

Pointwise approximation theorems for Meyer–König and Zeller–Durrmeyer operators

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Abstract. In this paper, we give the estimates of the second and fourth-order moments for the Meyer–König and Zeller–Durrmeyer type operators. Secondly, using the equivalence between the unified moduli of smoothness $\omega_{\varphi^\lambda}^2(f, t)$ and the Peetre’s K -functional $K_{\varphi^\lambda}^2(f, t^2)$ ($0 \leq \lambda \leq 1$), we obtain the direct, inverse and equivalence theorems for these operators.

1. Introduction

The Meyer–König and Zeller operators are given by [4], [8], [11], [12]

$$M_n(f, x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) m_{n,k}(x), \quad 0 \leq x < 1,$$

$$M_n(f, 1) = f(1), \quad m_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}.$$

There are several studies about these operators in approximation theory (cf. [1], [4], [7]–[9], [11], [12]). DITZIAN [5] introduced the unified moduli of smoothness $\omega_{\varphi^\lambda}^2(f, t)$ ($0 \leq \lambda \leq 1$) and gave an interesting direct result for the Bernstein operators which united the result with $\omega^2(f, t)$ and $\omega_\varphi^2(f, t)$. It is difficult to get the estimates of the moments for the Meyer–König and Zeller type operators (cf. [2], [3], [10]). The direct and inverse results for Durrmeyer-type modifications

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of the Meyer-König and Zeller operators are not perfect. In this paper, we will consider a new modification of Meyer-König and Zeller-Durrmeyer type operators $\widetilde{M}_n(f, x)$:

$$\widetilde{M}_n(f, x) = \sum_{k=0}^{\infty} \Phi_{n,k}(f) m_{n,k}(x), \quad f(x) \in C[0, 1],$$

where

$$\Phi_{n,k}(f) = \begin{cases} C_{n-2,k-1}^{-1} \int_0^1 f(t) m_{n-2,k-1}(t) dt, & k > 0, \\ f(0), & k = 0, \end{cases}$$

$$C_{n,k} = \int_0^1 m_{n,k}(t) dt = \frac{n+1}{(n+k+1)(n+k+2)}.$$

Let

$$\|f\| = \sup_{x \in [0,1]} |f(x)|, \quad \omega_{\varphi^\lambda}^2(f, t) = \sup_{0 < h \leq t} \|\Delta_{h\varphi^\lambda}^2\|,$$

where $\varphi(x) = \sqrt{x(1-x)}$ and

$$\Delta_{h\varphi^\lambda}^2 f(x) = \begin{cases} f(x + h\varphi^\lambda(x)) - 2f(x) + f(x - h\varphi^\lambda(x)), & \text{if } x \pm h\varphi^\lambda(x) \in [0, 1]; \\ 0, & \text{otherwise,} \end{cases}$$

$$K_{\varphi^\lambda}^2(f, t^2) = \inf_{g \in D} \{\|f - g\| + t^2 \|\varphi^{2\lambda} g''\|\},$$

where

$$D = \{g \mid g' \in A.C_{loc}, \|\varphi^{2\lambda} g''\| < \infty\}.$$

It is well known that $\omega_{\varphi^\lambda}^2(f, t) \sim K_{\varphi^\lambda}^2(f, t^2)$ (cf. [6]), i.e. there exists a positive constant C such that

$$C^{-1} K_{\varphi^\lambda}^2(f, t^2) \leq \omega_{\varphi^\lambda}^2(f, t) \leq C K_{\varphi^\lambda}^2(f, t^2).$$

Noting that it is absolutely superfluous to talk about $f(1)$, we can assume that the function space is $C_B[0, 1)$ (the set of bounded and continuous functions on $[0, 1)$). Now we state the direct and inverse theorems.

Theorem 1.1. *For $f \in C_B[0, 1)$, $0 \leq \lambda \leq 1$, there holds*

$$|\widetilde{M}_n(f, x) - f(x)| \leq C \omega_{\varphi^\lambda}^2(f, n^{-\frac{1}{2}} \varphi^{1-\lambda}(x)).$$

Theorem 1.2. For $f \in C_B[0, 1)$, $0 < \alpha < 2$, $0 \leq \lambda \leq 1$, $n \geq 2$, if

$$|\widetilde{M}_n(f, x) - f(x)| = O(n^{-\frac{\alpha}{2}} \varphi^{\alpha(1-\lambda)}(x)),$$

then $\omega_{\varphi^\lambda}^2(f, t) = O(t^\alpha)$.

Remark 1. In this paper the notation $f(x) = O(g(x))$ means there exists a positive constant C such that $f(x) \leq Cg(x)$.

Remark 2. Throughout this paper, C denotes a positive constant independent of n and x and not necessarily the same at each occurrence.

From Theorem 1.1, 1.2, we obtain the equivalence theorem.

Theorem 1.3. For $f \in C_B[0, 1)$, $0 < \alpha < 2$, $0 \leq \lambda \leq 1$, $n \geq 2$, one has

$$|\widetilde{M}_n(f, x) - f(x)| = O(n^{-\frac{\alpha}{2}} \varphi^{\alpha(1-\lambda)}(x)) \Leftrightarrow \omega_{\varphi^\lambda}^2(f, t) = O(t^\alpha).$$

2. The direct theorem

In order to prove Theorem 1.1, we shall need the following lemmas.

Lemma 2.1. For $x \in [0, 1)$ and sufficiently large n , there holds

$$\widetilde{M}_n(t - x, x) = 0, \quad (2.1)$$

$$\widetilde{M}_n((t - x)^2, x) = O\left(\frac{\varphi^2(x)}{n}\right), \quad (2.2)$$

$$\widetilde{M}_n((t - x)^4, x) = O\left(\frac{\varphi^4(x)}{n^2}\right). \quad (2.3)$$

PROOF. By an easy calculations, we can obtain (2.1) and (2.2). Next we estimate (2.3). Writing

$$\begin{aligned} \widetilde{M}_n(t^4, x) &= \sum_{k=1}^{\infty} \frac{k(k+1)(k+2)(k+3)}{(n+k)(n+k+1)(n+k+2)(n+k+3)} m_{n,k}(x) \\ &= \varphi^4(x) \sum_{k=0}^{\infty} \frac{(n+k-3)(n+k-2)(n+k-1)(n+k)(n+k+1)}{(k+1)(n-3)(n-2)(n-1)n} \\ &\quad \cdot \frac{(k+3)(k+4)(k+5)}{(n+k+3)(n+k+4)(n+k+5)} m_{n-4,k}(x), \end{aligned}$$

$$\begin{aligned}
x\widetilde{M}_n(t^3, x) &= x \sum_{k=1}^{\infty} \frac{k(k+1)(k+2)}{(n+k)(n+k+1)(n+k+2)} m_{n,k}(x) \\
&= \varphi^4(x) \sum_{k=0}^{\infty} \frac{(n+k-3)(n+k-2)(n+k-1)(n+k)(k+2)(k+3)}{(n-3)(n-2)(n-1)n(n+k+3)(n+k+2)} m_{n-4,k}(x), \\
x^2\widetilde{M}_n(t^2, x) &= \varphi^4(x) \sum_{k=1}^{\infty} \frac{(n+k-3)(n+k-2)(n+k-1)k(k+1)}{(n-3)(n-2)(n-1)n(n+k+1)} m_{n-4,k}(x), \\
x^4 &= \varphi^4(x) \sum_{k=0}^{\infty} \frac{(n+k-3)(n+k-2)(k-1)k}{(n-3)(n-2)(n-1)n} m_{n-4,k}(x),
\end{aligned}$$

recalling that

$$\widetilde{M}_n((t-x)^4, x) = \widetilde{M}_n(t^4, x) - 4x\widetilde{M}_n(t^3, x) + 6x^2\widetilde{M}_n(t^2, x) - 3x^4,$$

we can get the result. $\square$

Lemma 2.2. For $k \geq 1$ and sufficiently large n , one has

$$C_{n-2,k-1}^{-1} \int_0^1 (1-t)^{-2} m_{n-2,k-1}(t) dt \leq C \left(\frac{n}{n+k} \right)^{-2}, \quad (2.4)$$

$$C_{n-2,k-1}^{-1} \int_0^1 (1-t)^4 m_{n-2,k-1}(t) dt \leq C \left(\frac{n}{n+k} \right)^4, \quad (2.5)$$

$$C_{n-2,k-1}^{-1} \int_0^1 t^2 m_{n-2,k-1}(t) dt \leq C \left(\frac{k}{n+k} \right)^2. \quad (2.6)$$

PROOF. We only need to prove (2.4). The proof of (2.5)(2.6) is similar.

For $k \geq 1$, there holds

$$\int_0^1 t^{k-1} (1-t)^{n-3} = \frac{(n-3)!(k-1)!}{(n+k-3)!}.$$

By the observation

$$\frac{(n+k-2)(n+k-1)}{n-1} \frac{(n+k-3)!}{(n-2)!(k-1)!} \frac{(n-3)!(k-1)!}{(n+k-3)!} \left(\frac{n}{n+k} \right)^2 \leq 4,$$

the proof of (2.4) is completed. $\square$

Lemma 2.3 (cf. [11]). *For $l \in N_0, m \in Z$, there holds*

$$\sum_{k=1}^{\infty} m_{n,k}(x) \left(\frac{k}{n+k} \right)^{-l} \leq Mx^{-l}, \sum_{k=1}^{\infty} m_{n,k}(x) \left(\frac{n}{n+k} \right)^{-m} \leq M(1-x)^{-m}.$$

Remark 3. For $x \in E_n = [\frac{1}{n}, 1)$, one has

$$\sum_{k=1}^{\infty} m_{n,k}(x) \left(\frac{k}{n+k} \right)^2 \leq Mx^2.$$

Lemma 2.4. *For $f \in C[0, 1]$, one has $|\widetilde{M}_n(f, x)| \leq \|f\|$.*

PROOF. We can get the result by direct computations. $\square$

Lemma 2.5. *For $f \in D, 0 \leq \lambda \leq 1$, there holds $|\widetilde{M}_n(f, x) - f(x)| \leq Cn^{-1}\varphi^{2(1-\lambda)}(x)\|\varphi^{2\lambda}f''\|$.*

PROOF. It is known from the Taylor's expansion, that

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t f''(v)(t-v)dv,$$

one can write

$$\widetilde{M}_n(f, x) - f(x) = f'(x)\widetilde{M}_n(t-x, x) + \widetilde{M}_n(R_2(f, t, x), x),$$

where $R_2(f, t, x) = \int_x^t f''(v)(t-v)dv$.

Since $\widetilde{M}_n(t-x, x) = 0$, we have

$$\begin{aligned} |\widetilde{M}_n(f, x) - f(x)| &= \widetilde{M}_n(R_2(f, t, x), x) \\ &= \|f''\varphi^{2\lambda}\| \sum_{k=0}^{\infty} C_{n-2, k-1}^{-1} \int_0^1 \left| \int_x^t \frac{|t-v|}{\varphi^{2\lambda}(v)} dv \right| m_{n-2, k-1}(t) dt m_{n, k}(x). \end{aligned}$$

It is sufficient to show that

$$\widetilde{M}_n \left(\left| \int_x^t \frac{|t-v|}{\varphi^{2\lambda}(v)} dv \right|, x \right) = O \left(\frac{\varphi^{2-2\lambda}(x)}{n} \right). \quad (2.7)$$

For v between t and x , there holds [4]

$$\left| \int_x^t \frac{|t-v|}{\varphi^{2\lambda}(v)} dv \right| \leq g_x(t) = \begin{cases} \frac{(t-x)^2}{\varphi^{2\lambda}(x)}, & \text{for } t \leq x; \\ \frac{(t-x)^2}{(x(1-x)(1-t))^{\lambda}}, & \text{for } t > x. \end{cases}$$

Therefore,

$$\begin{aligned}
& \widetilde{M}_n \left(\left| \int_x^t \frac{|t-v|}{\varphi^{2\lambda}(v)} dv \right|, x \right) \\
&= \sum_{k=0}^{\infty} C_{n-2,k-1}^{-1} \int_0^1 \left| \int_x^t \frac{|t-v|}{\varphi^{2\lambda}(v)} dv \right| m_{n-2,k-1}(t) dt m_{n,k}(x) \\
&= \sum_{k=0}^{\infty} C_{n-2,k-1}^{-1} \int_0^x \frac{(t-x)^2}{\varphi^{2\lambda}(x)} m_{n-2,k-1}(t) dt m_{n,k}(x) \\
&\quad + \sum_{k=0}^{\infty} C_{n-2,k-1}^{-1} \int_x^1 \frac{(t-x)^2}{(x(1-x)(1-t))^{\lambda}} m_{n-2,k-1}(t) dt m_{n,k}(x) =: E_1 + E_2.
\end{aligned}$$

We will estimate E_1, E_2 separately. Using (2.2), we can deduce

$$\begin{aligned}
E_1 &\leq \sum_{k=0}^{\infty} C_{n-2,k-1}^{-1} \int_0^1 \frac{(t-x)^2}{\varphi^{2\lambda}(x)} m_{n-2,k-1}(t) dt m_{n,k}(x) \\
&\leq \varphi^{-2\lambda} \widetilde{M}_n((t-x)^2, x) \leq C n^{-1} \varphi^{2-2\lambda}(x).
\end{aligned}$$

By Lemma 2.2 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
E_2 &\leq \sum_{k=0}^{\infty} C_{n-2,k-1}^{-1} \int_0^1 \frac{(t-x)^2}{(x(1-x)(1-t))^{\lambda}} m_{n-2,k-1}(t) dt m_{n,k}(x) \\
&\leq \left(\sum_{k=0}^{\infty} C_{n-2,k-1}^{-1} \int_0^1 (t-x)^4 m_{n-2,k-1}(t) dt m_{n,k}(x) \right)^{\frac{1}{2}} \\
&\quad \cdot (x(1-x))^{-\lambda} \left(\sum_{k=0}^{\infty} C_{n-2,k-1}^{-1} \int_0^1 (1-t)^{-2\lambda} m_{n-2,k-1}(t) dt m_{n,k}(x) \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k=0}^{\infty} C_{n-2,k-1}^{-1} \int_0^1 (t-x)^4 m_{n-2,k-1}(t) dt m_{n,k}(x) \right)^{\frac{1}{2}} \\
&\quad \cdot (x(1-x))^{-\lambda} \left(\sum_{k=0}^{\infty} C_{n-2,k-1}^{-1} \int_0^1 (1-t)^{-2} m_{n-2,k-1}(t) dt m_{n,k}(x) \right)^{\frac{\lambda}{2}} \\
&\leq C n^{-1} \varphi^{2-2\lambda}(x).
\end{aligned}$$

From the above two inequalities, we can get (2.7) which completed Lemma 2.5. $\square$

PROOF OF THEOREM 1.1. From the definition of $K_{\varphi^\lambda}^2(f, t^2)$, we can choose $g \in D$ satisfying

$$\|f - g\| + t^2 \|\varphi^{2\lambda} g''\| \leq 2K_{\varphi^\lambda}^2(f, t^2). \quad (2.8)$$

By Lemma 2.4 and Lemma 2.5, there holds

$$\begin{aligned} \left| \widetilde{M}_n(f, x) - f(x) \right| &\leq \left| \widetilde{M}_n(f - g, x) \right| + |f(x) - g(x)| + \left| \widetilde{M}_n(g, x) - g(x) \right| \\ &\leq C_1 \left[\|f - g\| + n^{-1} \varphi^{2(1-\lambda)}(x) \|\varphi^{2\lambda} g''\| \right]. \end{aligned}$$

Hence by (2.8) and set $t = n^{-\frac{1}{2}} \varphi^{1-\lambda}(x)$, one has

$$\left| \widetilde{M}_n(f, x) - f(x) \right| \leq CK_{\varphi^\lambda}^2 \left(f, n^{-1} \varphi^{2(1-\lambda)}(x) \right).$$

Using the relation $K_{\varphi^\lambda}^2(f, t^2) \sim \omega_{\varphi^\lambda}^2(f, t)$, we get

$$\left| \widetilde{M}_n(f, x) - f(x) \right| \leq C \omega_{\varphi^\lambda}^2(f, n^{-\frac{1}{2}} \varphi^{1-\lambda}(x)). \quad \square$$

3. The inverse theorem

For the convenience of the proof of Theorem 1.2, we need some new notations.

Set

$$\begin{aligned} \|f\|_0 &= \sup_{x \in (0,1)} \left| \varphi^{\alpha(\lambda-1)}(x) f(x) \right|, \\ C_0 &= \{f \in C_B[0, 1], f(0) = f(1) = 0\}, \\ C_{\alpha, \lambda}^0 &= \{f \in C_0, \|f\|_0 < \infty\}, \\ \|f\|_2 &= \sup_{x \in (0,1)} \left| \varphi^{2+\alpha(\lambda-1)}(x) f''(x) \right|, \\ C_{\alpha, \lambda}^2 &= \{f \in C_0, \|f\|_2 < \infty, f' \in A.C_{loc}\}. \end{aligned}$$

Let us introduce a new K -functional, for $f \in C_0$

$$K_\lambda^\alpha(f, t^2) = \inf_{g \in C_{\alpha, \lambda}^2} \{\|f - g\|_0 + t^2 \|g\|_2\}.$$

For the proof of the inverse theorem, we also need the following lemmas, which will be proved in the next section.

Lemma 3.1 (cf. [10]). *Let $\varphi(x) = \sqrt{x}(1-x)$, $A_{n,2p}(x) = M_n((t-x)^{2p}, x)$, $p \in \mathbb{N}$, we have the estimates, for $n > 2p$,*

$$A_{n,2p}(x) \leq C \begin{cases} \frac{\varphi^{2p}(x)}{n^p}, & \text{for } x \geq \frac{1}{n}; \\ \frac{\varphi^2(x)(1-x)^{2p-2}}{n^{2p-1}}, & \text{for } x < \frac{1}{n}. \end{cases}$$

Lemma 3.2. *Let $n \in \mathbb{N}$, $n \geq 2$, $f \in C_{\alpha,\lambda}^0$, then $\|\widetilde{M}_n f\|_2 \leq C_1 n \|f\|_0$.*

Lemma 3.3. *Let $n \in \mathbb{N}$, $n \geq 2$, $f \in C_{\alpha,\lambda}^2$, then $\|\widetilde{M}_n f\|_2 \leq C_2 \|f\|_2$.*

Lemma 3.4 (cf. [4]). *For $0 \leq \beta < 1, 0 < h \leq \frac{1}{8}$, one has*

$$\iint_{-\frac{h}{2}}^{\frac{h}{2}} \frac{ds dt}{\varphi^{2\beta}(x+s+t)} \leq \frac{Mh^2}{\max\{\varphi(x \pm h), \varphi(x)\}^{2\beta}}, \quad (x \in [h, 1-h]).$$

PROOF OF THEOREM 1.2. From the definition of $K_\lambda^\alpha(f, t^2)$, for $\widetilde{M}_n(f, x) \in C_{\alpha,\lambda}^2$, we have

$$K_\lambda^\alpha(f, t^2) \leq \|\widetilde{M}_n(f, x) - f(x)\|_0 + t^2 \|\widetilde{M}_n(f, x)\|_2, \quad (3.1)$$

and setting $t = n^{-\frac{1}{2}}$, we can choose $g \in C_{\alpha,\lambda}^2$ such that

$$\|f - g\|_0 + n^{-1} \|g\|_2 \leq 2K_\lambda^\alpha(f, n^{-1}). \quad (3.2)$$

Noting the condition

$$\left| \widetilde{M}_n f - f \right| = O(n^{-\frac{\alpha}{2}} \varphi^{\alpha(1-\lambda)}(x)),$$

which implies that

$$\|\widetilde{M}_n f - f\|_0 \leq C n^{-\frac{\alpha}{2}}. \quad (3.3)$$

By Lemma 3.2, Lemma 3.3 and (3.2), one has

$$\begin{aligned} \|\widetilde{M}_n f\|_2 &\leq \|\widetilde{M}_n(f - g)\|_2 + \|\widetilde{M}_n g\|_2 \leq C(n\|f - g\|_0 + \|g\|_2) \\ &\leq Cn(\|f - g\|_0 + n^{-1}\|g\|_2) \leq 2CnK_\lambda^\alpha(f, n^{-1}). \end{aligned} \quad (3.4)$$

We have by (3.1), (3.3), (3.4)

$$K_\lambda^\alpha(f, t^2) \leq C \left(n^{-\frac{\alpha}{2}} + t^2 n K_\lambda^\alpha(f, n^{-1}) \right).$$

Applying the Berens–Lorentz Lemma [1, P122], we can get $K_\lambda^\alpha(f, t^2) \leq Ct^\alpha$.

For $\lambda = 1$, $K_\lambda^\alpha(f, t^2) = K_{\varphi^\lambda}^2(f, t^2)$, then $\omega_{\varphi^\lambda}^2(f, t) = O(t^\alpha)$.

For $0 \leq \lambda < 1$, now we estimate $|\Delta_{h\varphi^\lambda}^2 f(x)|$.

(i) Fixed $h \in (0, \frac{1}{8})$, $x \in [h, 1-h]$ and for any $f \in C_{\alpha, \lambda}^0$, one has

$$\begin{aligned} |\Delta_{h\varphi^\lambda}^2 f(x)| &\leq |f(x + h\varphi^\lambda(x))| + 2|f(x)| + |f(x - h\varphi^\lambda(x))| \\ &\leq 4\|f\|_0 M(x)^{\alpha(1-\lambda)}, \end{aligned}$$

where $M(x) := \max\{|\varphi(x + h\varphi^\lambda(x))|, |\varphi(x)|, |\varphi(x - h\varphi^\lambda(x))|\}$.

(ii) Using Lemma 3.4, for any $g \in C_{\alpha, \lambda}^2$, one has

$$\begin{aligned} |\Delta_{h\varphi^\lambda}^2 g(x)| &= \left| \iint_{-\frac{h\varphi^\lambda}{2}}^{\frac{h\varphi^\lambda}{2}} g''(x + s + t) ds dt \right| \\ &\leq \|g\|_2 \iint_{-\frac{h\varphi^\lambda}{2}}^{\frac{h\varphi^\lambda}{2}} |\varphi^{-2+\alpha(1-\lambda)}(x + s + t)| ds dt \\ &\leq C\|g\|_2 h^2 M(x)^{(\alpha-2)(1-\lambda)}. \end{aligned}$$

Using (i), (ii) and (3.2), there holds

$$\begin{aligned} |\Delta_{h\varphi^\lambda}^2 f(x)| &\leq |\Delta_{h\varphi^\lambda}^2 (f(x) - g(x))| + |\Delta_{h\varphi^\lambda}^2 g(x)| \\ &\leq C_1 \left[\|f - g\|_0 M(x)^{\alpha(1-\lambda)} + \|g\|_2 h^2 M(x)^{(\alpha-2)(1-\lambda)} \right] \\ &\leq C_2 M(x)^{\alpha(1-\lambda)} K_\lambda^\alpha \left(f, \frac{h^2}{M(x)^{2(1-\lambda)}} \right) \leq Ch^\alpha. \end{aligned}$$

If we take supremum over $h : 0 < h \leq t$, we have

$$\omega_{\varphi^\lambda}^2(f, t) = O(t^\alpha).$$

Noting that $\widetilde{M}_n(f, x) - f(x) = \widetilde{M}_n(f_0, x) - f_0(x)$, where $f_0(x) = f(x) - f(0)$, the proof of Theorem 1.2 is completed. $\square$

4. Proofs of Lemma 3.2 and Lemma 3.3

We will need the following two representations of derivations of $\widetilde{M}_n(f, x)$ which are given by [4] or simple computations.

$$\left(\widetilde{M}_n f \right)''(x) = \sum_{k=0}^{\infty} \Phi_{n,k}(f) r_{n,k} m_{n,k}(x), \quad x \in (0, 1), \quad (4.1)$$

where

$$r_{n,k} := \frac{1}{x^2} \left[\left(k - \frac{(n+1)x}{1-x} \right)^2 - \left(k - \frac{(n+1)x}{1-x} \right) \right] - \frac{n+1}{x(1-x)^2},$$

$$\left(\widetilde{M}_n f \right)''(x) = (1-x)^{-2} \sum_{k=0}^{\infty} \Delta_{n,k}(f) m_{n,k}(x), \quad x \in (0, 1), \quad (4.2)$$

where

$$\Delta_{n,k}(f) = (n+k+1)[(n+k+2)\Phi_{n,k+2}(f) - 2(n+k+1)\Phi_{n,k+1}(f) + (n+k)\Phi_{n,k}(f)].$$

PROOF OF LEMMA 3.2. Supposing that $E_n = [\frac{1}{n}, 1)$, we will estimate E_n and E_n^c separately.

Case I. For $x \in E_n$, in view of (4.1), by the Hölder inequality,

$$\begin{aligned} & \left| \varphi^{2+\alpha(\lambda-1)}(x) \left(\widetilde{M}_n f \right)''(x) \right| \\ & \leq \left| \varphi^{\alpha(\lambda-1)}(x) \sum_{k=0}^{\infty} \frac{(1-x)^2}{x} \left(k - \frac{(n+1)x}{1-x} \right)^2 \Phi_{n,k}(f) m_{n,k}(x) \right| \\ & \quad + \left| \varphi^{\alpha(\lambda-1)}(x) \sum_{k=0}^{\infty} \frac{(1-x)^2}{x} \left(k - \frac{(n+1)x}{1-x} \right) \Phi_{n,k}(f) m_{n,k}(x) \right| \\ & \quad + \left| \varphi^{\alpha(\lambda-1)}(x) \sum_{k=0}^{\infty} (n+1) \Phi_{n,k}(f) m_{n,k}(x) \right| := T_1 + T_2 + T_3. \end{aligned}$$

First using the Cauchy–Schwarz inequality, the Hölder inequality and the relation (2.6), we have

$$\begin{aligned} C_{n-2,k-1}^{-1} \int_0^1 \varphi^{\alpha(1-\lambda)}(t) m_{n-2,k-1}(t) dt & \leq \left(C_{n-2,k-1}^{-1} \int_0^1 \varphi^2(t) m_{n-2,k-1}(t) dt \right)^{\frac{\alpha(1-\lambda)}{2}} \\ & \leq \left(C_{n-2,k-1}^{-1} \int_0^1 t^2 m_{n-2,k-1}(t) dt \right)^{\frac{\alpha(1-\lambda)}{4}} \cdot \left(C_{n-2,k-1}^{-1} \int_0^1 (1-t)^4 m_{n-2,k-1}(t) dt \right)^{\frac{\alpha(1-\lambda)}{4}} \\ & \leq C \left(\frac{k}{n+k} \right)^{\frac{\alpha(1-\lambda)}{2}} \left(\frac{n}{n+k} \right)^{\alpha(1-\lambda)}. \end{aligned}$$

Next we estimate T_3 , using the Hölder inequality twice and Lemma 2.3, we obtain

$$T_3 = \left| \varphi^{\alpha(\lambda-1)}(x) \sum_{k=1}^{\infty} (n+1) \Phi_{n,k}(f) m_{n,k}(x) \right|$$

$$\begin{aligned}
&\leq Cn\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\sum_{k=1}^{\infty}C_{n-2,k-1}^{-1}\int_0^1\varphi^{\alpha(1-\lambda)}(t)m_{n-2,k-1}(t)dtm_{n,k}(x) \\
&\leq C_1n\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\sum_{k=1}^{\infty}\left(\frac{k}{n+k}\right)^{\frac{\alpha(1-\lambda)}{2}}\cdot\left(\frac{n}{n+k}\right)^{\alpha(1-\lambda)}m_{n,k}(x) \\
&\leq C_1n\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\left(\sum_{k=1}^{\infty}\left(\frac{k}{n+k}\right)^2m_{n,k}(x)\right)^{\frac{\alpha(1-\lambda)}{4}} \\
&\quad\cdot\left(\sum_{k=1}^{\infty}\left(\frac{n}{n+k}\right)^4m_{n,k}(x)\right)^{\frac{\alpha(1-\lambda)}{4}}\leq C_1n\|f\|_0.
\end{aligned}$$

For T_2 , using the Hölder inequality and Lemma 2.2, Lemma 2.3, there holds

$$\begin{aligned}
T_2 &= \left| \varphi^{\alpha(\lambda-1)}(x) \sum_{k=1}^{\infty} \frac{(1-x)^2}{x} \left(k - \frac{(n+1)x}{1-x} \right) \Phi_{n,k}(f) m_{n,k}(x) \right| \\
&\leq C\|f\|_0\varphi^{\alpha(\lambda-1)}(x) \sum_{k=1}^{\infty} \left| \frac{(1-x)^2}{x} \left(k - \frac{(n+1)x}{1-x} \right) \right| \\
&\quad\cdot C_{n-2,k-1}^{-1} \int_0^1 \varphi^{\alpha(1-\lambda)}(t) m_{n-2,k-1}(t) dt \cdot m_{n,k}(x) \\
&\leq \|f\|_0\varphi^{\alpha(\lambda-1)}(x) \sum_{k=1}^{\infty} \left| \frac{(1-x)^2}{x} \left(k - \frac{(n+1)x}{1-x} \right) \right| \left(\frac{k}{n+k} \right)^{\frac{\alpha(1-\lambda)}{2}} \\
&\quad\cdot \left(\frac{n}{n+k} \right)^{\alpha(1-\lambda)} m_{n,k}(x).
\end{aligned}$$

Noticing that

$$\frac{(1-x)^2}{x} \left(k - \frac{(n+1)x}{1-x} \right) m_{n,k}(x) \leq (n+2)(m_{n+2,k-1}(x) + m_{n,k}(x)),$$

we have

$$\begin{aligned}
T_2 &\leq \|f\|_0\varphi^{\alpha(\lambda-1)}(x) \sum_{k=1}^{\infty} (n+2)(m_{n+2,k-1}(x) + m_{n,k}(x)) \left(\frac{k}{n+k} \right)^{\frac{\alpha(1-\lambda)}{2}} \\
&\quad\cdot \left(\frac{n}{n+k} \right)^{\alpha(1-\lambda)} := Q_1 + Q_2.
\end{aligned}$$

The estimate of Q_2 is similar to T_3 , so we only need to estimate Q_1 .

$$Q_1 = \|f\|_0\varphi^{\alpha(\lambda-1)}(x) \sum_{k=1}^{\infty} (n+2)m_{n+2,k-1}(x) \left(\frac{k}{n+k} \right)^{\frac{\alpha(1-\lambda)}{2}} \cdot \left(\frac{n}{n+k} \right)^{\alpha(1-\lambda)}$$

$$\begin{aligned}
&\leq Cn\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\sum_{k=1}^{\infty}\left(\frac{k}{n+k}\right)^{\frac{\alpha(1-\lambda)}{2}}\cdot\left(\frac{n}{n+k}\right)^{\alpha(1-\lambda)}m_{n+2,k-1}(x) \\
&\leq Cn\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\left(\sum_{k=1}^{\infty}\left(\frac{k}{n+k}\right)^{\alpha(1-\lambda)}m_{n+2,k-1}(x)\right)^{\frac{1}{2}} \\
&\quad\cdot\left(\sum_{k=1}^{\infty}\left(\frac{n}{n+k}\right)^{2\alpha(1-\lambda)}m_{n+2,k-1}(x)\right)^{\frac{1}{2}} \\
&\leq Cn\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\left(\sum_{k=1}^{\infty}\left(\frac{k}{n+k}\right)^2m_{n+2,k-1}(x)\right)^{\frac{\alpha(1-\lambda)}{4}} \\
&\quad\cdot\left(\sum_{k=1}^{\infty}\left(\frac{n}{n+k}\right)^4m_{n+2,k-1}(x)\right)^{\frac{\alpha(1-\lambda)}{4}} \\
&= Cn\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\left(\sum_{k=0}^{\infty}\left(\frac{k+1}{n+k+1}\right)^2m_{n+2,k}(x)\right)^{\frac{\alpha(1-\lambda)}{4}} \\
&\quad\cdot\left(\sum_{k=0}^{\infty}\left(\frac{n}{n+k+1}\right)^4m_{n+2,k}(x)\right)^{\frac{\alpha(1-\lambda)}{4}}.
\end{aligned}$$

By Lemma 2.3 and Remark 3, we get

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left(\frac{k+1}{n+k+1}\right)^2m_{n+2,k}(x) \\
&= 2\sum_{k=1}^{\infty}\left(\frac{k}{n+k+2}\right)^2m_{n+2,k}(x) + \frac{1}{(n+1)^2}(1-x)^{n+3} \leq Cx^2 \quad (\text{for } x \in E_n),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left(\frac{n}{n+k+1}\right)^4m_{n+2,k}(x) \\
&= \sum_{k=1}^{\infty}\left(\frac{n}{n+k+1}\right)^4m_{n+2,k}(x) + \left(\frac{n}{n+1}\right)^4(1-x)^{n+3} \\
&\leq 2\sum_{k=1}^{\infty}\left(\frac{n+2}{n+k+2}\right)^4m_{n+2,k}(x) + (1-x)^4 \leq C(1-x)^4,
\end{aligned}$$

from which $Q_1 \leq Cn\|f\|_0$. Therefore, $T_2 \leq Cn\|f\|_0$.

Finally we estimate T_1 .

$$\begin{aligned}
T_1 &= \left| \varphi^{\alpha(\lambda-1)}(x) \sum_{k=0}^{\infty} \frac{(1-x)^2}{x} \left(k - \frac{(n+1)x}{1-x} \right)^2 \Phi_{n,k}(f) m_{n,k}(x) \right| \\
&\leq \|f\|_0 \varphi^{\alpha(\lambda-1)}(x) \sum_{k=1}^{\infty} \frac{(1-x)^2}{x} \left(k - \frac{(n+1)x}{1-x} \right)^2 \\
&\quad \cdot \frac{(n+k-2)(n+k-1)}{n-1} \int_0^1 \varphi^{\alpha(1-\lambda)}(t) m_{n-2,k-1}(t) dt m_{n,k}(x) \\
&\leq C \|f\|_0 \varphi^{\alpha(\lambda-1)}(x) \sum_{k=1}^{\infty} \frac{(1-x)^2}{x} \left(k - \frac{(n+1)x}{1-x} \right)^2 \left(\frac{k}{n+k} \right)^{\frac{\alpha(1-\lambda)}{2}} \\
&\quad \cdot \left(\frac{n}{n+k} \right)^{\alpha(1-\lambda)} m_{n,k}(x) \\
&\leq C \|f\|_0 \varphi^{\alpha(\lambda-1)}(x) \left\{ \sum_{k=1}^{\infty} \frac{1}{x} [k - (n+k)x]^2 \left(\frac{k}{n+k} \right)^{\frac{\alpha(1-\lambda)}{2}} \right. \\
&\quad \cdot \left(\frac{n}{n+k} \right)^{\alpha(1-\lambda)} m_{n,k}(x) + x \sum_{k=1}^{\infty} \left(\frac{k}{n+k} \right)^{\frac{\alpha(1-\lambda)}{2}} \\
&\quad \cdot \left. \left(\frac{n}{n+k} \right)^{\alpha(1-\lambda)} m_{n,k}(x) \right\} := P_1 + P_2.
\end{aligned}$$

Similarly we estimate P_1, P_2 respectively.

$$\begin{aligned}
P_1 &= C \|f\|_0 \varphi^{\alpha(\lambda-1)}(x) n^2 \sum_{k=1}^{\infty} \frac{1}{x} \left(\frac{k}{n+k} - x \right)^2 \left(\frac{k}{n+k} \right)^{\frac{\alpha(1-\lambda)}{2}} \\
&\quad \cdot \left(\frac{n}{n+k} \right)^{\alpha(1-\lambda)-2} m_{n,k}(x) \\
&\leq C \|f\|_0 \varphi^{\alpha(\lambda-1)}(x) \frac{n^2}{x} \left(\sum_{k=1}^{\infty} \left(\frac{k}{n+k} \right)^2 m_{n,k}(x) \right)^{\frac{\alpha(1-\lambda)}{4}} \\
&\quad \cdot \left(\sum_{k=1}^{\infty} \left(\frac{k}{n+k} - x \right)^8 m_{n,k}(x) \right)^{\frac{1}{4}} \\
&\quad \cdot \left(\sum_{k=1}^{\infty} \left(\frac{n}{n+k} \right)^{-8} m_{n,k}(x) \right)^{\frac{2-\alpha(1-\lambda)}{8}} \leq C n \|f\|_0,
\end{aligned}$$

and

$$P_2 = C\|f\|_0 \varphi^{\alpha(\lambda-1)}(x) x \sum_{k=1}^{\infty} \left(\frac{k}{n+k}\right)^{\frac{\alpha(1-\lambda)}{2}} \cdot \left(\frac{n}{n+k}\right)^{\alpha(1-\lambda)} m_{n,k}(x) \leq Cn\|f\|_0.$$

From the estimate of T_1 , T_2 and T_3 , we obtain the required result.

Case II. For $x \in E_n^c = (0, \frac{1}{n})$.

For $f \in C_{\alpha,\lambda}^0$, the representation (4.2) shows that

$$\begin{aligned} & \left| \varphi^{2+\alpha(\lambda-1)}(x) (\widetilde{M}_n f)''(x) \right| \\ &= \left| \varphi^{\alpha(\lambda-1)}(x) x \sum_{k=1}^{\infty} (n+k+1) [(n+k+2)\Phi_{n,k+2}(f) - 2(n+k+1)\Phi_{n,k+1}(f) \right. \\ & \quad \left. + (n+k)\Phi_{n,k}(f)] m_{n,k}(x) \right| \\ &\leq \varphi^{\alpha(\lambda-1)}(x) x \left[\left| \sum_{k=1}^{\infty} (n+k+1)(n+k+2)\Phi_{n,k+2}(f) m_{n,k}(x) \right| \right. \\ & \quad \left. + \left| \sum_{k=1}^{\infty} 2(n+k+1)^2 \Phi_{n,k+1}(f) m_{n,k}(x) \right| \right. \\ & \quad \left. + \left| \sum_{k=1}^{\infty} (n+k+1)(n+k)\Phi_{n,k}(f) m_{n,k}(x) \right| \right] := I_1 + I_2 + I_3. \end{aligned}$$

The methods of estimating I_1 , I_2 , I_3 are similar, we make an example of I_1 .

$$\begin{aligned} I_1 &= \left| \varphi^{\alpha(\lambda-1)}(x) x \sum_{k=1}^{\infty} (n+k+1)(n+k+2) C_{n-2,k+1}^{-1} \int_0^1 f(t) m_{n-2,k+1}(t) dt m_{n,k}(x) \right| \\ &\leq \|f\|_0 \varphi^{\alpha(\lambda-1)}(x) x \sum_{k=1}^{\infty} (n+k+1)(n+k+2) C_{n-2,k+1}^{-1} \int_0^1 \frac{m_{n-2,k+1}(t) dt}{\varphi^{\alpha(\lambda-1)}(t)} m_{n,k}(x) \\ &\leq \|f\|_0 \varphi^{\alpha(\lambda-1)}(x) x n(n+1) \sum_{k=1}^{\infty} C_{n,k+1}^{-1} \int_0^1 \varphi^{\alpha(1-\lambda)}(t) (1-t)^{-2} m_{n,k+1}(t) dt m_{n,k}(x) \\ &\leq C\|f\|_0 \varphi^{\alpha(\lambda-1)}(x) x n(n+1) \sum_{k=1}^{\infty} \left(\frac{k}{n+k}\right)^{\frac{\alpha(1-\lambda)}{2}} \cdot \left(\frac{n}{n+k}\right)^{\alpha(1-\lambda)-2} m_{n,k}(x). \end{aligned}$$

Using the Hölder inequality and Lemma 2.3, and noticing that $x < \frac{1}{n}$, $(1-x)^{-2} < (1 - \frac{1}{n})^{-2} \leq 4(n \geq 2)$, there holds

$$\begin{aligned} I_1 &\leq C \|f\|_0 \varphi^{\alpha(\lambda-1)}(x) x n(n+1) \left(\sum_{k=1}^{\infty} \frac{k}{n+k} m_{n,k}(x) \right)^{\alpha(1-\lambda)} \\ &\quad \cdot \left(\sum_{k=1}^{\infty} \left(\frac{n}{n+k} \right)^{-2} m_{n,k}(x) \right)^{\frac{2-\alpha(1-\lambda)}{2}} \leq C_1 n \|f\|_0. \end{aligned}$$

□

PROOF OF LEMMA 3.3. In view of (4.2) and

$$\begin{aligned} m'_{n,k}(x) &= \frac{n+1}{(1-x)^2} [m_{n+1,k-1}(x) - m_{n+1,k}(x)], \\ m''_{n,k}(x) &= \frac{1}{(1-x)^2} [(n+k)(n+k+1)m_{n,k}(x) - 2(n+k)^2 m_{n,k-1}(x) \\ &\quad + (n+k)(n+k-1)m_{n,k-2}(x)], \end{aligned}$$

using the integration by parts, we have

$$\Delta_{n,k}(f) = \int_0^1 f(t) n m''_{n,k+1}(t) dt = n \int_0^1 f''(t) m_{n,k+1}(t) dt.$$

Furthermore, there holds

$$(\widetilde{M}_n f)''(x) = (1-x)^{-2} \sum_{k=0}^{\infty} n \int_0^1 f''(t) m_{n,k+1}(t) dt m_{n,k}(x).$$

To prove the lemma, for $f \in C_{\alpha,\lambda}^2$, we estimate

$$\begin{aligned} \left| \varphi^{2+\alpha(\lambda-1)} (\widetilde{M}_n f)''(x) \right| &= \left| \varphi^{\alpha(\lambda-1)}(x) x \sum_{k=1}^{\infty} n \int_0^1 f''(t) m_{n,k+1}(t) dt m_{n,k}(x) \right| \\ &\leq \|f\|_2 \varphi^{\alpha(\lambda-1)}(x) x \sum_{k=1}^{\infty} n \int_0^1 \varphi^{-[2+\alpha(\lambda-1)]}(t) m_{n,k+1}(t) dt m_{n,k}(x) := H. \end{aligned}$$

Noticing that

$$\begin{aligned} &x \frac{n(n+1)}{(n+k+2)(n+k+3)} m_{n,k}(x) \\ &= \frac{(n+1)(n+k)(n+k-1)}{(n+k+2)(n+k+3)(n-1)} m_{n-1,k}(x) \varphi^2(x) \end{aligned}$$

$$\leq \frac{n+1}{n-1} m_{n-1,k}(x) \varphi^2(x) \leq 3m_{n-1,k}(x) \varphi^2(x), \quad (\text{for } n \geq 2),$$

we can deduce

$$\begin{aligned} H &\leq 3\|f\|_2 \varphi^{\alpha(\lambda-1)}(x) \sum_{k=1}^{\infty} C_{n,k+1}^{-1} \int_0^1 \varphi^{-[2+\alpha(\lambda-1)]}(t) m_{n,k+1}(t) dt m_{n-2,k}(x). \\ &\leq 3\|f\|_2 \varphi^{\alpha(\lambda-1)}(x) \sum_{k=1}^{\infty} \left(C_{n,k+1}^{-1} \int_0^1 t^{-2} m_{n,k+1}(t) dt \right)^{\frac{2-\alpha(1-\lambda)}{4}} \\ &\quad \cdot \left(C_{n,k+1}^{-1} \int_0^1 (1-t)^{-4} m_{n,k+1}(t) dt \right)^{\frac{2-\alpha(1-\lambda)}{4}} m_{n-2,k}(x). \end{aligned}$$

Using the method in the proof of Lemma 2.2, we also get

$$\begin{aligned} C_{n,k+1}^{-1} \int_0^1 t^{-2} m_{n,k+1}(t) dt &\leq C \left(\frac{k}{n+k-2} \right)^{-2}, \\ C_{n,k+1}^{-1} \int_0^1 (1-t)^{-4} m_{n,k+1}(t) dt &\leq C \left(\frac{n-2}{n+k-2} \right)^{-4}. \end{aligned}$$

Hence using the Hölder inequality and Lemma 2.3, one has

$$\begin{aligned} H &\leq C\|f\|_2 \varphi^{\alpha(\lambda-1)+2}(x) \sum_{k=1}^{\infty} \left(\frac{k}{n+k-2} \right)^{\frac{\alpha(1-\lambda)-2}{2}} \\ &\quad \cdot \left(\frac{n-2}{n+k-2} \right)^{\alpha(1-\lambda)-2} m_{n-2,k}(x) \\ &\leq C\|f\|_2 \varphi^{\alpha(\lambda-1)+2}(x) \left(\sum_{k=1}^{\infty} \left(\frac{k}{n+k-2} \right)^{\alpha(1-\lambda)-2} m_{n-2,k}(x) \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\sum_{k=1}^{\infty} \left(\frac{n-2}{n+k-2} \right)^{2(\alpha(1-\lambda)-2)} m_{n-2,k}(x) \right)^{\frac{1}{2}} \\ &\leq C\|f\|_2 \varphi^{\alpha(\lambda-1)+2}(x) \left(\sum_{k=1}^{\infty} \left(\frac{k}{n+k-2} \right)^{-2} m_{n-2,k}(x) \right)^{\frac{2-\alpha(1-\lambda)}{4}} \\ &\quad \cdot \left(\sum_{k=1}^{\infty} \left(\frac{n-2}{n+k-2} \right)^{-4} m_{n-2,k}(x) \right)^{\frac{2-\alpha(1-\lambda)}{4}} \\ &\leq C\|f\|_2 \varphi^{\alpha(\lambda-1)+2}(x) x^{-\frac{2-\alpha(1-\lambda)}{2}} (1-x)^{-(2-\alpha(1-\lambda))} \leq C\|f\|_2. \quad \square \end{aligned}$$

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