi) \quad (0 < \xi < 1). \quad (1.20)$$

Let  $\delta > 0$  be a constant. Then

$$\lim_{N \rightarrow \infty} \max_{\delta \leq \frac{k}{N} \leq 1-\delta} \left| \frac{1}{\binom{N}{k}} \sum_{\substack{n \in \mathcal{E}_{N,k} \\ n \leq q^N}} g(n) - M_{\eta_{N,k}} \right| = 0. \quad (1.21)$$

**Theorem E.** Let  $f \in \mathcal{A}_2$ , such that  $\sum f(2^j)$ ,  $\sum f^2(2^j)$  are convergent. Let  $\xi_0, \xi_1, \dots$  be independent random variables,  $P(\xi_{\nu} = 0) = 1 - \eta$ ,  $P(\xi_{\nu} = f(2^{\nu})) = \eta$ ,  $\Theta = \sum_{j=0}^{\infty} \xi_j$ ,

$$F_{\eta}(y) := P(\Theta < y). \quad (1.22)$$

Then

$$\lim_{N \rightarrow \infty} \max_{\delta \leq \frac{k}{N} \leq 1-\delta} \sup_{y \in \mathbb{R}} \left| \frac{1}{\binom{N}{k}} \# \{n \in \mathcal{E}_{N,k}, f(n) < y\} - F_{\frac{k}{N}}(y) \right| = 0. \quad (1.23)$$

Here  $\delta > 0$  is an arbitrary small constant.

In [6] we mentioned that we would be able to prove

**Theorem F.** *Let  $f \in \mathcal{A}_2$ ,  $f(2^j) = O(1)$ . Let  $h_N \in \mathcal{A}_2$  be defined by  $h_N(2^j) := f(2^j) - \frac{1}{N}A_N$ ,  $A_N = \sum_{j=0}^{N-1} f(2^j)$ ,  $\sigma_N^2(\eta) := (1-\eta)\eta \sum_{j=0}^{N-1} h_N^2(2^j)$ .*

*Assume that  $\lim_{N \rightarrow \infty} \sigma_N(1/2) = \infty$ .*

*Let  $0 < \delta < 1/2$  be a constant. Then*

$$\lim_{N \rightarrow \infty} \sup_{\frac{k}{N} \in [\delta, 1-\delta]} \sup_{y \in \mathbb{R}} \left| \frac{1}{\binom{N}{k}} \# \left\{ n \in \mathcal{E}_{N,k} \mid \frac{f(n) - \frac{k}{N}A_N}{\sigma_N\left(\frac{k}{N}\right)} < y \right\} - \Phi(y) \right| = 0.$$

Here we shall prove that for the fulfilment of (1.23) the convergence of  $\sum f(2^j)$ , and of  $\sum f^2(2^j)$  is necessary. Namely we shall prove the following

**Theorem 1.** *Let  $f \in \mathcal{A}_2$ . Assume that there exists a sequence of integers  $k_N$ ,  $\frac{k_N}{N} \rightarrow \xi$  ( $N \rightarrow \infty$ ),  $0 < \xi < 1$  such that*

$$\lim_{N \rightarrow \infty} \sup_{y \in \mathbb{R}} \left| \frac{1}{\binom{N}{k_N}} \# \{n \in \mathcal{E}_{N,k_N}, f(n) < y\} - G(y) \right| = 0$$

*with a suitable distribution function  $G(y)$ . Then both of the series (1.9), (1.10) are convergent and  $G(y) = F_\xi(y)$ ,  $F_\xi(y)$  is defined in Theorem E.*

**1.3. Additive functions.** We say that  $f : \mathbb{N} \rightarrow \mathbb{R}$  is additive if  $f(mn) = f(m) + f(n)$  holds for every coprime pairs of integers. We say that  $g : \mathbb{N} \rightarrow \mathbb{C}$  is multiplicative, if  $g(1) = 1$ , and  $g(mn) = g(m) \cdot g(n)$ , whenever  $(m, n) = 1$ . Let  $\mathcal{A}, \mathcal{M}$  be the sets of additive, and multiplicative functions, let  $\overline{\mathcal{M}} = \{g \in \mathcal{M} \mid |g(n)| = 1 \ (n \in \mathbb{N})\}$ . For the sake of brevity we shall write  $x_1 = \log x$ ,  $x_2 = \log x_1, \dots$ .

Let  $\Omega(n)$  = number of distinct prime powers of  $n$ ,  $\mathcal{N}_k = \{n \mid \Omega(n) = k\}$ ,

$$N_k(x) := \#\{n \leq x, n \in \mathcal{N}_k\}, \quad N_k(x \mid D) := \#\{n \leq x \mid (n, D) = 1, n \in \mathcal{N}_k\}.$$

According to a classical theorem of Erdős and Wintner, if  $f \in \mathcal{A}$  and the following three series

$$\sum_{|f(p)| < 1} \frac{f(p)}{p}, \quad \sum_{|f(p)| < 1} \frac{f^2(p)}{p}, \quad \sum_{|f(p)| \geq 1} 1/p \quad (1.24)$$

are convergent, then

$$\lim_x \frac{1}{x} \# \{n \leq x \mid f(n) < y\} = F(y) \quad (1.25)$$

exists at every continuity points of  $F$ , where  $F$  is a distribution function. They proved also that the convergence of the series in (1.24) is necessary for the existence of satisfying (1.25).

In [6] we proved the following two theorems.

**Theorem G.** Assume that  $f \in \mathcal{A}$ , the series (1.24) are convergent and  $k = k(x)$  satisfies the inequality

$$\left| \frac{k}{x_2} - 1 \right| < \delta_x, \quad (1.26)$$

where  $\delta_x \downarrow 0$ . Then

$$\lim_{x \rightarrow \infty} \frac{1}{N_k(x)} \# \{n \leq x, n \in \mathcal{N}_k, f(n) < y\} = F(y), \quad (1.27)$$

where  $F(y)$  is defined by (1.25).

**Theorem H.** Let  $g \in \overline{\mathcal{M}}$ , and assume that

$$\sum_p \frac{1 - g(p)}{p} \quad (1.28)$$

is convergent. Let  $k = k(x)$  be such a sequence for which (1.26) is satisfied. Then

$$\frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} g(n) = (1 + o_x(1))M(g),$$

$$M(g) = \prod_p e_p, \quad e_p = (1 - 1/p) \left( 1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \dots \right).$$

Here we shall prove

**Theorem 2.** Let  $g$  be as in Theorem H satisfying the conditions formulated there. Let  $\delta > 0$  be a fixed constant,  $\xi_{k,x} := \frac{k}{x_2}$ . Let

$$M_\eta(g) := \prod_p e_p(\eta), \quad e_p(\eta) = \left( 1 - \frac{\eta}{p} \right) \left( 1 + \frac{g(p)\eta}{p} + \frac{g(p^2)\eta^2}{p^2} + \dots \right).$$

We have

$$\lim_{x \rightarrow \infty} \sup_{\delta \leq \xi_{k,x} \leq 2-\delta} \left| \frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} g(n) - M_{\xi_{k,x}}(g) \right| = 0.$$

**Theorem 3.** Let  $f \in \mathcal{A}$ ,  $f(p^\alpha) = O(1)$  if  $p \in \mathcal{P}$ , and  $\alpha \in \mathbb{N}$ . Let  $A_x = \sum_{p \leq x} \frac{f(p)}{p}$ ,  $f^*(p^\alpha) = (p^\alpha) - \frac{\alpha}{x_2} A_x$ ,  $B_x^2 = \sum_{p \leq x} \frac{1}{p} (f^*(p))^2$ . Assume that  $f^*$  is extended to  $\mathbb{N}$  so that  $f^* \in \mathcal{A}$ . Let  $B_x \rightarrow \infty$ . Let  $\xi_{k,x} := \frac{k}{x_2}$ ,  $\delta \in (0, 1/2)$  be a constant. Then

$$\lim_{x \rightarrow \infty} \max_k \max_{\xi_{k,x} \in [\delta, 2-\delta]} \max_{y \in \mathbb{R}} \left| \frac{1}{N_k(x)} \# \left\{ n \leq x \mid \frac{f^*(n)}{B_x \sqrt{\xi_{k,x}}} < y, n \in \mathcal{N}_k \right\} - \Phi(y) \right| = 0.$$

**Theorem 4.** Assume that the conditions of Theorem 3 hold true. Let  $\delta, A$  be positive constants, so that  $0 < \delta < 1/2$ ,  $A > 2 + \delta$ . Then

$$\lim_{x \rightarrow \infty} \max_{\xi_{k,x} \in [2+\delta, A]} \max_{y \in \mathbb{R}} \left| \frac{1}{N_k(x)} \# \left\{ n \leq x \mid \frac{f^*(n)}{B_x \sqrt{2}} < y \right\} - \Phi(y) \right| = 0.$$

**Theorem 5.** Let  $f \in \mathcal{A}$ , and assume that the 3 series in (1.24) are convergent. For some  $\eta \in (0, 2)$  and  $p \in \mathcal{P}$  let  $\xi_p = \xi_p(\eta)$  be the random variable distributed by  $P(\xi_p = f(p^\alpha)) = \left(1 - \frac{\eta}{p}\right) \left(\frac{\eta}{p}\right)^\alpha$  ( $\alpha = 0, 1, 2, \dots$ ). Assume that  $\xi_p (p \in \mathcal{P})$  are completely independent,  $\Theta(\eta) := \sum \xi_p(\eta)$ .

Let  $F_\eta(y) := P(\Theta(\eta) < y)$ . Let furthermore

$$F_{k,x}(y) := \frac{1}{N_k(x)} \# \{n \leq x, n \in \mathcal{N}_k, f(n) < y\}.$$

Let  $0 < \delta < 1/2$ .

Then

$$\lim_{x \rightarrow \infty} \max_{\xi_{k,x} \in [\delta, 2-\delta]} \sup_{y \in \mathbb{R}} |F_{k,x}(y) - F_{\xi_{k,x}}(y)| = 0.$$

**Theorem 6.** Let  $g \in \overline{\mathcal{M}}$ , (1.28) is convergent. Assume furthermore that  $g(2^\alpha) = 1$  ( $\alpha = 1, 2, \dots$ ). Let  $A > 2 + \delta$  be constants. In the notations of Theorem 4 we have

$$\lim_{x \rightarrow \infty} \sup_{2+\delta \leq \xi_{k,x} \leq A} \left| \frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} g(n) - M_2^*(g) \right| = 0,$$

where

$$M_2^*(g) = \prod_{p>2} e_p(2).$$

**Theorem 7.** Let  $f \in \mathcal{A}$  be as in Theorem 7. Assume furthermore that  $f(2^\alpha) = 0$  ( $\alpha = 1, 2, \dots$ ). Then

$$\lim_{x \rightarrow \infty} \max_{2+\delta \leq \xi_{k,x} \leq A} |F_{k,x}(y) - F_2^*(y)| = 0,$$

where

$$F_2^*(y) = P\left(\sum_{p>2} \xi_p(2) < y\right).$$

Here  $\xi_{k,x} = \frac{k}{x_2}$ .

*Remark.* In Theorems 6 and 7 we have to assume something on the values  $g(2^\alpha)$  and on  $f(2^\alpha)$ , since for the function  $\nu(n)$  defined by  $2^{\nu(n)} \parallel n$ ,

$$\lim_{x \rightarrow \infty} \frac{1}{N_k(x)} \#\{n \leq x, \nu(n) < c, n \in \mathcal{N}_k\} = 0$$

for every fixed  $c$ .

In the proof of some of the theorems we use the following analogue of the Turán–Kubilius inequality.

**Theorem 8.** Let  $f \in \mathcal{A}$ ,  $A_x = \sum_{p \leq x} \frac{f(p)}{p}$ ,  $\tilde{B}_x^2(\eta) := \sum_{p^\alpha \leq \sqrt{x}} \frac{f^2(p^\alpha) \eta^{2\alpha}}{p^\alpha}$ . Assume that  $f(p^\alpha) = 0$  if  $p^\alpha > x^{1/4}$  or if  $p \in \mathcal{P}$  and  $\alpha > \sqrt{x_2}$ . Let  $\delta > 0$  be a constant,  $\xi_{k,x} := \frac{k}{x_2}$ . Then

$$\frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} (f(n) - \xi_{k,x} A_x)^2 \leq c \tilde{B}_x^2(\xi_{k,x}), \quad (1.29)$$

if  $\xi_{k,x} \in [\delta, 2 - \delta]$ . Here  $c$  is an absolute constant.

**Theorem 9.** Let  $f$  be as in Theorem 8. Assume that  $f(2^\alpha) = 0$  ( $\alpha = 1, 2, \dots$ ). Let  $\delta$  and  $A > 2 + \delta$  be constants. Then, for  $(2 + \delta)x_2 \leq k \leq Ax_2$ ,

$$\frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} (f(n) - 2A_x)^2 \leq c \tilde{B}_x^2(2), \quad (1.30)$$

where  $c$  is a constant that may depend on  $\delta$  and  $A$ .

*Remark.* In Theorem 9

$$\tilde{B}_x^2(2) = \sum_{\substack{p > 2 \\ p^\alpha \leq \sqrt{x}}} \frac{f^2(p^\alpha)}{p^\alpha}.$$

## §2. Some lemmas and proof of Theorem 1

Let  $f \in \mathcal{A}_2$ , and

$$Q_{k,N}(D) := \sup_{y \in \mathbb{R}} \#\{n \in \mathcal{E}_{N,k}, f(n) \in [y, y + D]\}.$$

**Lemma 1.** *Let  $D > 0$  be fixed. If  $\limsup_j |f(2^j)| = \infty$ , then*

$$\max_{\delta \leq k/N \leq 1-\delta} \frac{Q_{k,N}(D)}{\binom{N}{k}} \rightarrow 0 \quad (N \rightarrow \infty).$$

PROOF. By changing the sign of  $f$ , if needed, we may assume that  $\limsup f(2^j) = \infty$ .

Let  $l_1 < l_2 < \dots$  be such a sequence of integers for which:  $2D \leq f(2^{l_1})$ ,  $f(2^{l_{h+1}}) \geq 2f(2^{l_h})$ .

Let  $N$  be a large integer,  $T$  be defined such that  $l_T \leq N-1 < l_{T+1}$ . Let

$$U = \{l_1, l_2, \dots, l_T\}, \quad V = \{0, 1, \dots, N-1\} \setminus U.$$

Let

$$\alpha_1(n) = \sum_{s \in V} \varepsilon_s(n), \quad \alpha_2(n) = \sum_{t \in U} \varepsilon_t(n),$$

$$\mathcal{E}_h := \{n \in \mathcal{E}_{k,N}, \alpha_2(n) = h\}, \quad h = 0, 1, \dots, T.$$

Then

$$\mathcal{E}_{N,k} = \bigcup_{h=0}^T \mathcal{E}_h.$$

Assume that  $h \geq 1$ . Then

$$\mathcal{E}_h = \bigcup_{a_1, a_2, \dots, a_h} \mathcal{E}_h^{(a_1, \dots, a_h)},$$

where  $a_1, a_2, \dots, a_h$  run over all strictly monotonic sequences of length  $h$  from the set  $U$ ,

$$\mathcal{E}_h^{(a_1, \dots, a_h)} := \{n \in \mathcal{E}_{k,N}; \varepsilon_{a_\nu}(n) = 1$$

$$\text{if } \nu = 1, \dots, h; \varepsilon_b(n) = 0 \text{ if } b \in U \setminus \{a_1, \dots, a_h\}\}.$$

If  $n \in \mathcal{E}_h^{(a_1, \dots, a_h)}$ , then  $n = m + \rho_h$ , where

$$\rho_h = \sum_{\nu=1}^h 2^{a_\nu}, \quad m = \sum_{\substack{j=0 \\ j \in V}}^{N-1} \delta_j \cdot 2^j, \quad (\delta_j \in \{0, 1\}).$$

It is clear that  $|f(\rho_h^{(1)}) - f(\rho_h^{(2)})| > D$  if  $\rho_h^{(1)} \neq \rho_h^{(2)}$ .

Let  $y$  and  $h$  be fixed. Then, for a fixed  $m$ , no more than one  $\rho_h$  may exist for which  $f(\rho_h + m) \in [y, y + D]$ .

Thus

$$\#\{n \in \mathcal{E}_h \mid f(n) \in [y, y + D]\} \leq \binom{N-T}{k-h}.$$

This inequality holds for  $h = 0$  as well.

We have

$$\frac{\binom{N-T}{k-h}}{\binom{N}{k}} = \frac{(N-T)!k!(N-k)!}{N!(k-h)!(N-T-(k-h))!}.$$

It is clear that, if  $\{l_\nu\}$  satisfies the conditions stated above, then these conditions hold for every infinite subsequence of it. Therefore we may assume that  $T^2/N \rightarrow 0$  as  $N \rightarrow \infty$ , whence we can deduce that

$$\frac{\binom{N-T}{k-h}}{\binom{N}{k}} = (1 + o_N(1)) \frac{k^h \cdot (N-k)^{T-h}}{N^T},$$

and so

$$\begin{aligned} \frac{Q_{k,N}(D)}{\binom{N}{k}} &\leq (1 + o_N(1)) \sum_{h=0}^T \left(\frac{k}{N}\right)^h \left(1 - \frac{k}{N}\right)^{T-h} \\ &\leq cT \max \left\{ \left(1 - \frac{k}{N}\right)^T, \left(\frac{k}{N}\right)^T \right\} \leq cT(1-\delta)^T \rightarrow 0 \quad \text{as } T \rightarrow \infty. \end{aligned}$$

The proof of Lemma 1 is complete.  $\square$

**Lemma 2.** Let  $f \in \mathcal{A}_2$ ,  $f(2^j) = O(1)$ ,  $h_N \in \mathcal{A}_2$ ,

$$h_N(2^j) := f(2^j) - \frac{1}{N} A_N, \quad A_N = \sum_{j=0}^{N-1} f(2^j), \quad B_N^2 = \sum_{j=0}^{N-1} h_N^2(2^j).$$

Assume that  $\limsup_{N \rightarrow \infty} B_N^2 = \infty$ . Then

$$\lim_{N \rightarrow \infty} \max_{\frac{k}{N} \in [\delta, 2-\delta]} \frac{Q_{k,N}(D)}{\binom{N}{k}} = 0.$$

PROOF. The assertion is clear from Theorem F.  $\square$

PROOF OF THEOREM 1. Assume that the conditions hold. Then  $Q_{k_N,N}(D) > c \binom{N}{k_N}$  with  $c > 0$ , if  $\frac{k_N}{N} \in (\delta, 1-\delta)$ . Thus  $f(2^j) = O(1)$ , and  $B_N^2$  is bounded. One can prove simply that

$$\frac{1}{\binom{N}{k_N}} \sum_{\substack{n < 2^N \\ n \in \mathcal{E}_{N,k_N}}} h_N^2(n) = \frac{k_N}{N} \cdot \frac{(N-k_N)}{(N-1)} B_N^2, \quad (2.1)$$

whence

$$\frac{1}{\binom{N}{k_N}} \#\{n \in \mathcal{E}_{N,k_N} \mid |h_N(n)| > \Delta\} < \frac{c(\delta)}{\Delta^2}, \quad (2.2)$$

where  $c(\delta)$  is a constant, and  $\Delta$  is an arbitrary positive number. If  $f$  has a limit distribution on  $\mathcal{E}_{N,k_N}$ , then

$$\limsup_{N \rightarrow \infty} \frac{1}{\binom{N}{k_N}} \#\{n \in \mathcal{E}_{N,k_N} \mid |f(n)| > \Delta\} \leq \varepsilon(\Delta), \quad (2.3)$$

where  $\varepsilon(\Delta) \rightarrow 0$  as  $\Delta \rightarrow \infty$ .

From (2.2), (2.3) we obtain that  $|h_N(n) - f(n)| \leq 2\Delta$  holds for at least  $(1 - 2\varepsilon(\Delta) - \frac{c(\delta)}{\Delta^2})\binom{N}{k_N}$  integers  $n \in \mathcal{E}_{N,k_N}$ , whence we obtain that  $A_N = O(1)$ . Thus  $\sum f^2(2^j) < \infty$  holds.

Let  $M < N$ ,  $A_{M,N} = A_N - A_M$ .

Let  $0 < \eta < 1$ ,  $\xi_i(\eta)$  be independent random variables,

$$P(\xi_i(\eta)) = -\eta f(2^i) = 1 - \eta, \quad P(\xi_i(\eta) = (1 - \eta)f(2^i)) = \eta,$$

$$\Theta_M(\eta) := \xi_0(\eta) + \xi_1(\eta) + \dots + \xi_{M-1}(\eta).$$

Since  $\sum f^2(2^i) < \infty$ , therefore  $P(\Theta_M(\eta) < z)$  converges weakly to a distribution function as  $M \rightarrow \infty$ .

Let

$$G_{M,\eta}(y) = P(\Theta_M(\eta) < y) \rightarrow G_\eta(y) = P(\Theta_\infty(\eta) < y).$$

Let  $\tau \in \mathbb{R}$ ,  $g(n) = e^{i\tau f(n)}$ ,  $g_M(n) = \prod_{j=0}^{M-1} g(\varepsilon_j(n)2^j)$ ,

$$h(n) = \tau f(n), \quad h_M^*(n) = \sum_{j=M}^{N-1} h(\varepsilon_j(n) \cdot 2^j),$$

$$u_M(n) := \sum_{j=M}^{N-1} h(\varepsilon_j(n) \cdot 2^j).$$

Repeating the simple computation used in [5], we can deduce that

$$\begin{aligned} \frac{1}{\binom{N}{k_N}} \sum_{n \in \mathcal{E}_{N,k_N}} (h_M^*(n) - \eta \tau A_{M,N})^2 &\leq c_1(\delta) \sum_{j=M}^{N-1} h^2(2^j) \\ &+ \frac{c_2(\delta)}{N} \sum_{i,j=M}^{N-1} |h(2^i)| \cdot |h(2^j)| \leq c_3(\delta) \sum_{j=M}^{N-1} h^2(2^j), \end{aligned}$$

with suitable constants  $c_j(\delta)$ ,  $j = 1, 2, 3$ .

We have

$$g(n) = g_M(n)e^{ih_M^*(n)} = g_M(n)e^{i\tau\eta A_{M,N}} + g_M(n)\left(e^{ih_M^*(n)} - e^{i\eta\tau A_{M,N}}\right),$$

whence  $|g(n) - g_M(n)e^{i\eta\tau A_{M,N}}| \leq |h_M^*(n) - \eta\tau A_{M,N}|$ , and in the notations

$$M_{N, \frac{k}{N}}(\tau) := \frac{1}{\binom{N}{k}} \sum_{\substack{n < 2^N \\ n \in \mathcal{E}_{N,k}}} g(n),$$

$$\varphi_{M,\eta}(\tau) = \prod_{l=0}^{M-1} (\eta e^{i\tau(1-\eta)f(2^l)} + (1-\eta)e^{-i\tau\eta f(2^l)}),$$

we obtain that

$$\left| M_{N, k_N/N}(\tau) - e^{i \frac{k_N}{N} \tau A_{M,N}} \cdot \frac{1}{\binom{N}{k_N}} \sum_{n < 2^N} g_M(n) \right| \leq c_4(\delta) |\tau| \sqrt{\sum_{j \geq M} f^2(2^j)}.$$

Arguing as in [5], we can deduce that

$$\begin{aligned} \frac{1}{\binom{N}{k}} \sum_{n \in \mathcal{E}_{N,k}} g_M(n) &= (1 + o_N(1)) \prod_{j=0}^{M-1} \left( \left(1 - \frac{k}{N}\right) + \frac{k}{N} \cdot g(2^j) \right) \\ &= (1 + o_N(1)) e^{i\tau \frac{k}{N} A_M} \varphi_{M, \frac{k}{N}}(\tau), \end{aligned}$$

thus

$$\left| M_{N, k_N/N}(\tau) - e^{i \frac{k_N}{N} \tau A_N} \varphi_{M, \frac{k_N}{N}}(\tau) \right| \leq o_N(1) + c_5(\delta) \varepsilon_M |\tau|,$$

where

$$\varepsilon_M^2 = \sum_{j=M}^{\infty} f^2(2^j), \quad \varepsilon_M \rightarrow 0 \quad \text{as } M \rightarrow \infty.$$

Let  $\psi_\eta(\tau) = \lim_{N \rightarrow \infty} M_{N, \frac{k_N}{N}}(\tau)$ . From the condition we know that  $\psi_\eta$  exists. Furthermore  $\lim_{N \rightarrow \infty} \varphi_{M, \frac{k_N}{N}}(\tau) = \varphi_{M,\eta}(\tau)$  obviously holds (due to  $\frac{k_N}{N} \rightarrow \eta$ ). Finally, we shall prove that  $\lim A_N$  exists.

Assume indirectly that  $\alpha = \liminf A_N$ ,  $\beta = \limsup A_N$ ,  $\alpha \neq \beta$ ,  $N_\nu \nearrow \infty$ ,  $R_\mu \rightarrow \infty$ ,  $A_{N_\nu} \rightarrow \alpha$  ( $\nu \rightarrow \infty$ ),  $R_\mu \rightarrow \beta$  ( $\mu \rightarrow \infty$ ). Then

$$\begin{aligned} &\left| M_{N_\nu, \frac{k_{N_\nu}}{N_\nu}}(\tau) - M_{R_\mu, \frac{k_{R_\mu}}{R_\mu}}(\tau) - e^{i \frac{k_{N_\nu}}{N_\nu} \tau A_{N_\nu}} \varphi_{M, \frac{k_{N_\nu}}{N_\nu}}(\tau) - e^{i \frac{k_{R_\mu}}{R_\mu} \tau A_{R_\mu}} \varphi_{M, \frac{k_{R_\mu}}{R_\mu}}(\tau) \right| \\ &\leq o_{\min(N_\nu, R_\mu)}(1) + c_6(\delta) \varepsilon_M |\tau|. \end{aligned}$$

It is clear that  $\varphi_{M,\lambda}$  is continuous uniformly in  $\lambda \in [\delta, 1 - \delta]$ , and  $\lim_{M \rightarrow \infty} \varphi_{M,\lambda}(\tau)$  is continuous as well. Hence we obtain that  $|e^{i\alpha\tau} - e^{i\beta\tau}| = 0$ . This holds only if  $\alpha = \beta$ .  
The proof is completed.  $\square$

### §3. Some useful lemmas

The following two lemmas can be found in [7], pages 59 and 60.

**Lemma 3** (Wintner, Fréchet–Shohat). *Let  $F_n(z)$  ( $n = 1, 2, \dots$ ) be a sequence of distribution functions. For each non-negative integer  $k$  let*

$$\alpha_k = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} z^k dF_n(z)$$

*exist.*

*Then there is a subsequence  $F_{n_j}(z)$ , ( $n_1 < n_2 < \dots$ ), which converges weakly to a limiting distribution  $F(z)$  for which*

$$\alpha_k = \int_{-\infty}^{\infty} z^k dF(z) \quad (k = 0, 1, 2, \dots).$$

*Moreover, if the set of moments  $\alpha_k$  determine  $F(z)$  uniquely, then as  $n \rightarrow \infty$  the distributions  $F_n(z)$  converge weakly to  $F(z)$ .*

**Lemma 4.** *In the notations of Lemma 3 let the series*

$$\phi(t) = \sum_{l=0}^{\infty} \alpha_l \frac{(it)^l}{l!}$$

*converge absolutely in a disc of complex  $t$ -values  $|t| < \tau$ ,  $\tau > 0$ .*

*Then the  $\alpha_k$  determine the distribution function  $F(u)$  uniquely. Moreover, the characteristic function  $\phi(t)$  of this distribution had the above representation in the disc  $|t| < \tau$ , and can be analytically continued into the strip  $|\operatorname{Im}(t)| < \tau$ .*

*Remark.* The proof of Lemma 3 can be found in [3], while the proof of Lemma 4 is given in [7], (Vol. I, page 60).

*Remark.* The characteristic function  $\varphi(t) = e^{-t^2/2}$  of the standard normal distribution can be written as

$$\varphi(t) = \sum_{l=0}^{\infty} \frac{\mu_{2l}(it)^{2l}}{2^l l!}, \quad \mu_{2l} = \frac{(2l)!}{2^l \cdot l!}$$

( $l = 0, 1, 2, \dots$ ). The expansion is absolute convergent on the whole complex plane.

**Lemma 5** (Newton–Girard formulas). *Let  $\mathcal{B}$  be a finite set of primes,  $M = \#\mathcal{B}$ ,  $\psi : \mathcal{B} \rightarrow \mathbb{R}$ ,*

$$E_l = (-1)^l \sum_{\substack{p_1 < \dots < p_l \\ p_\nu \in \mathcal{B}}} \psi(p_1) \dots \psi(p_l), \quad s_h = \sum_{p \in \mathcal{B}} \psi^h(p).$$

Then

$$E_1 + s_1 = 0$$

$$2E_2 + E_1 s_1 + s_2 = 0$$

$$\vdots$$

$$rE_r + E_{r-1}s_1 + \dots + E_1 s_{r-1} + s_r = 0 \quad (r = 1, 2, \dots, M).$$

We shall use some of the results from the book of TENENBAUM [4] (Part II., Chapter II. 6).

Let

$$\nu(z) = \frac{1}{\Gamma(z+1)} \prod_p \left(1 - \frac{z}{p}\right)^{-1} (1 - 1/p)^z$$

be defined in  $|z| < 2$ . Since  $\nu(z)$  is analytic in the open set  $|z| < 2$ , therefore

$$\nu(z) = \sum_{m=0}^{\infty} \frac{\nu^{(m)}(0)}{m!} z^m, \quad \left| \frac{\nu^{(m)}(0)}{m!} \right| \leq \frac{c}{(2 - \delta/2)^m}$$

with any  $\delta > 0$  and a suitable constant  $c = c(\delta)$ .

Let

$$b_m := \frac{\nu^{(m)}(0)}{m!}$$

$$Q_{0,k}(y) = \sum_{l=0}^{k-1} \frac{1}{l!} b_{k-1-l} y^l.$$

For some polynomial  $P(x) \in \mathbb{R}[x]$ ,  $P(x) = \sum u_l x^l$ , let  $\|P\|(x) = \sum |u_l| x^l$ .

We have

$$Q_{0,k}(y + \lambda) - Q_{0,k}(y) = \sum_{\mu=1}^{k-1} \frac{1}{\mu!} Q_{0,k}^{(\mu)}(y) \cdot \lambda^\mu,$$

and so

$$\sum_{\mu=1}^{k-1} \frac{1}{\mu!} \|Q_{0,k}^{(\mu)}\|(y) = \sum_{\mu=1}^{k-1} \frac{1}{\mu!} \sum_{k-1 \geq l \geq \mu} \frac{1}{(l-\mu)!} |b_{k-1-l}| y^{l-\mu}$$

$$= \sum_{t=0}^{k-2} \frac{1}{t!} \left( \sum_{l=t+1}^{k-1} \frac{1}{(l-t)!} |b_{k-1-l}| \right) y^t = \sum_{t=0}^{k-2} d_t \cdot y^t.$$

It is clear that  $d_t \leq \frac{c}{t!}$  with a suitable constant  $c$ .

We formulate the above assertion as

**Lemma 6.** *We have*

$$\sum_{\mu=1}^{k-1} \frac{1}{\mu!} \|Q_{0,k}^{(\mu)}\|(y) = \sum_{t=0}^{k-2} d_t y^t, \quad d_t < \frac{c}{t!}$$

with a suitable constant  $c$ .

Let

$$N_k^*(x) = \frac{x}{x_1} Q_{0,k}(x_2).$$

**Lemma 7.** *Let  $\delta$  satisfy  $0 < \delta < 1$ . Then, for  $x \geq 3$ ,  $1 \leq k \leq (2 - \delta)x_2$*

$$N_k(x) = N_k^*(x) + O_\delta \left( \frac{x_2}{k} N_k^*(x) \cdot \frac{1}{x_1} \right).$$

(See TENENBAUM [4] Theorem 5 in p. 205.)

Let  $1 \leq D \leq x^{\varepsilon_x}$ , where  $0 < \varepsilon_x < 0, 1$ . Let  $\eta_D := \frac{\log D}{x_1}$ ,  $\Theta_D := \log(1 - \eta_D)$ ,

$$\psi_{k,D}(y) := \frac{1}{1 - \eta_D} \left\{ 1 + \Theta_D \cdot \frac{Q'_{0,k}(y)}{Q_{0,k}(y)} + \dots + \Theta_D^{k-1} \frac{Q_{0,k}^{(k-1)}(y)}{Q_{0,k}(y)} \right\}. \quad (3.1)$$

After easy computation we have

$$N_k^* \left( \frac{x}{D} \right) = \frac{N_k^*(x)}{D} \psi_{k,D}(x_2). \quad (3.2)$$

#### §4. Proof of Theorem 3

Assume that the conditions of Theorem 3 are satisfied.

Let  $h$  be completely additive,  $J_x = [K_x, x^{\varepsilon_x}]$ ,

$$h(p) = \begin{cases} \frac{f^*(p)}{B_x} & \text{if } p \in J_x \\ 0 & \text{if } p \notin J_x, \end{cases}$$

where  $\varepsilon_x \downarrow 0$ ,  $K_x \uparrow \infty$  so slowly that

$$\lim_{x \rightarrow \infty} \sup_{\frac{k}{x_2} \in [\delta, 2-\delta]} \frac{1}{N_k(x)} \# \left\{ n \in \mathcal{N}_k, n \leq x, \left| h(n) - \frac{f^*(n)}{B_x} \right| > \varepsilon \right\} = 0 \quad (4.1)$$

for each  $\varepsilon > 0$ .

$$\lim_{x \rightarrow \infty} \sup_{\frac{k}{x_2} \in [\delta, 2-\delta]} \frac{1}{N_k(x)} \# \{n \in \mathcal{N}_k \mid n \leq x, \exists p > K_x, p^2 \mid n\} = 0. \quad (4.2)$$

Let  $p < K_x$ , count those  $n \in \mathcal{N}_k$  for which  $p^\alpha \mid n$ . The size of those  $n$  is no more than

$$N_{k-\alpha} \left( \frac{x}{p^\alpha} \right) < \frac{cx}{p^\alpha x_1} \frac{x_2^{k-\alpha-1}}{(k-1-\alpha)!} \leq \frac{c_1(2-\delta/2)^{\alpha+1}}{p^\alpha} N_k(x), \quad (4.3)$$

assuming e.g. that  $p^\alpha \leq x_1$ . Hence we obtain that

$$\sup_{\frac{k}{x_2} \in [\delta, 2-\delta]} \frac{1}{N_k(x)} \# \left\{ n \in \mathcal{N}_k, \left| \sum_{\substack{p^\alpha \parallel n \\ p < K_x}} \frac{f^*(p^\alpha)}{B_x} \right| > \varepsilon \right\} \rightarrow 0 \quad (x \rightarrow \infty)$$

if  $K_x \uparrow \infty$  sufficiently slowly. Since the number of prime divisors  $p$  in  $(x^{\varepsilon_x}, x]$  of  $n$  is less than  $\frac{1}{\varepsilon_x}$  therefore (4.1) clearly holds.

(4.2) can be proved easily. We use (4.3) if  $K_x \leq p \leq x_1$  with  $\alpha = 2$ , and for  $p > x_1$  we use the obvious

$$\# \{n \in \mathcal{N}_k, n \leq x \mid \exists p^2 \mid n, p > x_1\} \leq \sum_{p > x_1} \frac{x}{p^2} \leq \frac{x}{x_1}$$

inequality.

Thus (4.2) is true.

We have

$$\frac{1}{B_x^2} \sum_{p < K_x} \frac{f^{*2}(p)}{p} \ll \frac{\log \log K_x}{B_x^2}, \quad \frac{1}{B_x^2} \sum_{x^{\varepsilon_x} < p < x} \frac{f^{*2}(p)}{p} \ll \frac{\log 1/\varepsilon_x}{B_x^2},$$

and so

$$\sum \frac{h^2(p)}{p} = 1 + H_x, \quad |H_x| \ll \frac{\log \log K_x + \log 1/\varepsilon_x}{B_x^2}. \quad (4.4)$$

Assuming that  $K_x$  and  $1/\varepsilon_x$  are increasing sufficiently slowly, we can and will assume that  $H_x \rightarrow 0$  ( $x \rightarrow \infty$ ).

To prove the theorem it is enough to show that for every  $r = 1, 2, \dots$ ,

$$\sup_{\frac{k}{x_2} \in [\delta, 2-\delta]} \left| \frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} \frac{h^r(n)}{\beta_k^r} - \mu_r \right| \rightarrow 0 \quad \text{as } x \rightarrow \infty,$$

and then apply the Frechet-Shohat theorem.

Let us consider the sum

$$U_{k,r}(x) := \frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} h^r(n). \quad (4.5)$$

Since  $h$  is completely additive, therefore

$$U_{k,r}(x) = \sum_{s=1}^r \sum_{l_1+\dots+l_s=r} \frac{c(r; l_1, \dots, l_s)}{N_k(x)} \sum_{p_1, p_2, \dots, p_s}^* h^{l_1}(p_1) \dots h^{l_s}(p_s) N_{k-s} \left( \frac{x}{p_1 \dots p_s} \right) \quad (4.6)$$

where star indicates that we sum over all those  $s$  tuples  $p_1, \dots, p_s$  of primes for which  $p_i \neq p_j$ , if  $i \neq j$ . Here  $c(r; l_1, \dots, l_s) = \frac{r!}{l_1! \dots l_s!}$ .

Let

$$V_{k,r}(x \mid l_1, \dots, l_s) = \frac{1}{N_{k-s}^*(x)} \sum_{p_1, \dots, p_s}^* h^{l_1}(p_1) \dots h^{l_s}(p_s) N_{k-s}^* \left( \frac{x}{p_1 \dots p_s} \right), \quad (4.7)$$

$$\tilde{U}_{k,r}(x) = \sum_{s=1}^r \frac{c(r; l_1, \dots, l_s) \cdot N_{k-s}^*(x)}{N_k(x)} V_{k,r}(x \mid l_1, \dots, l_s). \quad (4.8)$$

From Lemma 7 we can deduce simply that  $U_{k,r}(x) - \tilde{U}_{k,r}(x) \rightarrow 0$  ( $x \rightarrow \infty$ ) uniformly as  $\frac{k}{x} \in [\delta, 2-\delta]$ . We estimate  $V_{k,r}(x \mid l_1, \dots, l_s)$  by using (3.1), (3.2) with  $D = p_1 \dots p_s$ . We can write  $\psi_{k,D}(y)$  as a convergent power series of  $\eta_D$ .

We try to estimate

$$E(l_1, t_1; l_2, t_2; \dots; l_s, t_s) := \sum_{p_1, \dots, p_s}^* \frac{h^{l_1}(p_1)(\log p_1)^{t_1}}{p_1 x_1^{t_1}} \dots \frac{h^{l_s}(p_s) \cdot (\log p_s)^{t_s}}{p_s x_1^{t_s}}. \quad (4.9)$$

Let

$$\kappa(l, t) := \frac{1}{x_1^t} \sum_{p \in J_x} \frac{h^l(p)(\log p)^t}{p} \quad (l = 1, 2, \dots; t = 0, 1, \dots). \quad (4.10)$$

From (4.4) we have

$$\kappa(2, 0) = 1 + H_x, \quad |H_x| < \frac{c \log \log K_x + \log 1/\varepsilon_x}{B_x^2}. \quad (4.11)$$

We have

$$\begin{aligned} \kappa(1, 0) &= \frac{1}{B_x} \sum_{p \in J_x} \frac{1}{p} \left( f(p) - \frac{A_x}{x_2} \right) = \frac{1}{B_x} \sum_{p < x_2} \frac{1}{p} \left( f(p) - \frac{A_x}{x_2} \right) \\ &\quad - \frac{1}{B_x} \sum_{p < B_x} \frac{1}{p} \left( f(p) - \frac{A_x}{x_2} \right) - \frac{1}{B_x} \sum_{x^{\varepsilon_x} \leq p < x} \frac{1}{p} \left( f(p) - \frac{A_x}{x_2} \right) \\ &= \sum_1 - \sum_x - \sum_3. \end{aligned}$$

Since  $x_2 - \sum_{p < x_2} 1/p = O(1)$ , therefore

$$\sum_1 = \frac{1}{B_x} \left( A_x - \frac{A_x}{x_2} \sum_{p < x_2} 1/p \right) = \frac{A_x}{x_2 B_x} \left( x_2 - \sum_{p < x_2} 1/p \right) = O\left(\frac{1}{B_x}\right).$$

Furthermore

$$\sum_1 = O\left(\frac{\log \log K_x}{B_x}\right), \quad \sum_2 = O\left(\frac{\log 1/\varepsilon_x}{B_x}\right).$$

Consequently

$$|\kappa(1, 0)| \leq \frac{c(\log \log K_x + \log 1/\varepsilon_x)}{B_x}$$

with a suitable constant  $c$ .

It is known that

$$\sum_{p < y} \frac{(\log p)^s}{p} < c \frac{(\log y)^s}{s}$$

for  $s \geq 1$ .

Let  $\Lambda_x$  be defined by

$$\Lambda_x := \frac{c(\log \log K_x + \log 1/\varepsilon_x)}{B_x} + \frac{r}{K_x} \geq |\kappa(1, 0)| + \frac{r}{K_x}. \quad (4.12)$$

It is known that

$$\sum_{p < y} \frac{(\log p)^s}{p} < c \frac{(\log y)^s}{s},$$

for  $s \geq 1$ .

Hence, by using the Cauchy–Schwarz inequality,

$$\kappa(1, t) \leq \left( \sum_{p \in J_x} \frac{h^2(p)}{p} \right)^{1/2} \left( \frac{1}{x_1^{2t}} \sum_{p \leq x^{\varepsilon_x}} \frac{(\log p)^{2t}}{p} \right) \leq \frac{c\varepsilon_x^t}{\sqrt{t}}. \quad (4.13)$$

For  $l \geq 2$ ,  $t \geq 1$

$$|\kappa(l, t)| \leq c\varepsilon_x^t \kappa(l, 0), \quad (4.14)$$

$$|\kappa(l, 0)| \leq c \left( \frac{1}{B_x} \right)^{l-2}. \quad (4.15)$$

Assume first that there exists at least one  $(l_j, t_j) = (1, 0)$ . Assume that  $(l_j, t_j) = (1, 0)$  if  $j = 1, \dots, h$  and  $(l_j, t_j) \neq (1, 0)$  if  $j > h$ . We have

$$\begin{aligned} & E(l_1, t_1; \dots; l_s, t_s) \\ &= \sum_{p_{h+1}, \dots, p_s}^* \left( \prod_{\nu=h+1}^s \frac{h^{l_\nu}(p_\nu)}{p_\nu} \cdot \frac{(\log p_\nu)^{t_\nu}}{x_1^{t_\nu}} \right) \left\{ \sum_{p_1, \dots, p_h}^{**} \frac{h(p_1)}{p_1} \dots \frac{h(p_h)}{p_h} \right\} \end{aligned}$$

where  $*$  means that  $p_{h+1}, \dots, p_s$  are distinct primes, and  $**$  means that  $p_1, \dots, p_h$  are distinct primes, none of them belongs to the set  $\{p_{h+1}, \dots, p_s\}$ . First we estimate the inner sum. Let us apply Lemma 5 with  $\mathcal{B} = \{p \mid p < x^{\varepsilon_x}\} \setminus \{p_{h+1}, \dots, p_s\}$ ,  $\psi(p) = \frac{h(p)}{p}$ . In the notation of Lemma 5  $\sum_{p_1, \dots, p_h}^{**} \frac{h(p_1)}{p_1} \dots \frac{h(p_h)}{p_h} = (-1)^h h! E_h$ . Since  $|E_1| = \left| \sum_{p \in \mathcal{B}} \frac{h(p)}{p} \right| \leq \Lambda_x$  (see (4.12)), from the Newton–Girard formulas (by using induction on  $h$  e.g.) we obtain that  $|E_h| \leq c\Lambda_x$ , where  $c$  is a constant that may depend on  $r$  at most.

Thus

$$E(l_1, t_1; \dots; l_s, t_s) \leq c\Lambda_x \kappa(l_{h+1}, t_{h+1}) \dots \kappa(l_s, t_s).$$

By the inequalities (4.13), (4.14), (4.15) we have

$$E(l_1, t_1; \dots; l_s, t_s) \leq c_1 \Lambda_x \varepsilon_x^{t_1 + \dots + t_s} \prod_{l_j \geq 2} \left( \frac{1}{B_x} \right)^{l_j - 2}. \quad (4.16)$$

$c_1$  is a constant which may depend on  $r$ .

Similarly, if  $(l_j, t_j) \neq (1, 0)$  holds for every  $j$ , then

$$E(l_1, t_1, \dots, l_s, t_s) \leq c_1 \varepsilon_x^{t_1 + \dots + t_s} \prod_{l_j \geq 2} \left( \frac{1}{B_x} \right)^{l_j - 2}. \quad (4.17)$$

We can observe that the right hand side of (4.16), (4.17) tends to zero except the case, when for every  $j$ ,  $(l_j, t_j) = (2, 0)$ . This can be happen only if  $r = 2R$  is even. Observe that

$$E(2, 0; \dots, 2, 0) = \sum_{p_1, \dots, p_R}^* \frac{h^2(p_1)}{p_1} \dots \frac{h^2(p_R)}{p_R},$$

and hence we can deduce easily that

$$E(2, 0; \dots; 2, 0) = \kappa(2, 0)^R + o_x(1) = 1 + o_x(1). \quad (4.18)$$

Let us go back to (4.7). See furthermore (3.1):

$$V_{k,r}(x \mid l_1, \dots, l_s) = \sum_{p_1, \dots, p_s}^* \frac{h^{l_1}(p_1) \dots h^{l_s}(p_s)}{p_1 \dots p_s} T_{k-s}(\eta_{p_1 \dots p_s}),$$

where

$$\begin{aligned} T_{k-s}(W) &= \frac{1}{1-W} \left\{ 1 + \log(1-W) \cdot S_1 + \log^2(1-W) S_2 + \dots \right. \\ &\quad \left. + \log^{k-s-1}(1-W) \cdot S_{k-s-1} \right\} \\ S_j &:= \frac{Q_{0,k-s-1}^{(j)}(x_2)}{Q_{0,k-s-1}(x_2)}. \end{aligned}$$

Let

$$V_{k,r}^{(T)}(x \mid l_1, \dots, l_s) = \sum_{p_1, \dots, p_s}^* \frac{h^{l_1}(p_1) \dots h^{l_s}(p_s)}{p_1 \dots p_s} \left( \frac{\log p_1 \dots p_s}{x_1} \right)^T.$$

Then

$$V_{k,r}^{(T)}(x \mid l_1, \dots, l_s) = \sum_{t_1 + \dots + t_s = T} \frac{T!}{t_1! \dots t_s!} E(l_1, t_1; \dots, l_s, t_s).$$

In the case  $T = 0$  it was already proved that  $V_{k,r}^{(0)}(x \mid l_1, \dots, l_s) = o_x(1)$ , except the case when  $l_1 = l_2 = \dots = l_s = 2$ ,  $s = R$ ,  $r = 2R$ , when  $V_{k,2R}^{(0)}(x \mid 2, \dots, 2) = 1 + o_x(1)$ .

Let now  $T \geq 1$ . From (4.16), (4.17) we obtain that

$$V_{k,r}^{(T)}(x \mid l_1, \dots, l_s) \leq c\varepsilon_x^T. \quad (4.19)$$

Let  $u(w) = p_0 + p_1 w + \dots$  be a power series with nonnegative coefficients, and assume that it converges in the disc  $|w| < 1$ .

Since

$$\sum_{p_1, \dots, p_s}^* \frac{h^{l_1}(p_1) \dots h^{l_s}(p_s)}{p_1 \dots p_s} u\left(\frac{\log p_1 \dots p_s}{x_1}\right) = \sum_{T=0}^{\infty} p_T V_{k,r}^{(T)}(x \mid l_1, \dots, l_s),$$

from (4.19) we obtain that the left hand side of (4.20) is less than

$$\leq \sum_{T=0}^{\infty} p_T c \varepsilon_x^T = cu(\varepsilon_x).$$

Since the coefficients of the Taylor expansion of  $\frac{w}{1-w}$  and of  $(-1)^j \frac{(\log(1-w))^j}{1-w}$  is positive, and they converge for  $|w| < 1$ , therefore

$$\left| \sum_{p_1, \dots, p_s}^* \frac{h^{l_1}(p_1) \dots h^{l_s}(p_s)}{p_1 \dots p_s} u\left(\frac{\log p_1 \dots p_s}{x_1}\right) \right| \leq cu(\varepsilon_x)$$

holds, if

$$u(w) = \frac{(-1)^j \log^j(1-w)}{1-w}, \quad j = 1, \dots, k-s-1,$$

and if

$$u(w) = \frac{w}{1-w},$$

$l_1, \dots, l_s$  arbitrary, and in the case  $u(w) = 1$ ,  $(l_1, \dots, l_s) \neq (2, \dots, 2)$  the left hand side tends to 0.

Consequently, by Lemma 6  $V_{k,r}(x \mid l_1, \dots, l_s) \rightarrow 0$  ( $x \rightarrow \infty$ ) if  $(l_1, \dots, l_s) \neq (2, \dots, 2)$ , while for  $s = R$ ,  $r = 2R$ ,

$$V_{k,2R}(x \mid 2, \dots, 2) = 1 + o_x(1).$$

We are almost ready. We have to observe only that

$$\frac{N_{k-s}^*(x_2)}{N_k^*(x_2)} = \frac{k(k-1) \dots k-(s-1)}{x_2^s} = (1 + o_x(1)) \xi_{k,x_2}^s.$$

The proof is complete.

## §5. Proof of Theorem 4

**Theorem 10.** *Let  $0 < \delta$ ,  $A > 2 + \delta$  be constants. Then for all  $k \in [(2 + \delta)x_2, Ax_2]$  we have*

$$N_k(x) = \frac{cx_1}{2^k} \left\{ 1 + O_A(x_1^{-\delta^2/5}) \right\}.$$

See [4].

To prove the theorem we can use the argument of the proof of Theorem 5. Instead of (3.1), (3.2) we can use the formula

$$N_k^*(x) = \frac{c x x_1}{2^k}, \quad N_k^*\left(\frac{x}{D}\right) = \frac{1}{D} N_k^*(x) \left(1 - \frac{\log D}{x_1}\right).$$

We omit the details.

### §6. Proof of Theorem 8

If (1.29) holds for  $f_1, f_2 \in \mathcal{A}$ , then it holds for  $f = f_1 + f_2$ . Let  $\gamma < 1/4$  be a small positive constant,  $f_1(p^\alpha) = f(p^\alpha)$  if  $p^\alpha < x^\gamma$ , and  $f_1(p^\alpha) = 0$  if  $p^\alpha \geq x^\gamma$ , and let  $f_2(p^\alpha) = f(p^\alpha) - f_1(p^\alpha)$ .

We have

$$\begin{aligned} S &:= \sum_{\substack{n \in N_k \\ n \leq x}} f_2^2(n) \leq \sum_{p_1 \neq p_2} |f_2(p_1^{\alpha_1})| \cdot |f_2(p_2^{\alpha_2})| \cdot N_{k-\alpha_1-\alpha_2}\left(\frac{x}{p_1^{\alpha_1} p_2^{\alpha_2}}\right) \\ &\quad + \sum_{p^\alpha} |f_2(p^\alpha)| \cdot N_{k-\alpha}\left(\frac{x}{p^\alpha}\right). \end{aligned}$$

From the conditions of the theorem  $f_2(p_i^{\alpha_i}) = f(p_i^{\alpha_i}) = 0$  if  $p_i^{\alpha_i} > x^{1/4}$ , or if  $\alpha_i > \sqrt{x_2}$ .

Assume that  $p_i^{\alpha_i} \leq x^{1/4}$  and  $\alpha_i \leq \sqrt{x_2}$ . Then

$$N_{k-\alpha_1-\alpha_2}\left(\frac{x}{p_1^{\alpha_1} p_2^{\alpha_2}}\right) \leq \frac{c N_k(x)}{p_1^{\alpha_1} p_2^{\alpha_2}} \xi_{k,x}^{\alpha_1+\alpha_2}, \quad (6.1)$$

( $c$  is an absolute constant) and we deduce that

$$\frac{1}{N_k(x)} S \leq c \left( \sum_{x^\gamma < p^\alpha \leq x^{1/4}} \frac{|f_2(p^\alpha)|}{p^\alpha} \xi_{k,x}^\alpha \right)^2 + c \tilde{B}_x^2(\xi_{k,x}).$$

Since

$$\sum \frac{|f_2(p^\alpha)| \xi_{k,x}^\alpha}{p^\alpha} \leq \left( \sum \frac{\xi_{k,x}^\alpha}{p^\alpha} \right)^{1/2} \tilde{B}_x(\xi_{k,x}),$$

where in the right hand side  $x^\gamma < p^\alpha < x^{1/4}$ ,  $\alpha \leq \sqrt{x_2}$ , thus  $p = 2$  cannot occur.

Therefore

$$\sum_{p^\alpha, p \geq 2} \frac{\xi_{k,x}^\alpha}{p^\alpha} \text{ is bounded by an absolute constant, and so}$$

$$\frac{1}{N_k(x)} S \leq c \tilde{B}_x^2(\xi_{k,x}).$$

Let

$$U_x = \sum \frac{f_2(p)}{p} = \sum_{x^\gamma \leq p < x^{1/4}} \frac{f(p)}{p}.$$

Since

$$\frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} (f_2(n) - U_x)^2 \leq \frac{2}{N_k(x)} S + 2|U_x|^2,$$

and

$$|U_x|^2 \leq \left\{ \sum_{x^\gamma < p < x^{1/4}} \frac{1}{p} \right\} \tilde{B}_x^2\left(\frac{k}{x_2}\right),$$

therefore (1.29) holds for  $f_2$ .

Let now  $f_3$  be defined on prime powers  $p^\beta$  such that  $f_3(p^\beta) = f_1(p^\alpha) - f_1(p^{\alpha-1})$  ( $\alpha = 1, 2, \dots$ ). Then, with the classical meaning of summation,

$$f_1(n) = \sum_{p^\beta | n} f_3(p^\beta).$$

Let

$$f_4(n) = \sum_{p|n} f_3(p), \quad f_5(n) = \sum_{\substack{p^\beta | n \\ \beta \geq 2}} f_3(p^\beta).$$

Let us estimate first

$$\begin{aligned} S_1 &:= \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} f_5(n)^2 = \sum_{\substack{p_1^{\alpha_1}, p_2^{\alpha_2} \\ p_1 \neq p_2}} f_3(p_1^{\alpha_1}) f_3(p_2^{\alpha_2}) N_{k-\alpha_1-\alpha_2} \left( \frac{x}{p_1^{\alpha_1} p_2^{\alpha_2}} \right) \\ &+ \sum_{\substack{p \\ \alpha_1, \alpha_2}} f_3(p^{\alpha_1}) f_3(p^{\alpha_2}) N_{k-\max(\alpha_1, \alpha_2)} \left( \frac{x}{p^{\max(\alpha_1, \alpha_2)}} \right) = S_2 + S_3. \end{aligned}$$

Since  $f_3(p_i^{\alpha_i}) = 0$ , if  $p_k^{\alpha_i} > x^\gamma$ , or if  $\alpha_i = 1$ , or  $\alpha_i > \sqrt{x_2}$ , from (6.1) we deduce that

$$\frac{S_2}{N_k(x)} \leq \left( \sum_{\alpha \geq 2} \frac{|f_3(p^\alpha)|}{p^\alpha} \xi_{k,x} \right)^2 + 2 \sum_{\alpha_1=2}^{\infty} \sum_{\alpha_2=2}^{\alpha_1} \sum_p \frac{|f_3(p^{\alpha_1}) f_3(p^{\alpha_2})|}{p^{\alpha_1}} \xi_{k,x}^{\alpha_1}.$$

The first sum on the right hand side is less than  $c\tilde{B}_x^2(\xi_{k,x})$ . To estimate the second sum we start from

$$\left| \xi_{k,x}^{\alpha_1} f_3(p^{\alpha_1}) f_3(p^{\alpha_2}) \right| \leq 2f_3^2(p_1^\alpha) \xi_{k,x}^{2\alpha_1} + f_3^2(p_2^\alpha)$$

and deduce that it is less than

$$4\tilde{B}_x(\xi_{k,x}) + 4 \sum_{\alpha_2=2}^{\infty} \sum_p \frac{|f_3(p_2^\alpha)|^2}{p^{\alpha_2}} \sum \frac{1}{1-1/p} \leq 4\tilde{B}_x^2(\xi_{k,x}) + 8\tilde{B}_x^2(1).$$

Thus

$$\frac{S_2}{N_k(x)} \leq c_1 \tilde{B}_x^2(\xi_{k,x}) + 8\tilde{B}_x^2(1).$$

Finally we prove that

$$T := \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} (f_4(n) - \xi_{k,x_2} A_x^*)^2 \leq c B_x^2 N_k(x),$$

$$A_x^* = \sum_{p < x^\gamma} \frac{f_4(p)}{p}. \quad (6.2)$$

Let  $\rho_x := \sum_{p < x^\gamma} 1/p$ .

Let  $\tilde{f}_4(p) = f_4(p) - \frac{A_x^*}{\rho_x}$ ,  $\tilde{f}_4(n) = \sum_{p|n} \tilde{f}_4(p)$ . Then  $\tilde{f}_4(n) = f_4(n) - \frac{k}{\rho_x} A_x^*$  if  $n \in \mathcal{N}_k$ .

Let

$$\tilde{T} = \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} \tilde{f}_4(n)^2. \quad (6.3)$$

We shall prove that

$$\tilde{T} \leq c N_k(x) \sum_{p \leq x} \frac{\tilde{f}_4^2(p)}{p}. \quad (6.4)$$

Hence (6.2) easily follows.

We have

$$\tilde{T} = \sum_{p_1 \neq p_2} \tilde{f}_4(p_1) \tilde{f}_4(p_2) N_{k-2} \left( \frac{x}{p_1 p_2} \right) + \sum_p \tilde{f}_4^2(p) N_{k-1} \left( \frac{x}{p} \right).$$

From Lemma 7 we obtain that

$$\tilde{T} = \tilde{T}_1 + \tilde{T}_2 + \text{error, where}$$

$$\begin{aligned}\tilde{T}_1 &= \sum_{p_1, p_2} \tilde{f}_4(p_1) \tilde{f}_4(p_2) N_{k-2}^* \left( \frac{x}{p_1 p_2} \right), \\ \tilde{T}_2 &= \sum \tilde{f}_4^2(p) \left( N_{k-1} \left( \frac{x}{p} \right) - N_{k-2} \left( \frac{x}{p_2} \right) \right),\end{aligned}$$

where the error is clearly less than  $c N_{k-2}(x) \sum \frac{\tilde{f}_4^2(p)}{p}$ .

Let

$$E_l := \sum_{p < x^\gamma} \frac{\tilde{f}_4(p)}{p} \frac{(\log p)^l}{x_1^l}.$$

It is clear that  $E_0 = 0$ , and

$$|E_l| \leq \left( \sum \frac{\tilde{f}_4^2(p)}{p} \right)^{1/2} \left( \sum_{p < x^\gamma} \frac{(\log p)^{2l}}{p x_1^{2l}} \right)^{1/2} \leq 2 \left( \sum \frac{\tilde{f}_4^2(p)}{p} \right)^{1/2} \frac{\gamma^l}{\sqrt{2l}},$$

if  $x > x_0$ , and  $l \geq 0$ . Thus

$$\left| \sum_{p_1, p_2} \frac{\tilde{f}_4(p_1)}{p_1} \frac{\tilde{f}_4(p_2)}{p_2} \frac{(\log p_1 p_2)^\nu}{x_1^\nu} \right| = \left| \sum_{l=0}^{\nu-1} E_l E_{\nu-l} \cdot \binom{\nu}{l} \right| \leq 4(2\gamma)^l. \quad (6.5)$$

We have

$$\frac{\tilde{T}_1}{N_{k-2}(x)} = \sum_{p_1, p_2} \frac{\tilde{f}_4(p_1)}{p_1} \frac{\tilde{f}_4(p_2)}{p_2} \psi_{k-2, p_1 p_2}(x_2),$$

where  $\psi_{k-2, p_1 p_2}$  is defined in (3.1). By using Lemma 6 and (6.5), furthermore that  $\tilde{T}_2 \ll \xi_{k,x} \sum \frac{\tilde{f}_4^2(p)}{p} N_k(x)$ , we get (6.4).

Since  $\tilde{f}_4^2(p) \leq 2f_4^2(p) + 2\frac{|A_x^*|^2}{\rho_x^2}$ , therefore

$$\sum \frac{\tilde{f}_4^2(p)}{p} \leq 2B_x^2 + 2\frac{|A_x^*|^2}{\rho_x}, \quad A_x^{*2} \leq \sum \frac{1}{p} B_x^2,$$

and so

$$\sum \frac{\tilde{f}_4^2(p)}{p} \leq cB_x^2.$$

Finally  $f_4(n) - \xi_{k,x_2} A_x^* = \tilde{f}_4(n) + \left( \frac{k}{\rho_x} - \xi_{k,x_2} \right) A_x^*$ , and so

$$T \leq 2\tilde{T} + \left| \frac{k}{\rho_x} - \xi_{k,x_2} \right|^2 |A_x^*|^2 N_k(x).$$

Furthermore  $|A_x^*|^2 \leq B_X^2 \rho_x$ , and so

$$\left| \frac{k}{\rho_x} - \frac{k}{x_2} \right|^2 |A_x^*|^2 \leq \rho_x \left| \frac{k(x_2 - \rho_x)}{\rho_x x_2} \right|^1 B_x^2 = o_x(1) B_x^2.$$

Thus (6.2) holds true.

The proof of the theorem is complete.

### §7. Proof of Theorem 9

We can argue similarly as in §6. Since now  $N_k(x) = N_k^*(x)(1 + O_A(x_1^{-\delta^2/5}))$  (Lemma 8),  $N_k^*\left(\frac{x}{D}\right) = \frac{1}{N} N_k^*(x) - \frac{\log D}{D x_1} N_k^*(x)$ , we obtain our theorem easier than that of Theorem 10.

We omit the details.

### §8. Proof of Theorem 5

Assume that the conditions of the theorem hold. Let  $\mathcal{B}$  be such a sequence of primes for which  $\sum_{p \in \mathcal{B}} 1/p < \infty$ . Let  $\rho(Y) := \sum_{\substack{bp > Y \\ p \in \mathcal{B}}} 1/p$ . Then  $\rho(Y) \rightarrow 0$  as  $Y \rightarrow \infty$ .

Count

$$S_Y := \#\{n \leq x \mid n \in \mathcal{N}_k, p \mid n \text{ for some } p > Y, p \in \mathcal{B}\}.$$

Then

$$\begin{aligned} S_Y &\leq \sum_{\substack{Y < p < x^{1-\delta_x} \\ p \in \mathcal{B}}} N_{k-1}\left(\frac{x}{p}\right) + \sum_{\substack{\nu < x^{\delta_x} \\ \nu \in \mathcal{N}_{k-1}}} \pi\left(\frac{x}{\nu}\right) \leq \\ &\leq N_{k-1}(x) \sum_{\substack{Y < p < x^{1-\delta_x} \\ p \in \mathcal{B}}} \frac{1}{p} \frac{\log x}{(\log x - \log p)} + \frac{3x}{x_1} \sum_{\substack{\nu < x^{\delta_x} \\ \nu \in \mathcal{N}_{k-1}}} 1/\nu \\ &\leq N_{k-1}(x) \cdot \frac{1}{\delta_x} \rho(Y) + \frac{3x}{x_1} \cdot \frac{1}{(k-1)!} \left( \sum_{p < x^{\delta_x}} 1/p \right)^{k-1}, \end{aligned}$$

whence

$$\frac{S_Y}{N_k(x)} \leq \frac{k}{x_2} \cdot \frac{1}{\delta_x} \rho(Y) + 3 \left( \frac{x_2 - \log 1/\delta_x}{x_2} \right)^{k-1} \leq \frac{k}{x_2} \frac{1}{\delta_x} \rho(Y) + 3e^{-\frac{(k-1)}{x_2} \log \frac{1}{\delta_x}}.$$

The second sum is small if  $\delta_x$  is small, the first sum is small if  $\frac{\rho(Y)}{\delta_x}$  is small, i.e. if  $Y$  is large.

Thus, by choosing  $\delta_x = \sqrt{\rho(Y)}$  for example, we obtain that

$$\frac{S_Y}{N_k(x)} = o_Y(1).$$

From the convergence of the three series it is obvious that there is a sequence  $\rho_p \downarrow 0$  such that for the set  $\mathcal{B}_1 = \{p \mid |f(p)| > \rho_p\}$ ,  $\sum_{p \in \mathcal{B}_1} 1/p < \infty$ . Let  $\mathcal{B}_1$  be fixed. Let  $\mathcal{B}_2 = \{p^\alpha \mid p \in \mathcal{P}, \alpha \geq 2\}$ , and let

$$S_Y^* := \#\{n \leq x \mid n \in \mathcal{N}_k, p^\alpha \mid n \text{ for some } p^\alpha \in \mathcal{B}_2, p^\alpha > Y\}.$$

This is clear:

$$S_Y \leq \sum_{\substack{Y < p^\alpha \leq \sqrt{x} \\ p^\alpha \in \mathcal{B}_2}} N_{k-\alpha} \left( \frac{x}{p^\alpha} \right) + \sum_{x \geq p^\alpha \geq \sqrt{x}} \frac{x}{p^\alpha} \leq c N_k(x) \sum_{p^\alpha \geq Y} \left( \frac{k}{x_2} \right)^\alpha \frac{1}{p^\alpha} + cx^{3/4},$$

and so

$$\frac{S_Y^*}{N_k(x)} \leq c \sum_{2^\alpha \geq Y} \left( \frac{k}{x_2 \cdot 2} \right)^\alpha + \frac{1}{Y^{1/10}} \sum_{\substack{\alpha \geq 2 \\ p \geq 3}} \left( \frac{k}{x_2 \cdot p^{9/10}} \right)^\alpha + cx^{3/4}.$$

The first sum on the right hand side is  $\ll \frac{Y^{-\delta/2 \log 2}}{1 - \frac{k}{2x_2}}$ , the second sum after  $\frac{1}{Y^{1/10}}$  is bounded by an absolute constant.

Thus

$$\limsup_{x \rightarrow \infty} \sup_{\frac{k}{x_2} \in [\delta, 2-\delta]} \frac{S_Y^*}{N_k(x)} \leq \varepsilon(Y),$$

where  $\varepsilon(Y) \rightarrow 0$  as  $Y \rightarrow \infty$ .

Let  $Y = Y_x$  be tending to infinity slowly. For some  $n \leq x$  let  $n = A(n) \cdot B(n)$ , where  $A(n) = \prod_{\substack{p^\alpha \parallel n \\ p < Y}} p^\alpha$ , and  $B(n) = \frac{n}{A(n)}$ . Consider the set of integers  $n \in \mathcal{N}_k$  up to  $x$ . Let us drop those  $n$  for which  $p \mid n$  for some  $p \in \mathcal{B}_1$ ,  $p > Y$  and those for which  $p^\alpha \mid n$  for some  $p^\alpha \in \mathcal{B}_2$ ,  $p^\alpha > Y$ . The number of the dropped elements is  $\ll \varepsilon_1(Y) N_k(x)$ , where  $\varepsilon_1(Y) \rightarrow 0$  uniformly as  $\frac{k}{x_2} \in [\delta, 2-\delta]$ . Let  $f^* \in \mathcal{A}$  defined on prime powers  $p^\alpha$  as follows:

$$f^*(p^\alpha) = \begin{cases} 0 & \text{if } \alpha \geq 2 \\ 0 & \text{if } \alpha = 1, p \leq Y, \text{ if } p \geq \sqrt{x}, \text{ or if } p \in \mathcal{B}_1 \\ f(p) & \text{if } \alpha = 1, p \in (Y, \sqrt{x}]. \end{cases}$$

From Theorem 8 we have

$$\frac{1}{N_k(x)} \sum \left( f^*(n) - \xi_{k,x} \sum \frac{f^*(p)}{p} \right)^2 \leq c \sum_{Y \leq p \leq \sqrt{x}} \frac{f^{*2}(p)}{p}. \quad (8.1)$$

$$\limsup_x \sup_{\xi_{k,x} \in [\delta, 2-\delta]} \frac{1}{N_k(x)} \#\{n \leq x \mid n \in \mathcal{N}_k, |f^*(n)| \geq \lambda\} \leq \varepsilon_2(Y), \quad \varepsilon_2(Y) \rightarrow 0 \quad (8.2)$$

valid for every  $\lambda > 0$ .

Let  $\mathcal{M}_Y$  be the set of those  $m$ , the largest prime power factor of which is not larger than  $Y$ , and if  $p^\alpha \parallel m$ ,  $\alpha \geq 2$ , then  $p^\alpha \leq Y$ . From the estimation of  $S_Y^*$  we obtain that

$$\frac{1}{N_k(x)} \#\{n \leq x \mid n \in \mathcal{N}_k, A(n) \notin \mathcal{M}_Y\}$$

$$\limsup_{x \rightarrow \infty} \sup_{\frac{k}{x_2} \in [\delta, 2-\delta]} \frac{1}{N_k(x)} \#\{n \leq x \mid n \in \mathcal{N}_k, A(n) \notin \mathcal{M}_Y\} \leq \varepsilon_3(Y),$$

where  $\varepsilon_3(Y) \rightarrow 0$  as  $Y \rightarrow \infty$ .

Let  $\mathcal{D}_{m,k} := \{n \in \mathcal{N}_k, A(n) = m\}$  ( $m \in \mathcal{M}_Y$ ), and let  $h(n) := f(A(n))$ . Thus  $h(n)$  is constant on  $D_{m,k}$ , and from (8.2) we obtain that

$$\limsup_{x \rightarrow \infty} \sup_{\frac{k}{x_2} \in [\delta, -2\delta]} \frac{1}{N_k(x)} \#\{n \leq x \mid n \in \mathcal{N}_k, |f(n) - f(A(n))| > \lambda\} \leq \varepsilon_2(Y).$$

Now we compute the density of the set  $D_{m,k}$ .

Let  $\mathcal{N}_k(D) = \{n \in \mathcal{N}_k \mid n \in D\}$ . Starting from the generating function

$$\prod_{p \nmid D} \frac{1}{1 - \frac{z}{p^s}} = \prod_{p \mid D} \left(1 - \frac{z}{p^s}\right) \cdot \sum \frac{z^{\Omega(n)}}{n^s},$$

for  $N_k(x|D) = \sum_{\substack{n \leq x \\ (n,D)=1 \\ n \in \mathcal{N}_k}} 1$  we have

$$N_k(x, D) = \sum_{d \mid D} \mu(d) N_{k-\Omega(d)} \left(\frac{x}{D}\right).$$

Let  $K_Y = \prod_{p \leq Y} p$ .

From the convergence of the series in (1.24) we obtain that

$$\sum \frac{f^*(p)}{p} = \sum_{\substack{Y \leq p < \sqrt{x} \\ |f(p)| < 1}} \frac{f(p)}{p} = \sum_{\substack{Y \leq p < \sqrt{x} \\ \rho_p < |f(p)| < 1}} \frac{f(p)}{p}$$

tends to zero as  $x \rightarrow \infty$ . The right hand side of (8.1) tends to zero as well. Applying these relations, from (8.1) we obtain

Consequently

$$\#(D_m, k) = N_{k-\Omega(m)} \left( \frac{x}{m} \mid K_Y \right) = \sum_{d \mid K_Y} N_{k-\Omega(m)-\Omega(d)} \left( \frac{x}{md} \right) \mu(d),$$

and so

$$\frac{\#(D_m, k)}{N_k(x)} = (1 + o_x(1)) \frac{\xi_{k, x_2}^{\Omega(m)}}{m} \prod_{p \mid K_Y} \left( 1 - \frac{\xi_{k, x_2}}{p} \right) \quad (8.3)$$

uniformly as  $\frac{k}{x_2} \in [\delta, 1 - \delta]$ ,  $m \in \mathcal{M}_Y$  even if  $Y = Y_x \rightarrow \infty$  slowly. Hence the assertion easily follows.

### §9. Proof of Theorem 2

This can be carried over by a simple application of Theorem 7 and of (8.3).

Let  $f(p^\alpha) = \arg g(p^\alpha) \in [-\pi, \pi]$ ,  $f$  be extended so that  $f \in \mathcal{A}$ . Then  $g(n) = e^{if(n)}$ .

From the convergence of  $\sum \frac{1-g(p)}{p}$  we obtain that  $\sum \frac{f(p)}{p}$ ,  $\sum \frac{f^2(p)}{p}$  are convergent. For some  $n \in \mathcal{N}_k$  define  $g_Y(n) := g(A(n))$ . First we observe that  $\frac{1}{N_k(x)} \sum_{n \leq x} |g(n) - g_Y(n)| \leq \varepsilon_1(Y)$ , uniformly in  $\frac{k}{x_2} \in [\delta, 2 - \delta]$ , where  $\varepsilon_1(Y) \rightarrow 0$  if  $Y \rightarrow \infty$ . Furthermore

$$\sum_{m \in \mathcal{M}_Y} g_Y(m) \frac{\#(D_m, k)}{N_k(x)} = (1 + o_x(1)) \sum_{m \in \mathcal{M}_Y} \frac{g(m) \xi_{k, x_2}^{\Omega(m)}}{m} \prod_{p \mid K_Y} \left( 1 - \frac{\xi_{k, x_2}}{p} \right).$$

The right hand side clearly tends to  $M_{\xi_k, x_2}(g)$  defined in Theorem 4.

Since

$$\limsup_x \sup_k \frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k \\ A(n) \notin \mathcal{M}_Y}} |g(n)| \rightarrow 0 \text{ as } Y \rightarrow \infty,$$

our theorem immediately follows.

### §10. Proof of Theorem 6 and 7

The proof is completely analogous to that of Theorem 2 and 5. So we omit it.

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