Publ. Math. Debrecen
73/1-2 (2008), 133–144

Symmetric proportionally modular Diophantine inequalities

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Abstract. We show that a proportionally modular numerical semigroup not containing the integer 2 is symmetric if and only if it is the set of integer solutions of an inequality of the form $(ab-1)x \mod b^2 \leq (b-2)\,x$, where a and b are positive integers such that $a < b$ and $\gcd\{a,b\} = \gcd\{a-1,b\} = 1$. We also obtain an easy procedure to compute the number of symmetric proportionally modular numerical semigroups with a given Frobenius number.

1. Introduction

Given two integers m and n with $n \neq 0$, we denote by $m \mod n$ the remainder of the division of m by n. Following the terminology used in [11], a proportionally modular Diophantine inequality is an expression of the form $\alpha x \mod \beta \leq \gamma x$, where α, β and γ are positive integers. The set $\mathrm{S}(\alpha,\beta,\gamma)$ of integer solutions of this inequality is a numerical semigroup, that is, it is a subset of $\mathbb{N}$ (here $\mathbb{N}$ denotes the set of nonnegative integers) that is closed under addition, contains the zero and its complement in $\mathbb{N}$ is finite. We say that a numerical semigroup is proportionally modular if it is the set of integer solutions of a proportionally modular Diophantine inequality.

The greatest integer not belonging to a numerical semigroup S is its Frobenius number and we denote it by $\mathrm{g}(S)$ (see [5]). A numerical semigroup S is symmetric if $x \in \mathbb{Z}\backslash S$ implies $\mathrm{g}(S) - x \in S$ (here $\mathbb{Z}$ denotes the set of integers).

Mathematics Subject Classification: 11D75, 20M14.
The first two authors are supported by the project MTM2004-01446 and FEDER funds. This paper has been supported by the Luso-Espanhola action HP2004-0056. The authors want to thank P. A. Garcıá-Sánchez and the referee for their comments and suggestions.

This type of numerical semigroups has been widely studied in the literature (see for example [1], [2], [4], [6], [7] and [9]).

The proportionally modular Diophantine inequality $\alpha x \mod \beta \leq \gamma x$ is symmetric if $\mathrm{S}(\alpha, \beta, \gamma)$ is a symmetric numerical semigroup.

The contents of the present paper are organized as follows. Section 2 compiles some results of [11] and [13] that are essential to the development of the rest of the paper. In Section 3 we show that, if a and b are positive integers such that $a < \frac{b+1}{2}$ and $\gcd\{a, b\} = \gcd\{a-1, b\} = 1$, then $\mathrm{S}\left(ab-1, b^2, b-2\right)$ is a symmetric proportionally modular numerical semigroup with Frobenius number b. Moreover, we will see that every symmetric proportionally modular numerical semigroup not containing the integer 2 is of this form. Finally, Section 4 is devoted to count the number of symmetric proportionally modular numerical semigroups with a given Frobenius number.

2. Preliminaries

Given a nonempty subset A of $\mathbb{Q}_0^+$ (here $\mathbb{Q}_0^+$ denotes the set of nonnegative rational numbers), we will denote by $\langle A\rangle$ the submonoid of $(\mathbb{Q}_0^+, +)$ generated by A, that is, $\langle A\rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N}\backslash\{0\},\ \lambda_1, \ldots, \lambda_n \in \mathbb{N} \text{ and } a_1, \ldots, a_n \in A\}$. Note that $\langle A\rangle \cap \mathbb{N}$ is a submonoid of $\mathbb{N}$, denoted here by $\mathrm{S}(A)$. We call $\mathrm{S}(A)$ the submonoid of $\mathbb{N}$ associated to A.

Let $p < q$ be two positive rational numbers. We use the following notation: $[p, q] = \{x \in \mathbb{Q} \mid p \leq x \leq q\}$, $[p, q[= \{x \in \mathbb{Q} \mid p \leq x < q\}$, $]p, q] = \{x \in \mathbb{Q} \mid p < x \leq q\}$ and $]p, q[= \{x \in \mathbb{Q} \mid p < x < q\}$. Along this paper, I will denote an interval of one of these types.

In [11] it is proved that $\mathrm{S}([p, q])$ is a proportionally modular numerical semigroup and that every proportionally modular numerical semigroup is of this form. The following result is a reformulation of [11, Corollary 9].

Proposition 1. (1) *Let α, β and γ be positive integers such that $\gamma < \alpha < \beta$. Then $\mathrm{S}\big(\big[\frac{\beta}{\alpha}, \frac{\beta}{\alpha-\gamma}\big]\big) = \mathrm{S}(\alpha, \beta, \gamma)$.*

(2) *Conversely, if a_1, b_1, a_2 and b_2 are positive integers such that $\frac{b_1}{a_1} < \frac{b_2}{a_2}$, then $\mathrm{S}\big(\big[\frac{b_1}{a_1}, \frac{b_2}{a_2}\big]\big) = \mathrm{S}(a_1 b_2, b_1 b_2, a_1 b_2 - a_2 b_1)$.*

Since the inequality $\alpha x \mod \beta \leq \gamma x$ has the same solutions that the inequality $(\alpha \mod \beta)\, x \mod \beta \leq \gamma x$, we can assume that $\alpha < \beta$. Moreover, if $\gamma \geq \alpha$, then $\mathrm{S}(\alpha, \beta, \gamma) = \mathbb{N}$. Therefore, we can suppose that α, β and γ are positive

integers such that $\gamma < \alpha < \beta$. Consequently, the condition imposed in (1) of the above proposition is not restrictive.

It is proved in [13] that proportionally modular numerical semigroups can be defined by using any interval of $\mathbb{Q}_0^+$. The following result is deduced from Proposition 5 in [13].

Proposition 2. *If I is an interval of positive rational numbers, then $\mathrm{S}(I)$ is a proportionally modular numerical semigroup.*

Next two results, which are part of Theorems 11 and 20 in [13], respectively, are the key to the development of this paper.

Proposition 3. *Let $2 \leq a < b$ be integers. Then $\mathrm{S}\big(\big]\frac{b}{a}, \frac{b}{a-1}\big[\big)$ is a proportionally modular numerical semigroup with Frobenius number b.*

Proposition 4. *Let S be a proportionally modular numerical semigroup. Then S is symmetric if and only if either $S = \mathbb{N}$, $S = \langle 2, 3\rangle$ or $S = \mathrm{S}\big(\big]\frac{b}{a}, \frac{b}{a-1}\big[\big)$, where a and b are integers such that $2 \leq a < b$ and $\gcd\{a, b\} = \gcd\{a-1, b\} = 1$.*

We conclude this section with a result that will be used several times along this paper and which can be easily deduced from [13, Lemma 2].

Lemma 5. *Let I be an interval of positive rational numbers and let x be a positive integer. Then $x \in \mathrm{S}(I)$ if and only if there exists a positive integer y such that $\frac{x}{y} \in I$.*

3. Symmetric proportionally modular numerical semigroups

Along this section we shall suppose that a and b are integers such that $2 \leq a < b$ and $\gcd\{a, b\} = \gcd\{a-1, b\} = 1$. Hence there exist positive integers u, v, s and t such that $au - bv = 1$ and $bt - s(a-1) = 1$. Besides, we can assume that both u and s are smaller than b. Note also that b has to be odd.

Lemma 6. *Under the above hypothesis, we have the following inequalities: $\frac{b}{a} < \frac{u}{v} \leq \frac{2b}{2a-1} \leq \frac{s}{t} < \frac{b}{a-1}$. Moreover, if $a \neq \frac{b+1}{2}$, then $\frac{u}{v} < \frac{s}{t}$.*

Proof. Since $au - bv = 1$, we have $\frac{b}{a} < \frac{u}{v}$. Clearly, $\frac{u}{v} \leq \frac{2b}{2a-1}$ if and only if $u \geq 2$. But $u \geq 2$ holds, since if $u = 1$, then $a = bv + 1$, which contradicts the fact that $a < b$.

As $bt - s(a-1) = 1$, we have $\frac{s}{t} < \frac{b}{a-1}$. It is easy to see that $\frac{2b}{2a-1} \leq \frac{s}{t}$ if and only if $s \geq 2$. Observe that $s \geq 2$, since if $s = 1$, then $a = bt$, in contradiction with $a < b$.

It is clear that $\gcd\{u,v\} = \gcd\{s,t\} = 1$. Hence, if $\frac{u}{v} = \frac{s}{t}$, then $u = s$ and $v = t$. Since $bt - s(a-1) = 1$, we have $bv - u(a-1) = 1$ and consequently $u = 2$. As $1 < \frac{b}{a} < \frac{u}{v}$, we have $v < u$, and so $v = 1$. Therefore, from the equality $au - bv = 1$, we get that $2a - b = 1$, which is equivalent to $a = \frac{b+1}{2}$. □

If S is a numerical semigroup, then the smallest positive integer that belongs to S is an important invariant of S, called the multiplicity of S and denoted here by $\mathrm{m}(S)$ (see [1]).

The following result, which implicitly appears in the introduction of [8], is straightforward to prove.

Lemma 7. *Let g be an odd positive integer and let S be a numerical semigroup. Then S has multiplicity 2 and Frobenius number g if and only if $S = \langle 2, g+2\rangle$.*

Next result determines when a semigroup of the form $\mathrm{S}\big(\big]\frac{b}{a}, \frac{b}{a-1}\big[\big)$ has multiplicity 2.

Lemma 8. $\mathrm{S}\big(\big]\frac{b}{a}, \frac{b}{a-1}\big[\big) = \langle 2, b+2\rangle$ *if and only if* $a = \frac{b+1}{2}$.

Proof. Suppose that $2 \in \mathrm{S}\big(\big]\frac{b}{a}, \frac{b}{a-1}\big[\big)$. In view of Lemma 5 and the fact that $1 < \frac{b}{a}$, we deduce that $\frac{b}{a} < \frac{2}{1} < \frac{b}{a-1}$. Therefore $a = \frac{b+1}{2}$.

Assume that $a = \frac{b+1}{2}$. Let us see that $2 \in \mathrm{S}\big(\big]\frac{b}{a}, \frac{b}{a-1}\big[\big)$. To this end, by Lemma 5, it suffices to show that $\frac{b}{\frac{b+1}{2}} < \frac{2}{1} < \frac{b}{\frac{b+1}{2}-1}$, which can be easily checked. Since from Proposition 3 we know that b is the Frobenius number of $\mathrm{S}\big(\big]\frac{b}{a}, \frac{b}{a-1}\big[\big)$, by applying Lemma 7 we obtain that $\mathrm{S}\big(\big]\frac{b}{a}, \frac{b}{a-1}\big[\big) = \langle 2, b+2\rangle$. □

Given an integer x we denote by $\{x, \to\}$ the set of all integers greater than or equal to x.

Lemma 9. *Let α and β be integers such that $2 \le \alpha < \beta$. Then $\{\beta-1, \to\} \subseteq \mathrm{S}\big(\big[\frac{\beta}{\alpha}, \frac{\beta}{\alpha-1}\big]\big)$.*

Proof. From Proposition 1, we know that $\mathrm{S}\big(\big[\frac{\beta}{\alpha}, \frac{\beta}{\alpha-1}\big]\big) = \{x \in \mathbb{N} \mid \alpha x \bmod \beta \le x\}$. Since $\alpha x \bmod \beta \le \beta - 1$ for every $x \in \mathbb{N}$, we deduce that $\{\beta-1, \to\} \subseteq \mathrm{S}\big(\big[\frac{\beta}{\alpha}, \frac{\beta}{\alpha-1}\big]\big)$. □

We know from Lemma 5 that $x \in \mathrm{S}(I)$ if and only if there exists a positive integer y such that $\frac{x}{y} \in I$. Next, we prove that if $x < \mathrm{g}(\mathrm{S}(I))$, then the integer y is unique.

Lemma 10. *Let I be an interval of rational numbers greater than 1 and let x be a positive integer. If $x \in \mathrm{S}(I)$ and $x < \mathrm{g}(\mathrm{S}(I))$, then there exists a unique positive integer y such that $\frac{x}{y} \in I$.*

PROOF. The existence of a positive integer y such that $\frac{x}{y} \in I$ is guaranteed by Lemma 5. Assume to the contrary, that this integer is not unique. Then there exists an integer $z \geq 2$ such that $\frac{x}{z}, \frac{x}{z-1} \in I$. Thus $\left[\frac{x}{z}, \frac{x}{z-1}\right] \subseteq I$, and in view of Lemma 5 we deduce that $\mathrm{S}\left(\left[\frac{x}{z}, \frac{x}{z-1}\right]\right) \subseteq \mathrm{S}(I)$. By applying Lemma 9, we have that $\{x-1, \rightarrow\} \subseteq \mathrm{S}(I)$, which is in contradiction with the hypothesis $x < \mathrm{g}(\mathrm{S}(I))$. □

There can be several intervals $I \subseteq \left]\frac{b}{a}, \frac{b}{a-1}\right[$ such that $\mathrm{S}(I) = \mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$. The following lemma shows that if this is the case, I must contain the interval $\left[\frac{u}{v}, \frac{s}{t}\right]$.

Lemma 11. *Let I be an interval of rational numbers such that $I \subseteq \left]\frac{b}{a}, \frac{b}{a-1}\right[$. If $\mathrm{S}(I) = \mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$, then $\left[\frac{u}{v}, \frac{s}{t}\right] \subseteq I$.*

PROOF. By applying Lemmas 6 and 5, we deduce that $u, s \in \mathrm{S}(I)$. By using Lemma 5 again, we have that there exist positive integers x and y such that $\frac{u}{x}, \frac{s}{y} \in I$. Since $u < b$ and $s < b$, from Proposition 3 and Lemmas 6 and 10 we obtain $x = v$ and $y = t$. Therefore $\frac{u}{v}, \frac{s}{t} \in I$ and consequently $\left[\frac{u}{v}, \frac{s}{t}\right] \subseteq I$. □

Our next goal in this section is to prove Theorem 17, which shows that the converse to Lemma 11 also holds. Prior to this we need to introduce some concepts and results.

It is well-known (see for example [10]) that every numerical semigroup S is finitely generated and therefore there exists a finite subset A of $\mathbb{N}$ such that $S = \langle A \rangle$. We say that A is a minimal system of generators of S if no proper subset of A generates S. It is also well-known (see for instance [10]) that $S^* \setminus (S^* + S^*)$ is the unique minimal system of generators of S, with $S^* = S \setminus \{0\}$. Given a numerical semigroup S, we denote by $\mathrm{M}(S)$ the minimal system of generators of S and say that the elements of $\mathrm{M}(S)$ are the minimal generators of S. Note that $\mathrm{m}(S)$ is the minimum of $\mathrm{M}(S)$.

Next we show that if S is a symmetric numerical semigroup, then every element of $\mathrm{M}(S)$ is smaller than $\mathrm{g}(S)$, except for the semigroups $S = \langle 2, \mathrm{g}(S) + 2 \rangle$ and $S = \mathbb{N}$.

Lemma 12. *Let S be a symmetric numerical semigroup with $\mathrm{m}(S) \geq 3$. Then every minimal generator of S is smaller than $\mathrm{g}(S)$.*

PROOF. Suppose that $\mathrm{M}(S) = \{n_1 < n_2 < \cdots < n_p\}$. Since $n_p - n_1 \notin S$, from the definition of symmetric numerical semigroup we have that $\mathrm{g}(S) - (n_p - n_1) = \mathrm{g}(S) - n_p + n_1 \in S$. Taking into account that $\mathrm{m}(S) = n_1$ we deduce that either $\mathrm{g}(S) - n_p + n_1 \geq n_1$ or $\mathrm{g}(S) - n_p + n_1 = 0$. To conclude the proof we only have

to see that the last case is not possible. For that we distinguish two subcases, which are $p \geq 3$ and $p = 2$. If $p \geq 3$, then $n_p - n_2 \notin S$. Since S is symmetric we have that $\mathrm{g}(S) - n_p + n_2 \in S$ and, as $n_p = \mathrm{g}(S) + n_1$, we get that $n_2 - n_1 \in S$, which is not possible. If $p = 2$, then $S = \langle n_1, \mathrm{g}(S) + n_1 \rangle$. Since $n_1 \geq 3$ and $\mathrm{g}(S) + 1, \mathrm{g}(S) + 2 \in S$, we deduce that $\mathrm{g}(S) + 1, \mathrm{g}(S) + 2 \in \langle n_1 \rangle$, which is impossible. □

From the previous lemma, we obtain the following result.

Lemma 13. *If $a \neq \frac{b+1}{2}$, then every minimal generator of $\mathrm{S}\big(\big]\frac{b}{a}, \frac{b}{a-1}\big[\big)$ is smaller than b.*

Proof. In view of Propositions 3 and 4, we know that $\mathrm{S}\big(\big]\frac{b}{a}, \frac{b}{a-1}\big[\big)$ is a symmetric numerical semigroup with Frobenius number b. Besides, by Lemmas 8 and 7 we deduce that $\mathrm{S}\big(\big]\frac{b}{a}, \frac{b}{a-1}\big[\big)$ has multiplicity greater than or equal to 3. Now the proof follows from Lemma 12. □

The following result is [12, Lemma 4].

Lemma 14. *Let n_1, n_2, α and β be positive integers such that $n_2\alpha - n_1\beta = 1$. Then*

$$\langle n_1, n_2 \rangle = \{x \in \mathbb{N} \mid \alpha n_2 x \mod n_1 n_2 \leq x\}.$$

As an immediate consequence of Proposition 1 and Lemma 14 we have the following result.

Lemma 15. *Let n_1, n_2, α and β be positive integers such that $n_2\alpha - n_1\beta = 1$. Then $\mathrm{S}\big(\big[\frac{n_1}{\alpha}, \frac{n_2}{\beta}\big]\big) = \langle n_1, n_2 \rangle$.*

Lemma 16. *Let n_1, n_2, α and β be positive integers such that $n_2\alpha - n_1\beta = 1$. If $x \in \mathrm{S}\big(\big]\frac{n_1}{\alpha}, \frac{n_2}{\beta}\big[\big) \backslash \{0\}$, then $x > \max\{n_1, n_2\}$.*

Proof. If $x \in \mathrm{S}\big(\big]\frac{n_1}{\alpha}, \frac{n_2}{\beta}\big[\big)$, by Lemma 5 we obtain that $x \in \mathrm{S}\big(\big[\frac{n_1}{\alpha}, \frac{n_2}{\beta}\big]\big)$ and by Lemma 15 we have $x \in \langle n_1, n_2 \rangle$. Thus $x = \lambda n_1 + \mu n_2$, for some $\lambda, \mu \in \mathbb{N}$. Clearly the result holds if $\lambda \neq 0$ and $\mu \neq 0$.

Suppose that $x = \lambda n_1$. By Lemma 5, there exists a positive integer y such that $\frac{n_1}{\alpha} < \frac{\lambda n_1}{y} < \frac{n_2}{\beta}$. Since $\frac{n_1}{\alpha} = \frac{\lambda n_1}{\lambda \alpha}$ we deduce that $y < \lambda\alpha$ and so $\frac{\lambda n_1}{\lambda\alpha - 1} < \frac{n_2}{\beta}$. Hence $\lambda n_1 \beta < \lambda\alpha n_2 - n_2$ and consequently $n_2 < \lambda$. Therefore $x = \lambda n_1 > n_2 n_1 \geq \max\{n_1, n_2\}$.

Now if $x = \mu n_2$, by proceeding in a similar way to the previous case, we deduce that $\frac{n_1}{\alpha} < \frac{\mu n_2}{\mu\beta + 1}$. Consequently $n_1 < \mu$, and therefore $x = \mu n_2 > n_1 n_2 \geq \max\{n_1, n_2\}$. □

Now we are in conditions to prove the converse to Lemma 11.

Theorem 17. *Let $a \neq \frac{b+1}{2}$ and let I be an interval of rational numbers such that $I \subseteq \left]\frac{b}{a}, \frac{b}{a-1}\right[$. Then $\mathrm{S}(I) = \mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$ if and only if $\left[\frac{u}{v}, \frac{s}{t}\right] \subseteq I$.*

Proof. The necessary condition was shown in Lemma 11. Let us prove the sufficient condition. Note that it is enough to see that $\mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right) \subseteq \mathrm{S}\left(\left[\frac{u}{v}, \frac{s}{t}\right]\right)$, to which it suffices to prove that every minimal generator of $\mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$ belongs to $\mathrm{S}\left(\left[\frac{u}{v}, \frac{s}{t}\right]\right)$.

Let x be a minimal generator of $\mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$. Then, by Lemma 5, we know that there exists a positive integer y such that $\frac{b}{a} < \frac{x}{y} < \frac{b}{a-1}$. If $\frac{b}{a} < \frac{x}{y} < \frac{u}{v}$, then $x \in \mathrm{S}\left(\left]\frac{b}{a}, \frac{u}{v}\right[\right)$ and by applying Lemma 16 we have that $x > b$, which contradicts Lemma 13. If $\frac{s}{t} < \frac{x}{y} < \frac{b}{a-1}$, then $x \in \mathrm{S}\left(\left]\frac{s}{t}, \frac{b}{a-1}\right[\right)$ and by using Lemma 16 we obtain that $x > b$, contradicting again Lemma 13. Therefore $\frac{u}{v} \leq \frac{x}{y} \leq \frac{s}{t}$ and, in view of Lemma 5, $x \in \mathrm{S}\left(\left[\frac{u}{v}, \frac{s}{t}\right]\right)$. □

Let x and y be positive integers. It is straightforward to prove that $\frac{b}{a} < \frac{x}{y} < \frac{b}{a-1}$ if and only if $\frac{b}{b+1-a} < \frac{x}{x-y} < \frac{b}{b-a}$. Therefore, by Lemma 5, we obtain the following result.

Lemma 18. $\mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right) = \mathrm{S}\left(\left]\frac{b}{b+1-a}, \frac{b}{b-a}\right[\right)$.

Now we give a characterization for the symmetric proportionally modular numerical semigroups with multiplicity greater than or equal to 3.

Corollary 19. *S is a symmetric proportionally modular numerical semigroup with $\mathrm{g}(S) = b$ and $\mathrm{m}(S) \geq 3$ if and only if $S = \mathrm{S}\left(ab-1, b^2, b-2\right)$, where a and b are integers such that $2 \leq a < \frac{b+1}{2}$ and $\gcd\{a, b\} = \gcd\{a-1, b\} = 1$.*

Proof. *Necessity.* By Propositions 3 and 4, there exists an integer a such that $2 \leq a < b$, $\gcd\{a, b\} = \gcd\{a-1, b\} = 1$ and $S = \mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$. Moreover, taking into account Lemmas 8 and 18 we can suppose that $a < \frac{b+1}{2}$.

Let u, v, s and t be positive integers defined as in the beginning of this section. In view of Lemma 6, we have that $\frac{b}{a} < \frac{u}{v} < \frac{s}{t} < \frac{b}{a-1}$. It is easy to check that $\frac{b}{a} < \frac{b^2}{ab-1} < \frac{u}{v} < \frac{s}{t} < \frac{b^2}{b(a-1)+1} < \frac{b}{a-1}$. So, by applying Theorem 17, we have $S = \mathrm{S}\left(\left[\frac{b^2}{ab-1}, \frac{b^2}{b(a-1)+1}\right]\right)$. Therefore, by Proposition 1, $S = \mathrm{S}\left(ab-1, b^2, b-2\right)$.

Sufficiency. By applying Proposition 1, we get that $S = \mathrm{S}\left(\left[\frac{b^2}{ab-1}, \frac{b^2}{ab-1-b+2}\right]\right) = \mathrm{S}\left(\left[\frac{b^2}{ab-1}, \frac{b^2}{b(a-1)+1}\right]\right)$. Let u, v, s and t be positive integers defined as in the beginning of this section. Then $\frac{b}{a} < \frac{b^2}{ab-1} < \frac{u}{v} < \frac{s}{t} < \frac{b^2}{b(a-1)+1} < \frac{b}{a-1}$. By Theorem 17 we have $S = \mathrm{S}\left(\left]\frac{b}{a}, \frac{b}{a-1}\right[\right)$. In view of Propositions 3 and 4, S is a symmetric proportionally modular numerical semigroup with Frobenius number b.

Finally, from Lemmas 8 and 7, we deduce that S has multiplicity greater than or equal to 3. □

For a set A, we denote by $\#A$ its cardinality. Given a numerical semigroup S, its set of gaps is $\mathrm{H}(S) = \mathbb{N}\backslash S$. The cardinality of $\mathrm{H}(S)$ is usually known as the genus of S (see [3]) and also as the singularity degree of S (see [1]). It is well-known (see for example [2]) that a numerical semigroup is symmetric if and only if $\#\mathrm{H}(S) = \frac{\mathrm{g}(S)+1}{2}$. As an immediate consequence of Corollary 19 we obtain the following result.

Corollary 20. *Let a and b be integers such that $2 \leq a < \frac{b+1}{2}$ and $\gcd\{a, b\} = \gcd\{a-1, b\} = 1$. Then $\#\mathrm{H}\left(\mathrm{S}\left(ab-1, b^2, b-2\right)\right) = \frac{b+1}{2}$.*

4. The number of symmetric proportionally modular numerical semigroups with a given Frobenius number

Our first aim in this section is to prove Theorem 23. To this end, we need some preliminary results.

Lemma 21. *Let I be an interval of positive rational numbers and let x be a minimal generator of $\mathrm{S}(I)$. If y is an integer such that $\frac{x}{y} \in I$, then $\gcd\{x, y\} = 1$.*

Proof. Suppose that $\gcd\{x, y\} = d \neq 1$. Then $\frac{\frac{x}{d}}{\frac{y}{d}} \in I$ and, in view of Lemma 5, we have that $\frac{x}{d} \in \mathrm{S}(I)$, which contradicts the fact that x is a minimal generator of $\mathrm{S}(I)$. □

The next result is the key to prove Theorem 23.

Lemma 22. *Let I be an interval of positive rational numbers and let $\mathrm{M}(\mathrm{S}(I)) = \{n_1 < n_2 < \cdots < n_p\}$. If $\{\alpha, \beta\} = \{n_1, n_2\}$, $n_2 < \mathrm{g}(\mathrm{S}(I))$ and x and y are integers such that $\left[\frac{\alpha}{x}, \frac{\beta}{y}\right] \subseteq I$, then $\beta x - \alpha y = 1$.*

Proof. Since $\frac{\beta}{y} \in I$, by applying Lemma 21, we have that $\gcd\{\beta, y\} = 1$. Hence there exist positive integers β' and y' such that $\beta y' - \beta' y = 1$. Besides, we can assume that $\beta' < \beta$. Note that $\frac{\beta'}{y'} < \frac{\beta}{y}$.

Analogously, as $\frac{\alpha}{x} \in I$, there exist positive integers α' and x' such that $\alpha' x - \alpha x' = 1$ and $\alpha' < \alpha$. We also have $\frac{\alpha}{x} < \frac{\alpha'}{x'}$.

Let us see that $\frac{\beta'}{y'} < \frac{\alpha}{x} < \frac{\beta}{y} < \frac{\alpha'}{x'}$ cannot hold. If this were not the case, by Lemma 5 we would deduce that $\alpha \in \mathrm{S}\big(\big]\frac{\beta'}{y'}, \frac{\beta}{y}\big[\big)\backslash\{0\}$ and $\beta \in \mathrm{S}\big(\big]\frac{\alpha}{x}, \frac{\alpha'}{x'}\big[\big)\backslash\{0\}$ and, by Lemma 16, that $\alpha > \beta$ and $\beta > \alpha$, which is not possible.

Therefore either $\frac{\alpha}{x} \leq \frac{\beta'}{y'} < \frac{\beta}{y}$ or $\frac{\alpha}{x} < \frac{\alpha'}{x'} \leq \frac{\beta}{y}$ must hold.

If $\frac{\alpha}{x} \leq \frac{\beta'}{y'} < \frac{\beta}{y}$, then in view of Lemma 5 we have that $\beta' \in \mathrm{S}(I)$. Since $\beta' < \beta$ and $\{\alpha, \beta\}$ is the set formed by the two smallest minimal generators of $\mathrm{S}(I)$, we deduce that $\beta' = k\alpha$, for some $k \in \mathbb{N}\backslash\{0\}$. As $k\alpha = \beta' < \beta < \mathrm{g}(\mathrm{S}(I))$, by Lemma 10, we obtain $y' = kx$. Replacing this in the equality $\beta y' - \beta' y = 1$, we obtain that $\beta kx - k\alpha y = 1$, and therefore $k = 1$ and $\beta x - \alpha y = 1$.

The case $\frac{\alpha}{x} < \frac{\alpha'}{x'} \leq \frac{\beta}{y}$ is treated similarly. □

If m and n are integers such that $n \geq 2$ and $\gcd\{m, n\} = 1$, then we will denote by $m^{-1} \mod n$ the smallest positive integer u such that $mu \equiv 1 \,(\mod n)$.

Observe that if $\mathrm{M}(S) = \{n_1 < n_2 < \cdots < n_p\}$, then $n_1 = \mathrm{m}(S)$. We call n_2 the ratio of S and denote it by $\mathrm{r}(S)$.

Theorem 23. *Let a, a' and b be integers such that $2 \leq a < b$ and $2 \leq a' < b$. If $S = \mathrm{S}\big(\big]\frac{b}{a}, \frac{b}{a-1}\big[\big) = \mathrm{S}\big(\big]\frac{b}{a'}, \frac{b}{a'-1}\big[\big)$ and $\mathrm{r}(S) < b$, then either $a = a'$ or $a + a' = b + 1$.*

Proof. In view of Proposition 3 we have $\mathrm{g}(S) = b$. From Lemma 10, there exist unique positive integers x and y such that $\big\{\frac{\mathrm{m}(S)}{x}, \frac{\mathrm{r}(S)}{y}\big\} \subseteq \big]\frac{b}{a}, \frac{b}{a-1}\big[$. Note that, by Lemma 21, $\frac{\mathrm{m}(S)}{x} \neq \frac{\mathrm{r}(S)}{y}$. Now, we distinguish two cases depending on $\frac{\mathrm{m}(S)}{x} < \frac{\mathrm{r}(S)}{y}$ or $\frac{\mathrm{r}(S)}{y} < \frac{\mathrm{m}(S)}{x}$.

If $\frac{\mathrm{m}(S)}{x} < \frac{\mathrm{r}(S)}{y}$, then by applying Lemma 22 we have that $\mathrm{r}(S)\cdot x - \mathrm{m}(S)\cdot y = 1$. As $1 < \frac{b}{a} < \frac{\mathrm{m}(S)}{x}$, we obtain that $x < \mathrm{m}(S)$, and so $x = \mathrm{r}(S)^{-1} \mod \mathrm{m}(S)$.

If $\frac{\mathrm{r}(S)}{y} < \frac{\mathrm{m}(S)}{x}$, then by using Lemma 22 we obtain that $\mathrm{m}(S)\cdot y - \mathrm{r}(S)\cdot x = 1$. Therefore $x = (-\mathrm{r}(S))^{-1} \mod \mathrm{m}(S) = \mathrm{m}(S) - \big(\mathrm{r}(S)^{-1} \mod \mathrm{m}(S)\big)$.

As a consequence of the two previous paragraphs we get that either $\frac{b}{a} < \frac{\mathrm{m}(S)}{\mathrm{r}(S)^{-1} \mod \mathrm{m}(S)} < \frac{b}{a-1}$ or $\frac{b}{a} < \frac{\mathrm{m}(S)}{\mathrm{m}(S) - (\mathrm{r}(S)^{-1} \mod \mathrm{m}(S))} < \frac{b}{a-1}$. Similarly, we also have either $\frac{b}{a'} < \frac{\mathrm{m}(S)}{\mathrm{r}(S)^{-1} \mod \mathrm{m}(S)} < \frac{b}{a'-1}$ or $\frac{b}{a'} < \frac{\mathrm{m}(S)}{\mathrm{m}(S) - (\mathrm{r}(S)^{-1} \mod \mathrm{m}(S))} < \frac{b}{a'-1}$. To conclude the proof, observe that $\frac{b}{a} < \frac{\mathrm{m}(S)}{\mathrm{r}(S)^{-1} \mod \mathrm{m}(S)} < \frac{b}{a-1}$ if and only if $\frac{b}{b+1-a} < \frac{\mathrm{m}(S)}{\mathrm{m}(S) - (\mathrm{r}(S)^{-1} \mod \mathrm{m}(S))} < \frac{b}{b-a}$ (see the paragraph preceding Lemma 18). □

Following the notation in [13], a numerical semigroup S is a half-line if $S = \{\mathrm{m}(S), \to\} \cup \{0\}$, and it is opened modular if either S is a half-line or there exist integers a and b such that $2 \leq a < b$ and $S = \mathrm{S}\big(\big]\frac{b}{a}, \frac{b}{a-1}\big[\big)$.

Corollary 24. *Let S and S' be two opened modular numerical semigroups. Then $S = S'$ if and only if $\mathrm{g}(S) = \mathrm{g}(S')$, $\mathrm{m}(S) = \mathrm{m}(S')$ and $\mathrm{r}(S) = \mathrm{r}(S')$.*

PROOF. The necessary condition is obvious. Let us see the sufficient condition. It is clear that S is a half-line if and only if $\mathrm{g}(S) = \mathrm{m}(S) - 1$. Thus, if S is a half-line, as $\mathrm{g}(S') = \mathrm{g}(S) = \mathrm{m}(S) - 1 = \mathrm{m}(S') - 1$, S' is also a half-line. Hence $S = S'$.

If S and S' are not half-lines, by Proposition 3 and by definition of opened modular, we can assume that $S = \mathrm{S}\big(\big]\frac{\mathrm{g}(S)}{a}, \frac{\mathrm{g}(S)}{a-1}\big[\big)$ and $S' = \mathrm{S}\big(\big]\frac{\mathrm{g}(S)}{a'}, \frac{\mathrm{g}(S)}{a'-1}\big[\big)$, for some integers a and a' such that $2 \leq a < \mathrm{g}(S)$ and $2 \leq a' < \mathrm{g}(S)$. If $\mathrm{m}(S) = \mathrm{m}(S') = 2$ then by Lemma 7 we deduce that $S = S'$. If $\mathrm{m}(S) \geq 3$, then by Lemma 12 $\mathrm{r}(S) < \mathrm{g}(S)$. Now, in view of the proof of Theorem 23, it follows that either $a = a'$ or $a + a' = b + 1$. Finally, by Lemma 18, we obtain $S = S'$. □

Hence an opened modular numerical semigroup is determined by its two smallest minimal generators and its Frobenius number.

Corollary 25. *Let $b \geq 3$ be an integer. Then the number of symmetric proportionally modular numerical semigroups with Frobenius number b is equal to $\#\big\{a \in \big\{2, \ldots, \frac{b+1}{2}\big\} \mid \gcd\{a, b\} = \gcd\{a-1, b\} = 1\big\}$.*

PROOF. By applying Propositions 3 and 4 and Lemma 18, we deduce that S is a symmetric proportionally modular numerical semigroup with Frobenius number b if and only if $S = \mathrm{S}\big(\big]\frac{b}{a}, \frac{b}{a-1}\big[\big)$, where a is an integer such that $2 \leq a \leq \frac{b+1}{2}$ and $\gcd\{a, b\} = \gcd\{a-1, b\} = 1$. From Lemmas 7 and 8, we know that $\mathrm{m}(S) = 2$ if and only if $a = \frac{b+1}{2}$. We conclude the proof by using Lemma 12 and Theorem 23. □

We define $X(b) = \big\{a \in \{1, \ldots, b\} \mid \gcd\{a, b\} = \gcd\{a-1, b\} = 1\big\}$ and $\chi(b) = \#X(b)$, for any positive integer b. Note that $X(b) \neq \emptyset$ if and only if b is odd and in this case $\frac{b+1}{2} \in X(b)$. Under these settings, Corollary 25 can be reformulated in the following way.

Corollary 26. *Let $b \geq 3$ be an odd integer. Then the number of symmetric proportionally modular numerical semigroups with Frobenius number b is equal to $\frac{\chi(b)+1}{2}$.*

PROOF. We have that $\{1, b\} \cap X(b) = \emptyset$. Note that $a \in X(b)$ if and only if $b + 1 - a \in X(b)$. To conclude the proof we observe that $\frac{b+1}{2} \in X(b)$ and that $a = b + 1 - a$ if and only if $a = \frac{b+1}{2}$. □

Finally we show that the values of $\chi(b)$, for any odd integer $b \geq 3$, can be easily obtained by using the following results.

Proposition 27. *Let $p \geq 3$ be a prime number. Then $\chi\left(p^k\right) = p^{k-1}(p-2)$, for any positive integer k.*

PROOF. Let $p \geq 3$ be a prime number and let k be a positive integer. Note that $1 \notin X\left(p^k\right)$. Since p is a prime number, the only numbers from the set $\left\{1, \ldots, p^k\right\}$ that do not belong to $X\left(p^k\right)$, apart from 1, are those of the form $t \cdot p$ and also those of the form $t' \cdot p + 1$, where $1 \leq t \leq p^{k-1}$ and $1 \leq t' \leq p^{k-1} - 1$. It is clear that there are p^{k-1} numbers of the first type and $p^{k-1} - 1$ numbers of the second one. Therefore $\chi\left(p^k\right) = p^k - 1 - p^{k-1} - \left(p^{k-1} - 1\right) = p^k - 2p^{k-1} = p^{k-1}(p-2)$. □

Proposition 28. *Let $b \geq 3$ and $b' \geq 3$ be two coprime odd integers. Then $\chi(b \cdot b') = \chi(b) \cdot \chi(b')$.*

PROOF. The proof is a consequence of the Chinese Remainder Theorem and the fact that $a \in X(b \cdot b')$ if and only if $(a \mod b, a \mod b') \in X(b) \times X(b')$. □

In view of Corollary 26 and Propositions 27 and 28 we can easily compute the number of symmetric proportionally modular numerical semigroups with a given Frobenius number. We illustrate this with an example.

Example 29. The number of symmetric proportionally modular numerical semigroups with Frobenius number 7007 is

$$\frac{\chi(7007)+1}{2} = \frac{\chi\left(13 \cdot 11 \cdot 7^2\right)+1}{2} = \frac{11 \cdot 9 \cdot (7 \cdot 5)+1}{2} = \frac{3466}{2} = 1733.$$

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(Received June 28, 2007; revised December 3, 2007)