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Curvature of locally conformal cosymplectic manifolds

By MARIA FALCITELLI (Bari)

Abstract. Locally conformal cosymplectic manifolds are investigated from the point of view of the curvature. Particular attention to the $N(k)$-nullity condition is given and classification theorems in dimension $2n+1 \geq 5$ are stated. This also allows to classify locally conformal cosymplectic manifolds which are locally symmetric spaces.

1. Introduction

Locally conformal cosymplectic (l.c. cosymplectic) manifolds are characterized as the almost contact metric manifolds $(M, \varphi, \xi, \eta, g)$ which admit a closed 1-form ω such that the covariant derivative of φ with respect to the Levi–Civita connection ∇ acts as

$$(\nabla_X \varphi)Y = -\omega(\varphi Y)X + \omega(Y)\varphi X + g(X, \varphi Y)B - g(X, Y)\varphi B, \tag{1.1}$$

where $B = \sharp\omega$ is the vector field g-associated to ω [17]. The vanishing of ω in (1.1) is equivalent to the ∇-parallelism of φ, namely to the condition that M is cosymplectic. Moreover, ω is exact if and only if M is globally conformal cosymplectic. An f-Kenmotsu manifold, namely an l.c. cosymplectic manifold such that $\omega = -f\eta$, f being a smooth function, is locally realized as a warped product $]-\epsilon, \epsilon[\, \times_{h^2} F$, $\epsilon \in \mathbb{R}^{\star}_{+}$, where F is a Kähler manifold and h is a smooth positive function on $]-\epsilon, \epsilon[$. These manifolds are studied from the point of view of the curvature, too. In particular, a locally symmetric non-cosymplectic f-Kenmotsu manifold has negative constant sectional curvature [18].

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In this paper, we consider an l.c. cosymplectic manifold. Denoting by R the curvature of ∇, firstly we evaluate the "cosymplectic defect" $R(X,Y)\circ\varphi-\varphi\circ R(X,Y)$, for any vector fields X, Y, and obtain general properties of the Ricci and $*$-Ricci tensors. Then we focus our attention to the (k,μ)-condition, proving that, in the context of locally cosymplectic geometry, it is equivalent to the $N(k)$-condition. So, we consider $N(k)$-l.c. cosymplectic spaces, namely connected l.c. cosymplectic manifolds admitting a smooth function k such that $R(X,Y,\xi)=k(\eta(Y)X-\eta(X)Y)$. In particular, given an $N(k)$-space, the covariant derivative $\nabla\omega$ is a combination of $\omega\otimes\omega$, g and $\eta\otimes\eta$ by means of suitable functions depending on $\delta\omega$, $\|\omega\|$ and k. This allows to obtain useful differential equations involving $\delta\omega$, $\|\omega\|$ and k. We also relate the $N(k)$-condition to the concept of $C(\lambda)$-manifold introduced by Janssens and Vanhecke [14].

Classification theorems in dimension $2n+1\geq 5$ are stated. We prove that a non-cosymplectic $N(k)$-l.c. cosymplectic space is either globally conformal cosymplectic or f-Kenmotsu or, possibly, it is locally expressed as a warped product $N\times_{f^2}N'$, N being a 2-dimensional manifold of curvature k and N' a cosymplectic manifold isometric to a leaf of the distribution $\ker\omega\cap\operatorname{Ker}\eta$. Moreover, given a cosymplectic manifold N', we define a class of smooth functions on $\mathbb{R}^2$ making $\mathbb{R}^2\times_{f^2}N'$ an $N(k)$-globally conformal cosymplectic manifold such that k is non-constant.

Suitable $N(k)$-l.c. cosymplectic spaces can be locally realized as a warped product $]-\epsilon,\epsilon[\times_{h^2}F$, where F is a Kähler manifold isometric to a leaf of the distribution orthogonal to B. Examples of these manifolds are described, considering on the hyperbolic space $\mathbb{H}^{2n+1}$ a family of almost contact metric structures compatible with the metric of constant sectional curvature $-c^2$, $c>0$.

Finally, $N(k)$-l.c. cosymplectic, non-cosymplectic and locally symmetric spaces are considered. They are characterized as the l.c. cosymplectic (non-cosymplectic) manifolds with constant sectional curvature k. Any of these manifolds is either globally conformal cosymplectic or it is locally a warped product $]-\epsilon,\epsilon[\times_{h^2}F$, with $h^2=a\exp(-2\|\omega\|t)$, a constant, $\|\omega\|^2=-k$, F being a flat Kähler manifold.

2. Some curvature formulas

An l.c. cosymplectic manifold (M,φ,ξ,η,g) is an almost contact metric manifold admitting an open covering $\{U_i\}_{i\in I}$ and, for any i, a smooth function $\rho_i:U_i\to\mathbb{R}$ such that the almost contact metric structure defined in U_i by

$$\varphi_i=\varphi_{/U_i},\ \xi_i=\exp(-\rho_i)\xi_{/U_i},\ \eta_i=\exp(\rho_i)\eta_{/U_i},\ g_i=\exp(2\rho_i)g_{/U_i}\qquad(2.1)$$

is cosymplectic.

The manifold M is globally conformal cosymplectic (g.c. cosymplectic) if there exists a smooth function $\rho\colon\to\mathbb{R}$ such that $(\varphi,\exp(-\rho)\xi,\exp\rho\,\eta,\exp(2\rho)g)$ is a cosymplectic structure.
Several examples of l.c. cosymplectic manifolds are given in [7].

Now, we fix a $2n+1$-dimensional l.c. cosymplectic manifold (M,φ,ξ,η,g), so (1.1) is satisfied. By direct calculus we get $\omega=\nabla_\xi\eta+\frac{\delta\eta}{2n}\eta$. Hence ω is the Lee form and it is related to the local conformal change (2.1) by $\omega_{/U_i}=d\rho_i$. The Lee vector field $B=\sharp\,\omega$ is expressed as $B=\nabla_\xi\xi+\frac{\delta\eta}{2n}\xi$, and $\eta(B)=\omega(\xi)=\frac{\delta\eta}{2n}$. Moreover, for any vector field X, one has:

$$\nabla_X\xi=-\omega(\xi)X+\eta(X)B, \tag{2.2}$$

$$X(\omega(\xi))=(\nabla_X\omega)\xi-\omega(\xi)\omega(X)+\|\omega\|^2\eta(X). \tag{2.3}$$

We denote by R the curvature of ∇, $R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$. For the Riemannian curvature we use the convention $R(X,Y,Z,W)=g(R(X,Y,W),Z)$ and denote by ρ, ρ^* the Ricci and $*$-Ricci tensors, by τ, τ^* the scalar and $*$-scalar curvatures. Since $\rho^*(X,Y)$ is the trace of the operator $Z\to\varphi(R(X,Z,\varphi Y))$, with respect to an adapted local orthonormal frame $\{X_1,\dots,X_{2n},\xi\}$ we have $\rho^*(X,Y)=\sum_{1\le i\le 2n}R(X,X_i,\varphi Y,\varphi X_i)$, $\tau^*=\sum_{1\le i\le 2n}\rho^*(X_i,X_i)$.

Furthermore we recall that, given a symmetric $(0,2)$-tensor field Q, the Kulkarni–Nomizu product $g\wedge Q$ of g and Q acts as

$$\begin{aligned}(g\wedge Q)(X,Y,Z,W)=g(X,Z)Q(Y,W)+g(Y,W)Q(X,Z)\\ -g(X,W)Q(Y,Z)-g(Y,Z)Q(X,W).\end{aligned}$$

In particular, to simplify the notations, we put $\pi_1=\frac{1}{2}g\wedge g$, since it is the $(0,4)$-tensor field associated with the $(1,3)$-tensor field π_1 acting as

$$\pi_1(X,Y,Z)=g(Y,Z)X-g(X,Z)Y.$$

By direct calculus, applying (1.1), (2.2), (2.3), one proves the following statement.

Proposition 1. *The curvature R of (M,φ,ξ,η,g) satisfies, for any vector fields X, Y, Z:*

i) $$\begin{aligned}R(X,Y,\varphi Z)-\varphi(R(X,Y,Z))&=((\nabla_Y\omega)\varphi Z-\omega(Y)\omega(\varphi Z)+\|\omega\|^2g(Y,\varphi Z))X\\ &-((\nabla_X\omega)\varphi Z-\omega(X)\omega(\varphi Z)+\|\omega\|^2g(X,\varphi Z))Y-((\nabla_Y\omega)Z\\ &-\omega(Y)\omega(Z))\varphi X+((\nabla_X\omega)Z-\omega(X)\omega(Z))\varphi Y+(\omega(Y)g(X,\varphi Z)\\ &-\omega(X)g(Y,\varphi Z)+\omega(\varphi Y)g(X,Z)-\omega(\varphi X)g(Y,Z))B+g(Y,\varphi Z)\nabla_XB\\ &-g(X,\varphi Z)\nabla_YB-g(Y,Z)\nabla_X\varphi B+g(X,Z)\nabla_Y\varphi B,\end{aligned}$$

ii) $R(X,Y,\xi) = Y(\omega(\xi))X - X(\omega(\xi))Y - \eta(Y)(\omega(X)B - \nabla_X B)$
$\qquad + \eta(X)(\omega(Y)B - \nabla_Y B).$

To the given manifold (M,φ,ξ,η,g) we associate the $(0,2)$-tensor field

$$P = \nabla\omega - \omega\otimes\omega + \frac{1}{2}\|\omega\|^2 g.$$

Note that P is symmetric, ω being closed and $trP = -\delta\omega + \left(n - \frac{1}{2}\right)\|\omega\|^2$.

Proposition 2. *For any vector fields X, Y, Z, W we have:*

i) $R(X,Y,Z,W) - R(X,Y,\varphi Z,\varphi W) = (g\wedge P)(X,Y,Z,W) - (g\wedge P)(X,Y,\varphi Z,\varphi W)$,

ii) $(\rho - \rho^*)(X,Y) = 2(n-1)P(X,Y) - P(\varphi X,\varphi Y) + P(X,\xi)\eta(Y) + trPg(X,Y)$,

iii) $\tau - \tau^* = 2(\xi(\omega(\xi)) - (2n-1)\delta\omega + 2n(n-1)\|\omega\|^2)$.

Proof. We sketch the proof of i), which is obtained by a quite long calculus. Considering X, Y, Z, W tangent to M, firstly we write:

$$\begin{aligned} R(X,Y,Z,W) - R(X,Y,\varphi Z,\varphi W) = &-g(\varphi(R(X,Y,Z),\varphi W) \\ &+ g(R(X,Y,\varphi Z),\varphi W) + g(R(X,Y,\xi),Z)\eta(W). \end{aligned}$$

Then, we apply i), ii) in Proposition 1 and the following relation, which is a consequence of (1,1), (2.3):

$$\begin{aligned} g(\nabla_X\varphi B,\varphi W) &= g((\nabla_X\varphi)B,\varphi W) + g(\nabla_X B, W) - \eta(\nabla_X B)\eta(W) \\ &= \|\omega\|^2(g(X,W) - \eta(X)\eta(W)) - \omega(\varphi X)\omega(\varphi W) \\ &\quad - \omega(X)(\omega(W) - \omega(\xi)\eta(W)) + (\nabla_X\omega)W - (\nabla_X\omega)\xi\,\eta(W) \\ &= P(X,W) + \frac{1}{2}\|\omega\|^2 g(X,W) - \omega(\varphi X)\omega(\varphi W) - X(\omega(\xi))\eta(W). \end{aligned}$$

The relation ii) directly follows by i). Moreover, considering an adapted local orthonormal frame $\{X_1,\ldots,X_{2n},\xi\}$, since $\{\varphi X_1,\ldots,\varphi X_{2n},\xi\}$ is an orthonormal frame, also, from *ii)* we get:

$$\tau - \tau^* = (4n-1)trP - \sum_{1\le i\le 2n} P(\varphi X_i,\varphi X_i) + P(\xi,\xi) = 2(2n-1)trP + 2P(\xi,\xi).$$

Then iii) follows, since (2.3) entails: $P(\xi,\xi) = \xi(\omega(\xi)) - \frac{1}{2}\|\omega\|^2$. □

Corollary 3. *For any vector fields X, Y one has:*

i) $\rho^*(X,Y)-\rho^*(Y,X)=Y(\omega(\xi))\eta(X)-X(\omega(\xi))\eta(Y)$,

ii) $\rho^*(\varphi X,\varphi Y)=\rho^*(Y,X)-(X-\eta(X)\xi)(\omega(\xi))\eta(Y)$,

iii) $\rho^*(X,\xi)=0,\quad \rho^*(\xi,X)=(X-\eta(X)\xi)(\omega(\xi))$,

iv) $\rho(\varphi X,\varphi Y)=\rho(X,Y)+(\delta\omega-(2n-1)\|\omega\|^2)\eta(X)\eta(Y)$
$-(2n-1)((\nabla_X\omega)Y-\nabla_{\varphi X}\omega)\varphi Y-\omega(X)\omega(Y)+\omega(\varphi X)\omega(\varphi Y))$,

v) $\rho(X,\xi)=(2n-1)X(\omega(\xi))-\delta\omega\,\eta(X)$.

Proof. We apply ii) in Proposition 2, use the symmetry of ρ and P and then (2.3), so obtaining:

$$\begin{aligned}\rho^*(X,Y)-\rho^*(Y,X)&=-P(X,\xi)\eta(Y)+P(Y,\xi)\eta(X)\\&=-X(\omega(\xi))\eta(Y)+Y(\omega(\xi))\eta(X).\end{aligned}$$

Moreover, if $\{X_1,\dots,X_{2n},\xi\}$ is an adapted local orthonormal frame, via Proposition 1 we have:

$$\begin{aligned}\rho^*(\varphi X,\varphi Y)&=\sum_{1\le i\le 2n}R(\varphi X,X_i,\varphi^2Y,\varphi X_i)\\&=\sum_{1\le i\le 2n}\{R(Y,X_i,\varphi X,\varphi X_i)-\eta(Y)g(R(\varphi X,X_i,\xi),\varphi X_i)\}\\&=\rho^*(Y,X)-\eta(Y)\sum_{1\le i\le 2n}X_i(\omega(\xi))g(X,X_i),\end{aligned}$$

and ii) follows. Formula iii) is a consequence of i), ii). Furthermore, via Proposition 2 and the just stated formulas, we have:

$$\begin{aligned}&\rho(X,Y)-\rho(\varphi X,\varphi Y)=(\rho-\rho^*)(X,Y)-(\rho-\rho^*)(\varphi X,\varphi Y)+\rho^*(X,Y)\\&\quad-\rho^*(Y,X)\ +(Y-\eta(Y)\xi)(\omega(\xi))\eta(X)=(2n-1)(P(X,Y)-P(\varphi X,\varphi Y))\\&\quad+\eta(X)(P(\xi,\xi)\eta(Y)-P(\xi,Y)+trP\eta(Y))+Y(\omega(\xi))\eta(X)-\xi(\omega(\xi))\eta(X)\eta(Y).\end{aligned}$$

This, combined with (2.3), implies iv) and v). $\square$

Remark. If M is f-Kenmotsu, we have $df=\xi(f)\eta,\nabla\eta=f(g-\eta\otimes\eta)$ and $P=-\xi(f)\eta\otimes\eta-\frac{1}{2}f^2g$. So, the formulas stated in Propositions 1, 2 and Corollary 3 reduce to the ones proved in [18]. In particular, ρ^* is symmetric and $\rho^*=\rho+((2n-1)f^2+\xi(f))g+(f^2+(2n-1)\xi(f))\eta\otimes\eta$.

3. The $N(k)$-condition

In contact geometry the behaviour of the tensor field $h = \frac{1}{2}L_\xi\varphi$, L_ξ denoting the Lie derivative with respect to ξ, plays an important role for the classification of contact manifolds satisfying suitable curvature conditions [4]. The next result shows that the anti-Lee form $\theta = -\omega \circ \varphi$ specifies h in the locally conformal cosymplectic case.

Lemma 4. *Let $(M, \varphi, \xi, \eta, g)$ be an l.c. cosymplectic manifold with Lee form ω. For any vector field X, one has $h(X) = -\frac{1}{2}\omega(\varphi X)\xi$. Then, h vanishes if and only if M is f-Kenmotsu.*

Proof. Formulas (1.1), (2.2) imply:

$$2h(X) = (\nabla_\xi\varphi)X - \nabla_{\varphi X}\xi + \varphi(\nabla_X\xi) = -\omega(\varphi X)\xi.$$

Therefore, h vanishes if and only if $\omega\circ\varphi = 0$, namely if and only if ω is proportional to η. □

Proposition 5. *Let $(M, \varphi, \xi, \eta, g)$ be an l.c. cosymplectic manifold. Assume the existence of smooth functions k, μ on M such that*

$$R(X, Y, \xi) = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h(X) - \eta(X)h(Y)), \tag{3.1}$$

for any vector fields X, Y. Then, one has $\mu h = 0$.

Proof. By Lemma 4 and the hypothesis, one has:

$$0 = R(\xi, \varphi B, \xi, \xi) = -\mu g(h(\varphi B), \xi) = \frac{1}{2}\mu\,\omega(\varphi^2 B) = -\frac{1}{2}\mu g(\varphi B, \varphi B).$$

Therefore $\mu\varphi B$ vanishes on the whole M and Lemma 4 yields the statement. □

In [5] the authors call (k, μ)-manifold a contact metric manifold whose curvature satisfies (3.1), k, μ being suitable real numbers. In particular, if $\mu = 0$, one obtains the concept of $N(k)$-contact metric manifold. Condition (3.1) with k, μ smooth functions is considered in [16] and the authors prove that, if M is a (k, μ)-contact metric manifold with k, μ smooth functions and $\dim M \geq 5$, then either M is Sasakian ($k = 1$, $h = 0$) or k, μ are both constants. A local classification of non-Sasakian (k, μ)-contact manifolds, in any dimension, is due to E. Boecks [6]. Moreover, $N(k)$-almost cosymplectic manifolds are studied by Dacko [10] and a generalization of (3.1) is also studied in [11]. Proposition 5 clarifies that the concepts of (k, μ) and $N(k)$-manifold are equivalent in the context of locally conformal cosymplectic geometry.

Definition 6. An $N(k)$-l.c. cosymplectic space is a connected l.c. cosymplectic manifold (M,φ,ξ,η,g) admitting a smooth function k such that

$$R(X,Y,\xi) = k(\eta(Y)X - \eta(X)Y), \tag{3.2}$$

for any vector fields X, Y.

In [18] the authors prove that the curvature of an f-Kenmotsu manifold satisfies (3.2) with $k = -(\xi(f)+f^2)$. Hence every connected f-Kenmotsu manifold is an l.c. $N(k)$-cosymplectic space and k needs not to be constant.

Now we are going to state several formulas which are essential for the proof of the theorems in Section 4.

Proposition 7. *Let (M,φ,ξ,η,g) be a $2n+1$-dimensional $N(k)$-l.c. cosymplectic space with Lee form ω. Then, we have:*

i) $(2n-1)d\omega(\xi) = (2nk+\delta\omega)\eta$,

ii) $\nabla_X B = -\frac{k+\delta\omega}{2n-1}X + \omega(X)B + \big(\frac{(2n+1)k+2\delta\omega}{2n-1} - \|\omega\|^2\big)\eta(X)\xi, \quad X \in \mathcal{X}(M)$.

Proof. By (3.2) the Ricci tensor satisfies, for any X tangent to M, $\rho(X,\xi) = 2nk\eta(X)$. Then we apply v) in Corollary 3 and obtain i). Combining with (2.3), for any vector field X we also have:

$$g(\nabla_X B,\xi) = (\nabla_X\omega)\xi = \omega(\xi)\omega(X) + \left(\frac{\delta\omega+2nk}{2n-1} - \|\omega\|^2\right)\eta(X).$$

Since $\nabla\omega$ is symmetric, this implies:

$$\nabla_\xi B = \omega(\xi)B + \left(\frac{\delta\omega+2nk}{2n-1} - \|\omega\|^2\right)\xi.$$

Now we apply Proposition 1, equation i), the previous relation and use the hypothesis, so obtaining:

$$\begin{aligned}\nabla_X B &= -R(\xi,X,\xi) + X(\omega(\xi))\xi - \xi(\omega(\xi))X - \eta(X)(\omega(\xi)B - \nabla_\xi B) + \omega(X)B\\ &= k(X-\eta(X)\xi) + \left(\frac{2(2nk+2\delta\omega)}{2n-1} - \|\omega\|^2\right)\eta(X)\xi\\ &\quad - \frac{2nk+\delta\omega}{2n-1}X + \omega(X)B.\end{aligned}$$

Then ii) follows. □

Given a $(2n+1)$-dimensional $N(k)$-l.c. cosymplectic space with Lee form ω, we put:

$$\alpha = -\frac{k+\delta\omega}{2n-1}, \quad \beta = \frac{(2n+1)k+2\delta\omega}{2n-1} - \|\omega\|^2. \tag{3.3}$$

The functions α, β fulfill:

$$2\alpha+\beta+\|\omega\|^2 = k, \quad \alpha+\beta+\|\omega\|^2 = \xi(\omega(\xi)). \tag{3.4}$$

They allow to express $\nabla\omega$ by:

$$\nabla\omega = \omega\otimes\omega + \alpha g + \beta\eta\otimes\eta. \tag{3.5}$$

In particular, Proposition 7 and (1.1) entail:

$$\nabla_X \varphi B = (\|\omega\|^2+\alpha)\varphi X - \omega(\varphi X)B, \quad X\in\mathcal{X}(M), \tag{3.6}$$

$$d(\|\omega\|^2) = 2(\|\omega\|^2+\alpha)\omega + 2\beta\,\omega(\xi)\eta. \tag{3.7}$$

Useful curvature identities can be derived applying the results in Section 2. In fact, Proposition 1, (3.5), (3.6) entail, for any vector fields X, Y, Z:

$$\begin{aligned} R(X,Y,\varphi Z) - \varphi(R(X,Y,Z)) = (k-\beta)(&g(Y,\varphi Z)X - g(X,\varphi Z)Y - g(Y,Z)\varphi X \\ &+ g(X,Z)\varphi Y) - \beta\eta(Z)(\eta(Y)\varphi X - \eta(X)\varphi Y) \\ &+ \beta(\eta(X)g(Y,\varphi Z) - \eta(Y)g(X,\varphi Z))\xi. \end{aligned} \tag{3.8}$$

By (3.5) we also have $P = (\alpha+\frac{1}{2}\|\omega\|^2)g + \beta\eta\otimes\eta$ and also using (3.3), (3.4) we obtain:

$$\rho^* = \rho + (2(n-1)\beta - (2n-1)k)g - (2(n-1)\beta+k)\eta\otimes\eta, \tag{3.9}$$

$$\tau^* = \tau + 4n((n-1)\beta - nk). \tag{3.10}$$

In particular, ρ^* is symmetric and by Proposition 7 and Corollary 3 one has:

$$\rho^*(X,Y) = \rho^*(\varphi X,\varphi Y), \quad \rho(X,Y) = \rho(\varphi X,\varphi Y) + 2nk\eta(X)\eta(Y). \tag{3.11}$$

Lemma 8. *Let (M,φ,ξ,η,g) be a $2n+1$-dimensional $N(k)$-l.c. cosymplectic space. For any $X,Y\in\mathcal{X}(M)$ we have:*

i) $R(X,Y,B) = X(\alpha)Y - Y(\alpha)X + (X(\beta)\eta(Y) - Y(\beta)\eta(X))\xi + \alpha(\omega(Y)X$
$- \omega(X)Y) - \beta\omega(\xi)(\eta(Y)X - \eta(X)Y) + 2\beta(\eta(X)\omega(Y) - \eta(Y)\omega(X))\xi,$

ii) $R(X,Y,\varphi B) = X(\alpha)\varphi Y - Y(\alpha)\varphi X + (\beta-k)(\omega(\varphi Y)X - \omega(\varphi X)Y)$
$+ (\alpha+\beta-k)(\omega(Y)\varphi X - \omega(X)\varphi Y - 2\beta\omega(\xi)(\eta(Y)\varphi X$
$- \eta(X)\varphi Y) - \beta(\eta(X)\omega(\varphi Y) - \eta(Y)\omega(\varphi X))\xi,$

iii) $\rho(X,B) = -2nX(\alpha) - (X-\eta(X)\xi)(\beta) + 2(n\alpha+\beta)\omega(X)$
$- 2(n+1)\beta\omega(\xi)\eta(X)$

iv) $(X-\eta(X)\xi)(\alpha+\beta) = (\alpha+2\beta-k)(\omega(X) - \omega(\xi)\eta(X)).$

Proof. Formula i) is obtained by direct calculus, applying Proposition 7, (3.3) and (2.3). Formula ii) follows by i) and (3.8), as well as iii) is a direct consequence of i). Furthermore, with respect to an adapted local orthonormal frame $\{X_1,\dots,X_{2n},\xi\}$, by i) and (3.2) we have:

$$\begin{aligned}\rho(X,B) &= -\sum_{1\le i\le 2n} g(R(X,X_i,B),X_i)+g(R(X,\xi,\xi),B)\\ &= (2n-1)\alpha\,\omega(X)+(\alpha-2n\beta)\omega(\xi)\eta(X)-(2n-1)X(\alpha)-\eta(X)\xi(\alpha)\\ &\quad + k(\omega(X)-\omega(\xi)\eta(X)).\end{aligned}$$

Comparing with iii) we obtain iv). □

Proposition 9. *Let (M,φ,ξ,η,g) be an $N(k)$-l.c. cosymplectic space, with* $\dim M\ge 5$ *Then we have:*

$$\begin{aligned} d\alpha &= (\alpha+\beta-k)\omega-2\beta\omega(\xi)\eta,\\ dk &= \beta\,\omega+(\xi(\beta)-3\beta\,\omega(\xi))\eta.\end{aligned}$$

Proof. Putting $\dim M=2n+1$, by (3.11) and Lemma 8 we obtain:

$$2nk\omega(\xi)=\rho(B,\xi)=-2n\xi(\alpha)+2n(\alpha-\beta)\omega(\xi),$$

so that $\xi(\alpha)=(\alpha-\beta-k)\omega(\xi)$. By direct calculus, from ii) in Lemma 8, for any $X\in\mathcal{X}(M)$ one has:

$$\begin{aligned}\rho(\varphi X,\varphi B) = -X(\alpha)&+((2n-1)k-2(n-1)\beta+\alpha)\omega(X)\\ &+((2n-3)\beta-2nk)\omega(\xi)\eta(X).\end{aligned}$$

On the other hand (3.11) and iii), iv) in Lemma 8 entail:

$$\begin{aligned}\rho(\varphi X,\varphi B) = \rho(X,B)-2nk\,\omega(\xi)\,\eta(X) &= -(2n-1)X(\alpha)\\ &+((2n-1)\alpha+k)\omega(X)-((2n-1)\beta+2nk)\omega(\xi)\,\eta(X).\end{aligned}$$

Comparing the just stated formulas we have:

$$(n-1)(X(\alpha)-(\alpha+\beta-k)\omega(X)+2\beta\omega(\xi)\,\eta(X))=0,$$

then the first equation, since $n\ge 2$. The second equation follows by (3.4), (3.7) and iv) in Lemma 8. □

Proposition 10. *Let $(M, \varphi, \xi, \eta, g)$ be an $N(k)$-l.c. cosymplectic space with* $\dim M \geq 5$. *For any vector fields X, Y, U we have:*

i) $R(X, Y, B) = (k - \beta)(\omega(Y)X - \omega(X)Y) + \beta\omega(\xi)(\eta(Y)X - \eta(X)Y) + \beta(\eta(X)\omega(Y) - \eta(Y)\omega(X))\xi,$

ii) $(\nabla_U R)(X, Y, \xi) = U(k)(\eta(Y)X - \eta(X)Y) + \omega(\xi)(R - k\pi_1)(X, Y, U) + \beta\eta(U)((\omega(Y) - \omega(\xi)\eta(Y))X - (\omega(X) - \omega(\xi)\eta(X))Y) - \beta\eta(U)(\eta(X)\omega(Y) - \eta(Y)\omega(X))\xi$

Proof. Formula i) follows by Lemma 8 and Proposition 9. By direct calculus, applying (3.2) and (2.2) we have:

$$\begin{aligned}(\nabla_U R)(X, Y, \xi) &= U(k)(\eta(Y)X - \eta(X)Y) + k((\nabla_U\eta)Y\,X - (\nabla_U\eta)X\,Y)\\ -R(X, Y, \nabla_U\xi) &= U(k)(\eta(Y)X - \eta(X)Y) - k\omega(\xi)(g(Y, U)X - g(X, U)Y)\\ &\quad + k\eta(U)(\omega(Y)X - \omega(X)Y) + \omega(\xi)R(X, Y, U) - \eta(U)R(X, Y, B).\end{aligned}$$

Then, we apply i) and ii) follows. □

We end this section relating the $N(k)$-condition to the concept of $C(\lambda)$-manifold. In [14] the authors call almost $C(\lambda)$-manifold an almost contact metric manifold whose Riemannian curvature satisfies:

$$\begin{aligned}R(X, Y, Z, W) = R(X, Y, \varphi Z, \varphi W) - \lambda(g(X, Z)g(Y, W) - g(Y, Z)g(X, W)\\ - g(X, \varphi Z)g(Y, \varphi W) + g(X, \varphi W)g(Y, \varphi Z)),\end{aligned} \tag{3.12}$$

λ being a suitable real number.

In particular, if M is f-Kenmotsu and f is constant, M is a $C(-f^2)$-manifold. We also remark that, in accordance with the notation in [14], (3.12) is equivalent to:

$$\begin{aligned}R(X, Y, Z) = -\varphi(R(X, Y, \varphi Z)) + \lambda(g(Y, Z)X\\ - g(X, Z)Y + g(Y, \varphi Z)\varphi X - g(X, \varphi Z)\varphi Y),\end{aligned} \tag{3.13}$$

with $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

Proposition 11. *Let $(M, \varphi, \xi, \eta, g)$ be a connected l.c. cosymplectic manifold with* $\dim M = 2n + 1 \geq 5$. *The following conditions are equivalent:*

i) *M is an almost $C(k)$-manifold,*

ii) *M is an $N(k)$-space and $(2n - 1)\|\omega\|^2 = 2\delta\omega + (2n + 1)k$.*

Proof. In the hypothesis i), by (3.13) we obtain:

$$R(X,Y,\xi) = k(\eta(Y)X - \eta(X)Y), \quad k \in \mathbb{R},$$

so M is an $N(k)$-space, k being a constant function. We prove that the corresponding function β defined in (3.3) vanishes. In fact, given $p \in M$, we consider orthonormal vectors $X, Y \in T_pM$, both orthogonal to ξ, such that $g_p(X, \varphi Y) = 0$. By (3.8) we have, at p,

$$R(X,Y,\varphi X) = \varphi(R(X,Y,X)) + (k - \beta(p))\varphi Y.$$

On the other hand (3.13) entails, at p,

$$R(X,Y,\varphi X) = \varphi(R(X,Y,X)) + k\varphi Y$$

Then $\beta(p) = 0$. Since the vanishing of β is equivalent to $(2n-1)\|\omega\|^2 = 2\delta\omega + (2n+1)k$, we get ii).

Viceversa, in the hypothesis ii) β vanishes and, by Proposition 9, k is constant. Via (3.8) we also have, for any $X, Y, Z \in \mathcal{X}(M)$:

$$\begin{aligned} R(X,Y,\varphi Z) = \varphi(R(X,Y,Z)) + k(&g(Y,\varphi Z)X - g(X,\varphi Z)Y \\ &- g(Y,Z)\varphi X + g(X,Z)\varphi Y). \end{aligned}$$

Hence M is an almost $C(k)$-manifold. □

4. The main results

Let $(M, \varphi, \xi, \eta, g)$ be an l.c. cosymplectic manifold with Lee form ω and anti-Lee form $\theta = -\omega \circ \varphi$. Since $d\eta = \eta \wedge \omega$, the distribution $\mathcal{D}$ associated with the subbundle $\ker \eta$ of TM is integrable. When ω never vanishes, the distribution $\mathcal{D}_1$ associated with the subbundle $\ker \omega$ is integrable, also, and its orthogonal distribution $\mathcal{D}_1^\perp$ corresponds to the subbundle $\langle B \rangle$ of TM. We remark that $\mathcal{D}$ and $\mathcal{D}_1$ coincide if and only if M is f-Kenmotsu.

Assume that θ never vanishes, equivalently $\|\omega\|^2 > \omega(\xi)^2$ everywhere, then the vector fields ξ and B are pointwise linearly independent and the distribution $\mathcal{D}_2$ associated with $\langle B, \xi \rangle$ has rank two. Since ω is closed and $d\eta = \eta \wedge \omega$, the orthogonal distribution $\mathcal{D}_2^\perp$, which is associated with $\operatorname{Ker}\omega \cap \operatorname{Ker}\eta$, is integrable. By Proposition 9, if M is an $N(k)$-space, k is constant on each leaf of $\mathcal{D}_2^\perp$.
We are going to state several results which clarify the role of the mentioned distributions in the context of $N(k)$-l.c. cosymplectic spaces.

Theorem 12. *Let $(M, \varphi, \xi, \eta, g)$ be an $N(k)$-l.c. cosymplectic space with $\dim M = 2n+1 \geq 5$ Then, one of the following cases occurs:*

i) *M is globally conformal cosymplectic,*

ii) *M is cosymplectic,*

iii) *M is f-Kenmotsu and k is constant on each leaf of $\mathcal{D}$,*

iv) *the anti-Lee form θ never vanishes and the distribution $\mathcal{D}_2$ is totally geodesic. Each leaf F' of $\mathcal{D}_2^{\perp}$ is a totally umbilical submanifold of M with mean curvature B and inherits from M a cosymplectic structure. Moreover, (M, g) is locally a warped product $N \times_{f^2} N'$, N being a 2-dimensional manifold of Gaussian curvature k, f a positive smooth function and N' a cosymplectic manifold.*

Proof. Applying Proposition 9, (3.7) and (3.4) we have:

$$d(\alpha + \|\omega\|^2) = (\alpha + \|\omega\|^2)\omega. \tag{4.1}$$

Since ω is closed, locally ω can be expressed as $-d\log\tau$, for some strictly positive function τ and (4.1) implies the existence of $c \in \mathbb{R}$ such that $\alpha + \|\omega\|^2 = \frac{c}{\tau}$. Together with the connectedness of M, this means that either $\alpha + \|\omega\|^2 \neq 0$ everywhere or $\alpha + \|\omega\|^2 = 0$ [20].

Therefore, when $\alpha + \|\omega\|^2 \neq 0, \omega = d\log|\alpha + \|\omega\|^2|$ is exact and i) occurs.

Now, we assume $\alpha = -\|\omega\|^2$, apply (3.4), (3.7) and obtain:

$$d(\|\omega\|^2) = 2(k + \|\omega\|^2)\omega(\xi)\,\eta. \tag{4.2}$$

Since $(2n-1)\|\omega\|^2 = -(2n-1)\alpha = k + \delta\omega$, i) in Proposition 7 yields:

$$d(\omega(\xi)) = (k + \|\omega\|^2)\eta, \tag{4.3}$$

and using (4.2) we have $d(\|\omega\|^2 - \omega(\xi)^2) = 0$. Therefore $\|\theta\|^2 = \|\omega\|^2 - \omega(\xi)^2$ is constant, so either θ vanishes or θ never vanishes. Note that $\theta = 0$ is equivalent to $\omega = \omega(\xi)\eta$ and, in this case, either M is cosymplectic or M is f-Kenmotsu, $f = -\omega(\xi)$. Moreover, if M is f-Kenmotsu, by Proposition 9 we have: $dk = (\xi(\beta) - 2\beta\omega(\xi))\eta$ and then $dk \wedge \eta = 0$, namely k is constant on any leaf of $\mathcal{D}$. Thus, the case $\alpha = -\|\omega\|^2 = -\omega(\xi)^2$ yields ii) or iii).

Finally, we examine the case $\alpha = -\|\omega\|^2, C^2 = \|\omega\|^2 - \omega(\xi)^2$, with $C \in \mathbb{R}^*_+$. Firstly, we prove that the distribution $\mathcal{D}_2$, which has rank 2, is totally geodesic. In fact, by (3.4), (3.5) we have:

$$\xi(\omega(\xi)) = \beta = k + \|\omega\|^2, \tag{4.4}$$

$$\nabla_X B = \omega(X)B - \|\omega\|^2 X + (k + \|\omega\|^2)\eta(X)\xi, \tag{4.5}$$

for any $X \in \mathcal{X}(M)$.

In particular, $\nabla_\xi B = \omega(\xi)B + k\xi$ and $\nabla_B B = (k + \|\omega\|^2)\omega(\xi)\xi$.

Since moreover $\nabla_B \xi = 0, \nabla_\xi \xi = B - \omega(\xi)\xi$, $\mathcal{D}_2$ is totally geodesic and integrable. Note that $\{\xi, C^{-1}(B - \omega(\xi)\xi)\}$ are orthonormal vector fields in $\mathcal{D}_2$ and the curvature of a leaf of $\mathcal{D}_2$, which is a totally geodesic submanifold of M, is

$$R(\xi, C^{-1}(B - \omega(\xi)\xi), \xi, C^{-1}(B - \omega(\xi)\xi)) = k.$$

Now, considering $X \in \mathcal{D}_2^\perp$, by (2.2) and (4.5) we have $\nabla_X \xi = -\omega(\xi)X$,

$$\nabla_X B = -\|\omega\|^2 X.$$

Then the Weingarten operators a_ξ, a_B of a leaf F' of $\mathcal{D}_2^\perp$ act as $a_\xi = \omega(\xi)I_{TF'}$, $a_B = \|\omega\|^2 I_{TF'}$ and the second fundamental form is given by $\alpha(X,Y) = g(X,Y)B$. Hence F' is a totally umbilical submanifold of M with parallel mean curvature B. Note that, given X tangent to F', $\varphi X + C^{-2}\omega(\varphi X)\varphi^2 B$ is tangent to F', also. Putting

$$\varphi' = (\varphi + C^{-2}\varphi^2 B \otimes \omega \circ \varphi)_{/TF'},\ \xi' = C^{-1}\varphi B_{/F'},\ \eta' = -C^{-1}\omega \circ \varphi_{/TF'}, \tag{4.6}$$

it is easy to verify that (φ', ξ', η') is an almost contact structure and the metric g' induced by g is compatible with (φ', ξ', η').

We prove that φ' is parallel with respect to the Levi–Civita connection ∇' on (F', g'). In fact, considering two vector fields X, Y on F', by the Gauss equation $\nabla_X Y = \nabla'_X Y + g'(X,Y)B$, we have:

$$\begin{aligned}(\nabla'_X \varphi')Y &= \nabla_X(\varphi' Y) - g'(X, \varphi' Y)B - \varphi(\nabla'_X Y) - C^{-2}\omega(\varphi(\nabla'_X Y)) \\ &= (\nabla_X \varphi)Y - g(X, \varphi Y)B + g(X,Y)\varphi B + C^{-2}\omega(\varphi Y)\nabla_X(\varphi^2 B) \\ &\quad + C^{-2}((\nabla_X \omega)(\varphi Y) + \omega((\nabla_X \varphi)Y))\varphi^2 B.\end{aligned}$$

Then $(\nabla'_X \varphi')Y$ vanishes, since (1.1), (4.5), (2.2) and (2.3) imply:

$$(\nabla_X \varphi)Y - g(X, \varphi Y)B + g(X,Y)\varphi B = -\omega(\varphi Y)X, \tag{4.7}$$

$$\nabla_X(\varphi^2 B) = (\|\omega\|^2 - \omega(\xi)^2)X = C^2 X, \quad (\nabla_X \omega)\varphi Y + \omega((\nabla_X \varphi)Y) = 0. \tag{4.8}$$

Finally, we point out that the stated properties of $\mathcal{D}_2$ and $\mathcal{D}_2^\perp$ allow to consider (M,g), locally, as a warped product (9.104 [2]). More precisely, given

$p_0 \in M$, there exist an open neighborhood U of p_0, two Riemannian manifolds $(N, G), (N', G')$ with $TN \simeq \langle B, \xi \rangle, TN' \simeq \mathrm{Ker}\,\omega \cap Ker\eta$, a positive smooth function f on N and an isometry $\psi : (U, g) \to (N \times N', G + f^2 G')$.

Then $\pi = p_1 \circ \psi$, p_1 denoting the projection onto the first factor, is a Riemannian submersion with fibres isometric to the leaves of $\mathcal{D}_2^{\perp}$. Hence $(N', f^2 G')$ is equipped with the cosymplectic structure corresponding to the one defined in (4.6). Moreover, $B_{|U}$ projects onto the vector field $-\frac{1}{f}\mathrm{grad}f$, $\mathrm{grad}f$ denoting the gradient of f with respect to G. So, given a basic vector field X π-related to X', we have $\omega(X) = -G(\frac{1}{f}\mathrm{grad}f, X') \circ \pi = -d\log f(X') \circ \pi$. Since ω vanishes on the vertical distribution, we get $\omega_{|U} = -\pi^*(d\log f)$. □

The following construction provides examples of $N(k)$-spaces which fall in the class iv) of Theorem 12, k being a non-constant function.

Example. Given a positive smooth function $h : \mathbb{R} \to \mathbb{R}$, we consider the functions λ, μ, $\nu : \mathbb{R}^2 \to \mathbb{R}$ such that, for any $(x, y) \in \mathbb{R}^2$ $\lambda(x, y) = \frac{\exp(-x)}{h(y)} - y$, $\mu(x, y) = \frac{\exp(-x)}{h(y)}$, $\nu(x, y) = -\frac{\exp(-x)h'(y)}{h(y)^2} - 1$.

Then the vector fields $e_1 = \lambda\frac{\partial}{\partial x}$, $e_2 = \nu\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial y}$ are linearly independent at each point, so we can consider the Riemannian metric G which makes $\{e_1, e_2\}$ an orthonormal frame on $\mathbb{R}^2$. Since $[e_1, e_2] = -e_1 - \lambda e_2$, the Levi–Civita connection D on $(\mathbb{R}^2, G)$ is determined by:

$$D_{e_1}e_1 = e_2, \quad D_{e_1}e_2 = -e_1, \quad D_{e_2}e_1 = \lambda e_2,\ D_{e_2}e_2 = -\lambda e_1. \tag{4.9}$$

It follows that $(\mathbb{R}^2, G)$ has non-constant Gaussian curvature $R'(e_1, e_2, e_1, e_2) = -(\lambda^2 + e_1(\lambda) + 1)$.

Let $(F', \varphi', \xi', \eta', G')$ be a cosymplectic manifold and denote by φ the $(1,1)$-tensor field on $M = \mathbb{R}^2 \times F'$ such that

$$\begin{gathered} \varphi e_1 = 0, \quad \varphi e_2 = \mu\xi', \quad \varphi\xi' = -\frac{1}{\mu}e_2, \quad \varphi X = \varphi' X, \\ X \in TF' \quad \text{with} \quad \eta'(X) = 0. \end{gathered} \tag{4.10}$$

It is easy to verify that $(\varphi, e_1, e^1 = \flat e_1, g = G + \mu^{-2}G')$ is an almost contact metric structure on M. Since F' is a totally umbilical submanifold of M with mean curvature $\mathrm{grad}\log\mu$, the Levi–Civita connection ∇ on (M, g) satisfies, for any vector field X on F':

$$\begin{aligned} &\nabla_X Y = \nabla'_X Y + g(X, Y)\mathrm{grad}\log\mu, Y \in \mathcal{X}(F'), \\ &\nabla_X e_1 = \nabla_{e_1} X = \lambda X, \quad \nabla_X e_2 = \nabla_{e_2} X = -X. \end{aligned} \tag{4.11}$$

Recalling that $\mathbb{R}^2$ is totally geodesic, by a direct calculus, also applying (4.9), (4.11), one proves that $(M, \varphi, e_1, e^1, g)$ is g.c. cosymplectic with Lee form $\omega = d\log\mu$. The anti-Lee form never vanishes, since $\|\varphi B\| = \|\mu\xi'\| = 1$. Furthermore, (4.9), (4.11) entail the curvature formulas, for any X, Y tangent to F':

$$R(X,Y,e_1) = 0, \quad R(X,e_2,e_1) = 0, \quad R(X,e_1,e_1) = -(\lambda^2 + e_1(\lambda) + 1)X,$$
$$R(e_1,e_2,e_1) = (\lambda^2 + e_1(\lambda) + 1)e_2.$$

Therefore, M is an $N(k)$-space with $k = -(\lambda^2 + e_1(\lambda) + 1)$.

It is known that the $C(k)$-condition for an f-Kenmotsu manifold M with $\dim M \geq 5$ entails $k = -f^2$. Hence k is constant and either M is cosymplectic or $k < 0$ [18]. This fits in the following more general result.

Proposition 13. *Let $(M, \varphi, \xi, \eta, g)$ be an $N(k)$-l.c. cosymplectic space with Lee form ω. Assume that $\dim M = 2n+1 \geq 5$ and $(2n-1)\|\omega\|^2 = 2\delta\omega + (2n+1)k$. Then k is constant and one of the following cases occurs:*

i) *$\|\omega\|^2 + k$ never vanishes, ω is exact and $\tilde{g} = (\|\omega\|^2 + k)^2 g$ is a cosymplectic metric with curvature $R - k\pi_1$. In particular, $(M, \tilde{g})$ is flat if and only if (M, g) has constant sectional curvature,*

ii) *$k = 0$ and M is cosymplectic,*

iii) *$k = -\|\omega\|^2 < 0$, M is f-Kenmotsu, f is constant and $f^2 = -k$,*

iv) *$k = -\|\omega\|^2 < 0$, the anti-Lee form never vanishes and (M, g) is locally a warped product $N \times_{f^2} N'$, N being a 2-dimensional manifold of constant curvature and N' cosymplectic. Moreover, (M, g) has constant sectional curvature if and only if each leaf of $\mathcal{D}_2^\perp$ is flat.*

Proof. By Proposition 11 the hypothesis is equivalent to the request that M is an almost $C(k)$-manifold. In particular, k is constant. By (3.4) we have: $\alpha = \frac{k - \|\omega\|^2}{2}$ and (4.1) yields:

$$d(k + \|\omega\|^2) = (k + \|\omega\|^2)\omega. \tag{4.12}$$

Arguing as in the proof of Theorem 12, the connectedness of M and (4.10) imply that either $k + \|\omega\|^2$ never vanishes or $k = -\|\omega\|^2$. So, if $k + \|\omega\|^2 \neq 0$, $\omega = d\log|k + \|\omega\|^2|$ is exact and the Levi–Civita connection $\tilde{\nabla}$ of the cosymplectic structure $(\varphi, |k + \|\omega\|^2|^{-1}\xi, |k + \|\omega\|^2|\eta, \tilde{g} = (k + \|\omega\|^2)^2 g)$ acts as:

$$\tilde{\nabla}_X Y = \nabla_X Y + \omega(Y)X + \omega(X)Y - g(X,Y)B.$$

Since $\nabla\omega = \omega \otimes \omega + \frac{k-\|\omega\|^2}{2} g$, by direct calculus we obtain: $\tilde{R} = R - k\pi_1$, $\tilde{R}$ denoting the curvature of $(M, \tilde{g})$. Hence i) occurs. Now, we discuss the case $k = -\|\omega\|^2$. Obviously, M is cosymplectic if and only if $k = 0$. Assuming $k = -\|\omega\|^2 < 0$, we observe that $\omega(\xi)$ is constant. In fact $\delta\omega = -2nk$ and Proposition 7 entails $d(\omega(\xi)) = 0$. Hence, when $\|\omega\| = |\omega(\xi)|$, we have that M is f-kenmotsu and $f^2 = \|\omega\|^2 = -k$, so that case iii) occurs.

Finally, we consider the case $k = -\|\omega\|^2, \|\omega\|^2 > \omega(\xi)^2$. Then the anti-Lee form, which has constant norm, never vanishes. By Theorem 12, (M, g) is locally a warped product $N \times_{f^2} N'$, where N is isometric to a leaf of $\mathcal{D}_2$ and N' to a leaf of $\mathcal{D}_2^\perp$ equipped with the structure defined in (4.6). By Proposition 10, also applying the first Bianchi identity and (3.2), for any $X, Y \in \mathcal{X}(M)$, we have:

$$R(X, Y, B) = k\pi_1(X, Y, B), \quad R(B, \xi, X) = k\pi_1(B, \xi, X). \tag{4.13}$$

Let (F', g') be a leaf of $\mathcal{D}_2^\perp$. Considering $X \in \mathcal{X}(F')$, since $[X, \xi] = [X, B] = 0$, we have $\nabla_B X = \nabla_X B = kX$, $\nabla_\xi X = \nabla_X \xi = -\omega(\xi)X$. Then, also applying the Gauss equation $\nabla_X Y = \nabla'_X Y + g'(X, Y)B$, we obtain, for any $X, Y, Z \in \mathcal{X}(F')$:

$$\begin{aligned} &R(\xi, X, Y) = k\pi_1(\xi, X, Y), \quad R(B, X, Y) = k\pi_1(B, X, Y), \\ &R(X, Y, Z) = R'(X, Y, Z) + k\pi_1(X, Y, Z), \end{aligned} \tag{4.14}$$

R' denoting the curvature of F'. Formulas (3.2), (4.13), (4.14) imply that the flatness of each leaf of $\mathcal{D}_2^\perp$ is equivalent to the condition $R = k\pi_1$. $\square$

Remark. Proposition 13 also allows to describe almost $C(0)$-l.c. cosymplectic spaces. In fact, we assume that the curvature of a connected l.c. cosymplectic manifold $(M, \varphi, \xi, \eta, g)$ with $\dim M = 2n + 1 \geq 5$ satisfies, for any vector fields X, Y, Z, W : $R(X, Y, Z, W) = R(X, Y, \varphi Z, \varphi W)$. By Proposition 11, M is an $N(0)$-space and $(2n - 1)\|\omega\|^2 = 2\delta\omega$. Therefore M satisfies the hypothesis of Proposition 13. It follows that either M is cosymplectic or ω never vanishes. Furthermore, if $\omega \neq 0$, we have $\omega = d \log \|\omega\|^2$ and R coincides with the curvature of the cosymplectic metric $\tilde{g} = \|\omega\|^4 g$.

Now, we state another classification of the spaces considered in Proposition 13 for which $k = -\|\omega\|^2$.

Proposition 14. *Let $(M, \varphi, \xi, \eta, g)$ be an $N(k)$-l.c. cosymplectic space with $\dim M = 2n + 1 \geq 5$. Assume that $\delta\omega = 2n\|\omega\|^2$ and $k = -\|\omega\|^2 < 0$. Then, any leaf of the distribution $\mathcal{D}_1$ is a totally umbilical submanifold of M with mean curvature the Lee vector field B and inherits from M a Kähler structure. Furthermore, the following equivalences hold:*

a) (M, g) *has constant sectional curvature if and only if all the leaves of* $\mathcal{D}_1$ *are flat.*

b) (M, g) *is Einstein if and only if all the leaves of* $\mathcal{D}_1$ *are Ricci-flat.*

Proof. By Proposition 13, k is constant. Moreover, for any $X \in \mathcal{X}(M)$ we have:

$$\nabla_X B = \omega(X)B - \|\omega\|^2 X, \tag{4.15}$$

and the Weingarten operator a_B of a leaf N' of $\mathcal{D}_1$ acts as $a_B X = \|\omega\|^2 X$.

So, the second fundamental form α is given by $\alpha(X, Y) = g'(X, Y)B$, g' denoting the metric induced by g on N'.

Moreover, if $\omega = \omega(\xi)\eta$, then $(N', J' = \varphi_{|TN'}, g')$ is a Kähler manifold [18].

Now, we consider the case $\|\omega\|^2 > \omega(\xi)^2$. Then $C' = \|\omega\| + \omega(\xi)$ is a nonzero constant and, for any $X \in \mathcal{D}_1$, $\varphi X - \frac{1}{C'}\omega(\varphi X)\big(\frac{B}{\|\omega\|} + \xi\big)$ is in $\mathcal{D}_1$, also. Let N' be a leaf of $\mathcal{D}_1$ and J' the endomorphism of TN' defined by:

$$J'X = \varphi X - \frac{1}{C'}\left(\eta(X)\varphi B + \omega(\varphi X)\left(\frac{B}{\|\omega\|} + \xi\right)\right)_{|N'}. \tag{4.16}$$

We prove that (J', g') is an almost Hermitian structure.

In fact, since $\xi'_0 = \|\omega\|\xi - \frac{\omega(\xi)}{\|\omega\|}B$ is orthogonal to B and φB, $\xi'_{0|N'}$ and $\varphi B_{|N'}$, simply denoted by $\xi'_0, \varphi B$, are mutually orthogonal, $\|\xi'_0\| = \|\varphi B\|$ and $J'(\varphi B) = \xi'_0, J'(\xi'_0) = -\varphi B$. Considering X tangent to N' and orthogonal to ξ'_0 and φB one has: $\eta(X) = 0, J'X = \varphi X$, hence $\|J'X\| = \|X\|$ and $J'^2 X = -X$. Therefore J' is an almost complex structure and g' is compatible with J'.

Now, we prove that J' is parallel with respect to the Levi–Civita connection ∇' of N'. Firstly, we point out that $\nabla_X B = -\|\omega\|^2 X$, $\nabla'_X \varphi B = 0$, for any $X \in TN'$. Hence we have: $\nabla'_X \xi'_0 = \|\omega\|(\nabla_X \xi - \eta(X)B + \omega(\xi)X) = 0$, and then $(\nabla'_X J')\xi'_0 = 0, (\nabla'_X J')\varphi B = 0$. Furthermore, given a vector field Y on N' orthogonal to ξ'_0 and φB, $\nabla'_X Y$ is orthogonal to ξ'_0 and φB, also. Then $(\nabla'_X J')Y = \nabla'_X \varphi Y - \varphi(\nabla'_X Y) = (\nabla_X \varphi)Y - g'(X, \varphi Y)B + g'(X, Y)\varphi B = 0$. Therefore, (N', J', g') is Kähler.

Finally, we state the equivalences a) and b). By the Codazzi equation, the curvature R' of a leaf N' of $\mathcal{D}_1$ acts as:

$$R'(X, Y, Z, W) = R(X, Y, Z, W) - k(g'(X, Z)g'(Y, W) - g'(Y, Z)g'(X, W)),$$

for any $X, Y, Z, W \in TN'$.

By Proposition 10, for any $X, Y \in \mathcal{X}(M)$, we have:

$$R(X, Y, B) = k(\omega(Y)X - \omega(X)Y).$$

Then a) follows, taking account of the symmetries of R.

Moreover, the Ricci tensor ρ' of a leaf N' of $\mathcal{D}_1$ acts as:

$$\rho'(X,Y) = \rho(X,Y) - 2nkg'(X,Y), \quad X,Y \in TN'. \tag{4.17}$$

and we have, also:

$$\rho(X,B) = 2nk\omega(X), \quad X \in TM. \tag{4.18}$$

Then the scalar curvature of a leaf N' of $\mathcal{D}_1$ is given by:

$$\tau' = \tau - 2n(2n+1)k. \tag{4.19}$$

Assume that (M,g) is Einstein. By (4.19) and (4.17) each leaf N' of $\mathcal{D}_1$ is an Einstein manifold with Ricci tensor $\rho' = \frac{\tau'}{2n+1}g'$. Since $\dim N' = 2n$ we also have: $\tau' = trace\rho' = \frac{2n}{2n+1}\tau'$. Then $\tau' = 0$ and N' is Ricci-flat.

The converse statement in b) follows by (4.17),(4.18). □

Theorem 15. *Let (M,φ,ξ,η,g) be an $N(k)$-l.c. cosymplectic space with $\dim M = 2n+1 \geq 5$. Assume that the Lee form ω satisfies: $\delta\omega = 2n\|\omega\|^2$ and $k = -\|\omega\|^2 < 0$. Then M is locally a warped product $]-\epsilon,\epsilon[\,\times_{h^2} F$, where F is a Kähler manifold, $h^2 = a\exp(-2\|\omega\|t)$, for some positive constant a, t denoting the Euclidean coordinate.*

Proof. We know that $TM = \langle B\rangle \oplus \ker\omega$, the corresponding distributions $\mathcal{D}_1^\perp, \mathcal{D}_1$ are integrable, the integral manifolds of D_1 are totally umbilical Kähler manifolds with second fundamental form $\alpha = g \otimes B$ and (4.15) entails that $\mathcal{D}_1^\perp$ is totally geodesic. Then, as a manifold, M is locally a product $]-\epsilon,\epsilon[\,\times F$, where $T(\,]-\epsilon,\epsilon[\,) = \langle B\rangle$ and F is Kähler. We can choose a neighborhood with coordinates $(t,x^1,\dots,x^{2n})$ such that $\pi_*B = \|\omega\|\frac{\partial}{\partial t}$, $\pi : \,]-\epsilon,\epsilon[\,\times F \to \,]-\epsilon,\epsilon[$ being the first projection. Then π is a C^∞-submersion with vertical distribution $\mathcal{V} = TF$ and horizontal distribution $\mathcal{H} = T(\,]-\epsilon,\epsilon[\,)$. The splitting $\mathcal{V}\oplus\mathcal{H}$ is orthogonal and π preserves the length of horizontal vectors, so π is a Riemannian submersion. The vector field $H = 2nB$ is basic, the O'Neill tensor A vanishes and, by direct calculus, the trace-free part T^0 of the O'Neill tensor T vanishes. Hence $]-\epsilon,\epsilon[\,\times F$, and then M, is locally a warped product by a smooth function h^2, $h > 0$. Moreover $H = 2nB$ is π-related to $-\frac{2n}{h}\mathrm{grad}_0 h$, where the gradient is evaluated with respect to the Euclidean metric [2]. Since B is π-related to $\|\omega\|\frac{\partial}{\partial t}$, we have: $d\log h = -\|\omega\|dt$, $\|\omega\|$ being constant. Hence $h = \lambda\exp(-\|\omega\|t), \lambda$ constant, and the warped metric is locally given by $dt\otimes dt + \lambda^2\exp(-2\|\omega\|t)\tilde{g}$, $\tilde{g}$ being a Kähler metric. □

Example. On the hyperbolic space $\mathbb{H}^{2n+1}=\{(x^1,\dots,x^{2n+1})\in\mathbb{R}^{2n+1};\ x^1>0\}$ we consider the metric $g_c=\frac{1}{(cx^1)^2}\sum_{1\le i\le 2n+1}dx^i\otimes dx^i$ of constant sectional curvature $-c^2, c>0$. For any $i\in\{1,\dots,2n+1\}$ we put $E_i=cx^1\frac{\partial}{\partial x^i}$, and the action of the Levi–Civita connection with respect to the orthonormal frame $\{E_1,\dots,E_{2n+1}\}$ is given by:

$$\nabla_{E_i}E_j=c(\delta_{ij}E_1-\delta_{1j}E_i),\quad i,j\in\{1,\dots,2n+1\}\,. \tag{4.20}$$

Let (φ,ξ,η,g_c) be an almost contact metric structure on $\mathbb{H}^{2n+1}$ such that φ and ξ have constant components with respect to $\{E_i\}_{1\le i\le 2n+1}$ and assume $n\ge 2$. We claim that $(\mathbb{H}^{2n+1},\varphi,\xi,\eta,g_c)$ is an $N(-c^2)$-l.c. cosymplectic space whose Lee form satisfies $\|\omega\|^2=c^2,\ \delta\omega=2n\|\omega\|^2$. In fact the $N(-c^2)$-condition holds since g_c has constant sectional curvature $-c^2$. We know that $(\mathbb{H}^{2n+1},\varphi,\xi,\eta,g_c)$ is in the Chinea–Gonzales class $\mathcal{C}_4\oplus\mathcal{C}_5\oplus\mathcal{C}_{12}$ [8]. We remark that, as stated in [12], in dimension $2n+1\ge 5$, any manifold in this class is l.c. cosymplectic if and only if the codifferential of the fundamental form Φ, $\Phi(X,Y)=g(X,\varphi Y)$, satisfies:

$$\delta\Phi\circ\varphi+(2n-1)\nabla_\xi\eta=0. \tag{4.21}$$

Condition (4.21) is fulfilled by $\mathbb{H}^{2n+1}$.

By (4.20), we have: $\nabla_\xi\xi=-c\varphi^2(E_1)$, $\sum_{1\le i\le 2n+1}(\nabla_{E_i}\varphi)E_i=-(2n-1)c\varphi E_1$, so $\delta\Phi(\varphi X)=\sum_{1\le i\le 2n+1}g_c((\nabla_{E_i}\varphi)E_i,\varphi X)=(2n-1)cg_c(\varphi^2E_1,X)=-(2n-1)(\nabla_\xi\eta)X$.

Since moreover $\delta\eta=-\sum_{1\le i\le 2n+1}g_c(\nabla_{E_i}\xi,E_i)=2ncg_c(\xi,E_1)$, the Lee form acts as $\omega(X)=cg_c(E_1,X)$. It follows that $B=cE_1$ is the Lee vector field and $\omega=\frac{1}{x^1}dx^1$, so $\|\omega\|=c$. By (4.20) we also have $\delta\omega=2nc^2$. Hence $(\mathbb{H}^{2n+1},\varphi,\xi,\eta,g_c)$ satisfies the hypothesis of Proposition 14, the distribution $\mathcal{D}_1$ corresponding to the subbundle spanned by $\{E_2,\dots,E_{2n+1}\}$. Then the leaves of $\mathcal{D}_1$ carry a flat Kähler structure and by Theorem 15 g_c can be locally expressed as $dt\otimes dt+a\exp(-2\|\omega\|t)g_0$, t being the Euclidean coordinate, $a\in\mathbb{R}^*_+$ and g_0 a Kähler metric. In this case, the previous expression holds everywhere, considering $t=\frac{1}{c}\log x^1, a=\frac{1}{c^2}, g_0=\sum_{2\le i\le 2n+1}dx^i\otimes dx^i$.

Finally, we describe the action of the complex structure J' on a leaf F of $\mathcal{D}_1$. If $\omega=\omega(\xi)\eta$, namely if $\xi=E_1$ or $\xi=-E_1$, then $J'=\varphi_{|TF}$, so J' has constant components with respect to a suitable orthonormal frame. If $\xi=\xi^iE_i$, with $\xi^i\ne 0$ for some $i\ge 2$, as in Proposition 14 we consider the vector field $\xi_0'=\|\omega\|\xi-\frac{\omega(\xi)}{\|\omega\|}B=c(\xi-\xi^1E_1)$, which is tangent to F and orthogonal to φB. With respect to an orthonormal frame $\{Y_1,\dots,Y_{2n-2},\xi_0',\varphi B\}$, J' acts as $J'(Y_i)=\varphi Y_i$, $i\in\{1,\dots,2n-2\}$, $J'(\xi_0')=\varphi E_1$, $J'(\varphi E_1)=-\xi_0'$.

The previous example ensures the existence of $N(k)$-l.c. conformal cosymplectic spaces which are locally symmetric. We also recall that, for any $c > 0$, the hyperbolic space $(\mathbb{H}^{2n+1}, g_c)$ is the local model of non-cosymplectic f-Kenmotsu locally symmetric spaces, $f = -c^2$ [18].

This fits in the following more general result.

Theorem 16. *Let* $(M, \varphi, \xi, \eta, g)$ *be a connected, non-cosymplectic, l.c. cosymplectic manifold with* $\dim M = 2n + 1 \geq 5$. *The following conditions are equivalent:*

i) M *has constant sectional curvature* k,

ii) M *is a locally symmetric* $N(k)$*-l.c. cosymplectic space.*

Moreover, if one of the previous conditions holds, either $k + \|\omega\|^2 \neq 0$ *everywhere and* $\tilde{g} = (\|\omega\|^2 + k)^2 g$ *is a flat cosymplectic metric on* M, *or* $k = -\|\omega\|^2$ *and* (M, g) *is locally a warped product* $]-\epsilon, \epsilon[\, \times_{h^2} F$, *where* F *is a flat Kähler manifold,* $h^2 = a \exp(-2\|\omega\| t)$, t *being the Euclidean coordinate,* $a = const > 0$.

Proof. The statement i) $\rightarrow$ ii) is obvious. Viceversa, assuming *ii*), by (3.11), for any vector field X, one has $\rho(X, \xi) = 2nk\eta(X)$, hence $(\nabla_X \rho)(\xi, \xi) = 2nX(k)$. Then k is constant, ρ being ∇-parallel. By Proposition 9 and Lemma 8 the function β defined in (3.3) satisfies:

$$\beta(\omega - \omega(\xi)\eta) = 0, \quad d\beta = 2\beta\omega(\xi)\,\eta. \tag{4.22}$$

Then, applying Proposition 10 and the hypothesis, for any $U, X, Y \in \mathcal{X}(M)$ we have:

$$0 = (\nabla_U R)(X, Y, \xi) = \omega(\xi)\,(R - k\pi_1)(X, Y, U).$$

Since $R - k\pi_1$ is ∇-parallel, $\omega(\xi)$ is constant and then, if $\omega(\xi) \neq 0$, M has constant sectional curvature k. Now, we assume $\omega(\xi) = 0$, apply (4.22) and obtain $\beta\omega = 0$, β being constant. Hence, since M is non-cosymplectic, β vanishes, namely $(2n-1)\|\omega\|^2 = 2\delta\omega + (2n+1)k$. By Proposition 11 M is an almost $C(k)$-manifold, that is, for any vector fields X, Y, Z, one has:

$$\begin{aligned} R(X, Y, \varphi Z) - \varphi(R(X, Y, Z)) = k(&g(Y, \varphi Z)X - g(X, \varphi Z)Y \\ &- g(Y, Z)\varphi X + g(X, Z)\varphi Y). \end{aligned}$$

Therefore the covariant derivative ∇R satisfies:

$$\begin{aligned} (\nabla_U R)(X, Y, \varphi Z) - \varphi((\nabla_U R)(X, Y, Z)) &= (\nabla_U \varphi)(R(X, Y, Z)) \\ &- R(X, Y, (\nabla_U \varphi)Z) - k(g(Y, Z)(\nabla_U \varphi)X - g(X, Z)(\nabla_U \varphi)Y \\ &+ g(X, (\nabla_U \varphi)Z)Y - g(Y, (\nabla_U \varphi)Z)X). \end{aligned} \tag{4.23}$$

By i) in Proposition 10, the hypothesis and (4.23) we also obtain:

$$R(X,Y,(\nabla_U\varphi)B) = -k(g(X,(\nabla_U\varphi)B)Y - g(Y,(\nabla_U\varphi)B)X),$$

and, by (1.1), we have: $\|\omega\|^2(R-k\pi_1)(X,Y,\varphi U) = 0$.

Since moreover $(R-k\pi_1)(X,Y,\xi) = 0$, we get: $\|\omega\|^2(R-k\pi_1) = 0$. Since M is non-cosymplectic and $R-k\pi_1$ is parallel, we obtain $R = k\pi_1$, namely M has constant sectional curvature k.

Finally, assuming i), equivalently ii), M is a $C(k)$-space, so the function α defined in (3.3) satisfies: $2\alpha = k - \|\omega\|^2$ and (3.7) reduces to $d(\|\omega\|^2) = (k + \|\omega\|^2)\omega$. Therefore, either $k+\|\omega\|^2$ never vanishes or $k+\|\omega\|^2 = 0$.

If $k+\|\omega\|^2 \neq 0$, ω is exact and $\tilde{g} = (k+\|\omega\|^2)^2 g$ is a cosymplectic metric with curvature $\tilde{R} = R - k\pi_1 = 0$.

If $k = -\|\omega\|^2$, we also have $\delta\omega = 2n\|\omega\|^2$. By Theorem 15, (M,g) is locally a warped product $]-\epsilon,\epsilon[\times_{h^2} F$, $h^2 = a\exp(-2\|\omega\|t)$, $a>0$, and F is a Kähler manifold. The flatness of F follows by Proposition 14, since F is isometric to a leaf of $\mathcal{D}_1$ and M has constant sectional curvature. □

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MARIA FALCITELLI
DIPARTIMENTO DI MATEMATICA
CAMPUS UNIVERSITARIO
VIA E. ORABONA, 4
70125 BARI
ITALIA

E-mail: falci@dm.uniba.it

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