Publ. Math. Debrecen
73/3-4 (2008), 421–460

Generalized convex functions and a solution of a problem of Zs. Páles

By JANUSZ MATKOWSKI (Zielona Góra)

Abstract. Following an idea of Beckenbach, given a real function α defined on a convex subset U of a linear space and $t \in (0;1)$, we define: convexity, t-convexity, Jensen convexity, affinity, t-affinity and Jensen affinity of a function $f : U \to \mathbb{R}$ with respect to α. Some generalizations of Berstein–Doetsch and Sierpiński theorems are proved. Natural generalizations of Jensen and Cauchy functional equations are considered. A three variable functional equation on α which is a necessary condition for the existence of discontinuous Jensen affine functions with respect to α is presented. In one-dimensional case the explicit form of all Jensen affine functions with respect to α, involving the homographic functions, are determined. Applying this result we obtain a complete solution of a problem posed by Zs. Páles. Moreover, without any regularity assumptions, some functional equations are solved.

1. Introduction

Let U be a convex subset of a real linear space X and let $\alpha : U \to \mathbb{R}$ be a fixed function. A function $f : U \to \mathbb{R}$ is called *convex with respect to* α if for all $x, y \in U$ there exist some real numbers $b = b(x,y)$, $c = c(x,y)$ such that for all $t \in [0;1]$,

$$f(tx + (1-t)y) \leq b(x,y)\alpha(tx + (1-t)y) + c(x,y)$$

and for $t = 1$ and $t = 0$ this inequality becomes equality, i.e.

$$f(x) = b(x,y)\alpha(x) + c(x,y), \quad f(y) = b(x,y)\alpha(y) + c(x,y).$$

Mathematics Subject Classification: 26E30, 39B22.
Key words and phrases: Beckenbach convexity, convex function with respect a given function, affine (Jensen affine) function with respect to a given function, generalized Cauchy equation, homographic functions, Tchebycheff system.

If $\alpha(x) \neq \alpha(y)$ then the numbers

$$b(x,y) = \frac{f(x)-f(y)}{\alpha(x)-\alpha(y)}, \quad c(x,y) = f(x) - \frac{f(x)-f(y)}{\alpha(x)-\alpha(y)}\alpha(x)$$

are uniquely determined, and the above inequality can be written in the form

$$f(tx+(1-t)y) \leq \frac{\alpha\left(tx+(1-t)y\right)-\alpha(y)}{\alpha(x)-\alpha(y)}f(x) + \frac{\alpha(x)-\alpha\left(tx+(1-t)y\right)}{\alpha(x)-\alpha(y)}f(y).$$

In the case when $U \subset \mathbb{R}$ is an interval, the above definition can be obtained from the usual geometric interpretation of a convex function, replacing the family of chords $U \ni x \to bx+c$, by the family of curves $b\alpha + c$ for all real b, c. In particular, if $\alpha(x) = bx + c$, $(x \in U)$, for some $b, c \in \mathbb{R}$, $b \neq 0$, the last inequality reduces to the ordinary definition of a convex function.

In one-dimensional case the definition introduced by E. F. Beckenbach [2] is more general. The weaker variants of this definition: *t-convexity with respect to α*, *$\frac{1}{2}$-convexity with respect to α* (*Jensen convexity with respect to α*), considered in this paper, generalize the corresponding classical ones.

In Section 2, assuming some monotonicity type conditions on α, we prove that t-convexity with respect to α of a function implies its Jensen convexity with respect to α (Theorem 1). On the other hand, Jensen convexity of a function with respect to α implies its $\frac{k}{2^n}$-convexity with respect to α for all $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, 2^n\}$ (Lemma 1). Thus the continuity and Jensen convexity with respect to α of a function imply its convexity with respect to α.

Let U be an open and convex set in a linear topological space. Suppose that $\alpha : U \to \mathbb{R}$ is continuous, non-constant in a neighbourhood of any point of U, and, for all $x, y \in U$, the function $s \to \alpha(sx+(1-s)y)$ is monotonic in the interval $[0;1]$. Under these conditions, in Section 3, we show that the Jensen convexity with respect to α and boundedness above in a neighbourhood of a point of a function $f : U \to \mathbb{R}$ imply its local boundedness in a neighbourhood of every point of U (Theorem 3). Similar result holds true if $U \subset \mathbb{R}^m$ and f is Lebesgue measurable (Theorem 4). The main result of these section (Theorem 5) reads as follows. *Let $U \subset R^m$ be an open and convex set, let a mapping $A : U \to R^m$, $A = (\alpha_1, \ldots, \alpha_m)$ be a local C^1-diffeomorphism in U. Assume that at least one of the coordinate functions α_j is not constant in a neighbourhood of any point and, for all $x, y \in U$, the function $s \to \alpha_j(sx+(1-s)y)$ is monotonic in $[0;1]$. If a function $f : U \to R$ is Jensen convex with respect to α_j for each $j \in \{1, \ldots, m\}$, and Lebesgue measurable or bounded above in a neighbourhood of a point then it is continuous.*

In Section 4 we present the conditions on α (including some continuity and monotonicity properties) under which every continuous at least at one-point real function f defined on an open and convex subset of a normed space and Jensen affine with respect to α must be of the form $f = b\alpha + c$, for some real b, c (Theorem 6). We also show that the continuity of f at least at one point can be replaced by the boundedness above at a point or the Lebesgue measurability of f. (In one-dimensional case a counterpart of Theorem 6 for strictly monotonic functions holds true (Theorem 8 in Section 5)). In the next section, an important result (Theorem 7) says that, if there exists a discontinuous at least at one point function which is Jensen affine with respect to α, then, necessarily, α must satisfy the following functional equation

$$\frac{\alpha\left(\frac{x+2y+z}{4}\right)-\alpha(y)}{\alpha\left(\frac{x+z}{2}\right)-\alpha(y)}\,\frac{\alpha\left(\frac{x+z}{2}\right)-\alpha(z)}{\alpha(x)-\alpha(z)}=\frac{\alpha\left(\frac{x+2y+z}{4}\right)-\alpha\left(\frac{y+z}{2}\right)}{\alpha\left(\frac{x+y}{2}\right)-\alpha\left(\frac{y+z}{2}\right)}\,\frac{\alpha\left(\frac{x+y}{2}\right)-\alpha(y)}{\alpha(x)-\alpha(y)}.$$

Thus, a Jensen affine function with respect to α can be discontinuous only in the case when α is a solution of this functional equation of three variables.

In Section 7, using this fact, we prove Theorem 8 which says that, in one-dimensional case, a discontinuous Jensen affine function with respect to α exists if, and only if, α is a homographic function. Moreover the form of all Jensen affine functions with respect to α is given. In Section 8 we apply Theorem 8 to determine more general class of Beckenbach affine functions. In particular we obtain a solution of a problem posed by Zs. Páles [8]. In the next section we show that Theorems 7 and 8 are applied to solve some functional equations without any regularity assumptions.

If α is a homogeneous function of a given order then the generalized Jensen functional equation (2) leads to the functional equation (18) which can be treated as a generalization of the Cauchy functional equation. Section 9 is devoted to this equation. The conditions on α under which every solution of the respective Cauchy equation has to be continuous on an open set are given, and, moreover, the functions α for which the respective Cauchy equation has everywhere discontinuous solutions are determined (Theorem 11).

The classical method to determine the solutions of the Jensen equation is based on the Cauchy functional equation (that is, on the theory of additive functions). The key idea due to Cauchy is to derive from the functional equation of two variables, inductively, the suitable equations involving arbitrary finite number of variables (cf. for instance J. Aczél [1], M. Kuczma [6]). In this paper an opposite approach is presented. To determine the solutions of the (generalized) Cauchy type equations we first examine the (generalized) Jensen type equations, and we do not use the idea of Cauchy (cf. the proof of Theorem 6).

We end the paper with an open question concernig the relation between a modified (M, N)-convexity and the Beckenbach convexity.

2. Convex functions with respect to a given function and their basic properties

The following definitions generalize the classical notions of convex (concave, affine) functions; t-convex (t-concave, t-affine) functions; and Jensen convex (Jensen concave, Jensen affine) functions (cf. M. Kuczma [6], p. 111).

Definition 1. Let U be a convex set in a real linear space X, $\alpha : U \to \mathbb{R}$ be an arbitrary function and $t \in (0;1)$ be fixed. A function $f : U \to \mathbb{R}$ is called:

(1) t-convex with respect to α, briefly: t-convex with respect to α, if the inequality

$$f(tx+(1-t)y) \leq \frac{\alpha\left(tx+(1-t)y\right)-\alpha(y)}{\alpha(x)-\alpha(y)}f(x) + \frac{\alpha(x)-\alpha\left(tx+(1-t)y\right)}{\alpha(x)-\alpha(y)}f(y) \tag{1}$$

holds true for all $x, y \in U$ such that $\alpha(x) \neq \alpha(y)$; and strictly t-convex with respect to α, if this inequality is strict;

(2) t-concave with respect to α, if $-f$ is t-convex with respect to α;

(3) t-affine with respect to α if it is both t-convex and t-concave with respect to α.

In the case $t = \frac{1}{2}$ we say, respectively, that f is Jensen convex with respect to α, Jensen concave with respect to to α, and Jensen affine with respect to α. In particular f is Jensen affine with respect to α if

$$f\left(\frac{x+y}{2}\right) = \frac{\alpha\left(\frac{x+y}{2}\right)-\alpha(y)}{\alpha(x)-\alpha(y)}f(x) + \frac{\alpha(x)-\alpha\left(\frac{x+y}{2}\right)}{\alpha(x)-\alpha(y)}f(y) \tag{2}$$

for all $x, y \in U$ such that $\alpha(x) \neq \alpha(y)$.

A function $f : U \to \mathbb{R}$ is called convex (concave, affine) with respect to α if f is t-convex (t-concave, t-affine) for every $t \in (0,1)$.

The following is easy to verify.

Remark 1. Let U be a convex set in a real linear space, $\alpha : U \to \mathbb{R}$.

(1) A function $f : U \to \mathbb{R}$ is t-affine with respect to α iff

$$\begin{vmatrix} f(x) & f(tx+(1-t)y) & f(y) \\ \alpha(x) & \alpha(tx+(1-t)y) & \alpha(y) \\ 1 & 1 & 1 \end{vmatrix} = 0, \quad x, y \in U.$$

(2) For arbitrarily fixed $b, c \in \mathbb{R}$, the function $f = b\alpha + c$ is affine with respect to α.

(3) If $f : U \to \mathbb{R}$ is convex (Jensen convex) with respect to α, then for all $b, c \in \mathbb{R}$, $b > 0$, the function f is convex (Jensen convex) with respect to $b\alpha + c$.

(4) If $f, g : U \to \mathbb{R}$ are convex (Jensen convex) with respect to α, then so is $f+g$.

Remark 2. Let $U \subset \mathbb{R}$ be an interval and $\alpha : U \to \mathbb{R}$ be strictly monotonic. Then

(1) the range of function $w_\alpha : U \times U \times I \to \mathbb{R}$ be defined by

$$w_\alpha(x, y, t) := \begin{cases} \dfrac{\alpha\left(tx+(1-t)y\right) - \alpha(y)}{\alpha(x) - \alpha(y)} & \text{for } x \neq y \\ t & \text{for } x = y \end{cases}$$

is contained in the interval $(0; 1)$;

$$w_{b\alpha+c} = w_\alpha, \quad b, c \in \mathbb{R}, b \neq 0;$$

if α is differentiable at a point $x \in I$ and $\alpha'(x) \neq 0$, then

$$\lim_{y \to x} w_\alpha(x, y, t) = t;$$

(2) If α is increasing then $f : U \to \mathbb{R}$ is convex with respect to α iff the function

$$(U^2 \setminus \{(x, x) : x \in U\}) \ni (x, y) \to \frac{f(x) - f(y)}{\alpha(x) - \alpha(y)}$$

is increasing with respect to each variable.

(3) Suppose that $f : U \to \mathbb{R}$ is strictly monotonic. If f and α are of the same (different) type monotonicity then f is convex with respect to α iff α is concave with respect to f (convex with respect to f).

Remark 3. If $U \subset \mathbb{R}$ is an interval, and α is continuous and strictly monotonic then the convexity of a function f with respect to α is equivalent to Beckenbach's convexity with respect to the two-parameter family of functions

$$\mathcal{F} := \{b\alpha + c : b, c \in \mathbb{R}\}.$$

For a real linear space X denote by X' the set of all real linear functionals on X, and for a real linear topological space X denote by X^* the set of continuous linear functionals on X.

Remark 4. Let U be a convex set in a real linear space X, let $\alpha \in X'$, and let $t \in (0;1)$. Then, obviously, $f : U \to \mathbb{R}$ is t-convex with respect to α if, and only if,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad x, y \in U,\ \alpha(x-y) \neq 0;$$

in particular, f is Jensen convex with respect to α if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}, \quad x, y \in U,\ \alpha(x-y) \neq 0.$$

Proposition 1. *Let U be a convex set in a real linear space X, let $f : U \to \mathbb{R}$ and let $t \in (0;1)$.*

1) *The function f is t-convex, that is*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad x, y \in U,$$

iff f is t-convex with respect to α for every linear functional $\alpha \in X'$, (In particular, f is convex iff, for each $\alpha \in X'$, it is convex with respect to α.)

2) *If t is a rational number then f is t-convex iff f is t-convex with respect to α for every additive function $\alpha : X \to \mathbb{R}$.*

3) *If X is a normed space then f is t-convex iff f is t-convex with respect to α for each $\alpha \in X^*$. (Here X^* can be replaced by an arbitrary total set of functionals).*

Proof. 1) If $x = y$ there is nothing to prove. Suppose that every linear functional $\alpha \in X'$ the function $f : U \to \mathbb{R}$ is t-convex with respect to α and take arbitrary $x, y \in U$, $x \neq y$. By the Hamel base argument, there is an $\alpha \in X'$ such that $\alpha(x-y) \neq 0$. In view of Remark 1,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

which proves that f is t-convex. The converse implication is obvious. We omit similar arguments for the remaining parts. □

Remark 5. In this proposition the linear functionals can be replaced by the functionals of the form $\alpha + c$ where α is a linear functional and $c \in \mathbb{R}$ is constant (that is by the affine functionals).

Remark 6. Let U be a convex set of a real linear space, let $\alpha : U \to \mathbb{R}$ and let $t \in (0;1)$. Then, obviously, $f : U \to \mathbb{R}$ is t-convex with respect to α, iff

$$\begin{aligned}[\alpha(x) - \alpha(y)]\, f(tx + (1-t)y) \\ \leq [\alpha\,(tx + (1-t)y) - \alpha(y)]f(x) + [\alpha(x) - \alpha\,(tx + (1-t)y)]f(y)\end{aligned}$$

for all $x, y \in U$ such that $\alpha(x) > \alpha(y)$, and

$$[\alpha(x) - \alpha(y)]\, f(tx + (1-t)y) \geq [\alpha(tx + (1-t)y) - \alpha(y)]f(x) + [\alpha(x) - \alpha(tx + (1-t)y)]f(y)$$

for all $x, y \in U$ such that $\alpha(x) < \alpha(y)$.

Assuming that U is a subset of a normed space, $\alpha, f : U \to \mathbb{R}$ are continuous and $\operatorname{int} \alpha^{-1}(c) = \emptyset$ for every $c \in \mathbb{R}$, we infer that

$$[\alpha(tx + (1-t)y) - \alpha(x)][f(x) - f(y)] = 0$$

for all $x, y \in U$ such that $\alpha(x) = \alpha(y)$.

Remark 7. Let U be a convex subset of a real linear space. Suppose that $\alpha : U \to \mathbb{R}$ is such that for all $x, y \in U$ the function $[0;1] \ni s \to \alpha(sx + (1-s)y)$ is monotonic.

If $\alpha(x - y) \neq 0$ then, for every $t \in (0;1)$,

$$0 \leq \frac{\alpha(tx + (1-t)y) - \alpha(y)}{\alpha(x) - \alpha(y)} \leq 1,$$

the inequalities are strict if the monotonicity is strict, and

$$\frac{\alpha(tx + (1-t)y) - \alpha(y)}{\alpha(x) - \alpha(y)} + \frac{\alpha(x) - \alpha(tx + (1-t)y)}{\alpha(x) - \alpha(y)} = 1.$$

Theorem 1. *Let U be a convex set in a real linear space and let $t \in (0;1)$ be fixed. Suppose that $\alpha : U \to \mathbb{R}$ is such that for all $x, y \in U$ the function*

$$[0;1] \ni s \to \alpha(sx + (1-s)y), \tag{3}$$

is monotonic and strictly monotonic if $\alpha(x) \neq \alpha(y)$.

If a function $f : U \to \mathbb{R}$ is t-convex with respect to α, then it is Jensen convex with respect to α.

Proof. Suppose that $f : U \to \mathbb{R}$ is t-convex with respect to α for a $t \in (0;1)$. Take arbitrary $x, y \in U$ such that $\alpha(x) \neq \alpha(y)$ and put

$$a := \alpha\left(\frac{x+y}{2}\right), \quad b := \alpha\left(tx + (1-t)\frac{x+y}{2}\right),$$
$$c := \alpha\left(t\frac{x+y}{2} + (1-t)y\right), \quad d := \alpha(y), \quad e := \alpha(x).$$

By the Daróczy–Páles identity [5] we have

$$\frac{x+y}{2} = t\left(t\frac{x+y}{2} + (1-t)y\right) + (1-t)\left(tx + (1-t)\frac{x+y}{2}\right).$$

Hence, applying twice the t-convexity with respect to α of the function f, we get

$$f\left(\frac{x+y}{2}\right) \le \frac{a-b}{c-b}f\left(t\frac{x+y}{2} + (1-t)y\right) + \frac{c-a}{c-b}f\left(tx + (1-t)\frac{x+y}{2}\right)$$
$$\le \frac{a-b}{c-b}\left[\frac{c-d}{a-d}f\left(\frac{x+y}{2}\right) + \frac{a-c}{a-d}f(y)\right] + \frac{c-a}{c-b}\left[\frac{b-a}{e-a}f(x) + \frac{e-b}{e-a}f\left(\frac{x+y}{2}\right)\right]$$
$$= \left(\frac{a-b}{c-b}\frac{c-d}{a-d} + \frac{c-a}{c-b}\frac{e-b}{e-a}\right)f\left(\frac{x+y}{2}\right) + \frac{c-a}{c-b}\frac{b-a}{e-a}f(x) + \frac{a-b}{c-b}\frac{a-c}{a-d}f(y).$$

Since

$$1 - \frac{a-b}{c-b}\frac{c-d}{a-d} - \frac{c-a}{c-b}\frac{e-b}{e-a} = \frac{(e-d)(a-c)(a-b)}{(a-d)(a-e)(b-c)},$$

we can write the above inequality in the following form

$$\frac{(e-d)(a-c)(a-b)}{(a-d)(a-e)(b-c)}f\left(\frac{x+y}{2}\right) \le \frac{c-a}{c-b}\frac{b-a}{e-a}f(x) + \frac{a-b}{c-b}\frac{a-c}{a-d}f(y).$$

Taking into account that the points

$$y, \quad tx + (1-t)\frac{x+y}{2}, \quad \frac{x+y}{2}, \quad t\frac{x+y}{2} + (1-t)y, \quad x,$$

appear on trajectory: $[0;1] \ni s \to sx+(1-s)y$ of the endpoints y and x, according to the increasing order of the parameter s, by the definitions of a, b, c, d, e and the monotonicity of the function (3), we have

$$\frac{(e-d)(a-c)(a-b)}{(a-d)(a-e)(b-c)} > 0.$$

Dividing both sides of the previous inequality by this number we obtain

$$f\left(\frac{x+y}{2}\right) \le \frac{a-d}{e-d}f(x) + \frac{e-a}{c-d}f(y),$$

which coincides with inequality (2). This completes the proof. □

Put

$$T_n := \left\{\frac{k}{2^n} : k = 0, 1, \ldots, 2^n\right\} \quad \text{for } n \in \mathbb{N}, \qquad T := \bigcup_{n=1}^{\infty} T_n. \tag{4}$$

Lemma 1. *Let U be a convex set in a real linear space X and $\alpha : U \to \mathbb{R}$ be such that, for all $x, y \in U$, the function*

$$[0;1] \ni s \to \alpha(sx + (1-s)y),$$

is monotonic, and strictly monotonic if $\alpha(x) \neq \alpha(y)$.

If a function $f : U \to \mathbb{R}$ is Jensen convex (Jensen-concave, Jensen-affine) with respect to α, then, for every $t \in T$, it is t-convex (respectively, t-concave, t-affine) with respect to α.

Proof. We shall show that

$$\begin{aligned} f\left(\frac{kx + (2^n - k)y}{2^n}\right) &\leq \frac{\alpha\left(\frac{kx+(2^n-k)y}{2^n}\right) - \alpha(y)}{\alpha(x) - \alpha(y)} f(x) \\ &+ \frac{\alpha(x) - \alpha\left(\frac{kx+(2^n-k)y}{2^n}\right)}{\alpha(x) - \alpha(y)} f(y) \end{aligned} \tag{5}$$

for all $n \in \mathbb{N}$, $k \in \{0, 1, \ldots, 2^n\}$ and $x, y \in U$, $\alpha(x) \neq \alpha(y)$.

In the case when $n = 1$ it is obvious for $k \in \{0, 2\}$ and it follows from (2) for $k = 1$. Suppose that inequality (5) holds true for some positive integer n. Replacing x by $\frac{x+y}{2}$ in (5) and making use of (2) and the monotonicity of α. We get

$$\begin{aligned} f\left(\frac{kx + (2^{n+1} - k)y}{2^{n+1}}\right) &= f\left(\frac{k\frac{x+y}{2} + (2^n - k)y}{2^n}\right) \\ &\leq \frac{\alpha\left(\frac{k\frac{x+y}{2}+(2^n-k)y}{2^n}\right) - \alpha(y)}{\alpha\left(\frac{x+y}{2}\right) - \alpha(y)} f\left(\frac{x+y}{2}\right) + \frac{\alpha\left(\frac{x+y}{2}\right) - \alpha\left(\frac{k\frac{x+y}{2}+(2^n-k)y}{2^n}\right)}{\alpha\left(\frac{x+y}{2}\right) - \alpha(y)} f(y) \\ &\leq \frac{\alpha\left(\frac{k\frac{x+y}{2}+(2^n-k)y}{2^n}\right) - \alpha(y)}{\alpha\left(\frac{x+y}{2}\right) - \alpha(y)} \left[\frac{\alpha\left(\frac{x+y}{2}\right) - \alpha(y)}{\alpha(x) - \alpha(y)} f(x) + \frac{\alpha(x) - \alpha\left(\frac{x+y}{2}\right)}{\alpha(x) - \alpha(y)} f(y)\right] \\ &\quad + \frac{\alpha\left(\frac{x+y}{2}\right) - \alpha\left(\frac{k\frac{x+y}{2}+(2^n-k)y}{2^n}\right)}{\alpha\left(\frac{x+y}{2}\right) - \alpha(y)} f(y) \\ &= \frac{\alpha\left(\frac{kx+(2^{n+1}-k)y}{2^{n+1}}\right) - \alpha(y)}{\alpha(x) - \alpha(y)} f(x) + \frac{\alpha(x) - \alpha\left(\frac{kx+(2^{n+1}-k)y}{2^{n+1}}\right)}{\alpha(x) - \alpha(y)} f(y) \end{aligned}$$

for all $x, y \in U$, $\alpha(x) \neq \alpha(y)$, and $k \in \{0, 1, \ldots, 2^n\}$. Similarly, replacing y by $\frac{x+y}{2}$ in (5) and making use of (2) we get

$$f\left(\frac{(k + 2^n)x + [2^{n+1} - (k + 2^n)]y}{2^{n+1}}\right) = f\left(\frac{kx + (2^n - k)\frac{x+y}{2}}{2^n}\right)$$

$$\leq \frac{\alpha\left(\frac{kx+(2^n-k)\frac{x+y}{2}}{2^n}\right)-\alpha\left(\frac{x+y}{2}\right)}{\alpha(x)-\alpha\left(\frac{x+y}{2}\right)}f(x)+\frac{\alpha(x)-\alpha\left(\frac{kx+(2^n-k)\frac{x+y}{2}}{2^n}\right)}{\alpha(x)-\alpha\left(\frac{x+y}{2}\right)}f\left(\frac{x+y}{2}\right)$$

$$\leq \frac{\alpha\left(\frac{kx+(2^n-k)\frac{x+y}{2}}{2^n}\right)-\alpha\left(\frac{x+y}{2}\right)}{\alpha(x)-\alpha\left(\frac{x+y}{2}\right)}f(x)$$

$$+\frac{\alpha(x)-\alpha\left(\frac{kx+(2^n-k)\frac{x+y}{2}}{2^n}\right)}{\alpha(x)-\alpha\left(\frac{x+y}{2}\right)}\left[\frac{\alpha\left(\frac{x+y}{2}\right)-\alpha(y)}{\alpha(x)-\alpha(y)}f(x)+\frac{\alpha(x)-\alpha\left(\frac{x+y}{2}\right)}{\alpha(x)-\alpha(y)}f(y)\right]$$

$$=\frac{\alpha\left(\frac{(k+2^n)x+[2^{n+1}-(k+2^n)]y}{2^{n+1}}\right)-\alpha(y)}{\alpha(x)-\alpha(y)}f(x)$$

$$+\frac{\alpha(x)-\alpha\left(\frac{(k+2^n)x+[2^{n+1}-(k+2^n)]y}{2^{n+1}}\right)}{\alpha(x)-\alpha(y)}f(y)$$

for all $x,y\in U$, $\alpha(x)\neq\alpha(y)$, and $k\in\{0,1,\ldots,2^n\}$.

Thus

$$f\left(\frac{kx+(2^{n+1}-k)y}{2^{n+1}}\right)\leq\frac{\alpha\left(\frac{kx+(2^{n+1}-k)y}{2^{n+1}}\right)-\alpha(y)}{\alpha(x)-\alpha(y)}f(x)$$
$$+\frac{\alpha(x)-\alpha\left(\frac{kx+(2^{n+1}-k)y}{2^{n+1}}\right)}{\alpha(x)-\alpha(y)}f(y)$$

for all $x,y\in I$, $\alpha(x)\neq\alpha(y)$, and $k\in\left\{0,1,\ldots,2^{n+1}\right\}$, which proves that (5) holds true for $n+1$. By induction (5) holds true for all $n\in\mathbb{N}$. Now the first part of the lemma is a consequence of the definition (4) of the set T.

Since the second part is a consequence of the first one, the proof is complete. □

As an immediate consequence of Lemma 1 and the density of the set T we get the following

Theorem 2. *Let U be a convex set in a real linear topological space X and $\alpha:U\to\mathbb{R}$ be an arbitrary continuous function.*

(1) *If $f:U\to\mathbb{R}$ is continuous and Jensen convex with respect to α, then f is convex with respect to α.*

(2) *If $f:U\to\mathbb{R}$ is continuous and Jensen affine with respect to α, then f is affine with respect to α.*

3. A generalization of Berstein–Doetsch and Sierpiński theorems

Theorem 3. *Let U be an open and convex set in a linear topological space. Suppose that $\alpha : U \to \mathbb{R}$ is continuous,*

$$\operatorname{int} \alpha^{-1}\{c\} = \emptyset, \quad c \in \mathbb{R}, \tag{6}$$

(that is α is not constant in a neighbourhood of any point of U), and the function

$$[0;1] \ni s \to \alpha(sx + (1-s)y) \quad \text{is monotonic for all } x, y \in U. \tag{7}$$

If $f : U \to \mathbb{R}$ is Jensen convex with respect to α and bounded above in a neighbourhood of a point of the set U, then it is locally bounded in a neighbourhood of every point of U.

Proof. Let $f : U \to \mathbb{R}$ be Jensen convex with respect to α. Assume that for some $x_0 \in U$ there are a neighbourhood $B(x_0) \subset U$ of x_0 and $M \in \mathbb{R}$ such that

$$f(u) \leq M, \quad u \in B(x_0).$$

Take an arbitrary $x \in U$, $x \neq x_0$. We may assume, without any loss of generality, that $\alpha(x) \neq \alpha(x_0)$ as, if necessary, we could replace x_0 by a point $z_0 \in B(x_0)$ such that $\alpha(x) \neq \alpha(z_0)$. The existence of z_0 follows from the continuity of α, and the conditions (6) and (7). Since U is open and T is dense in the interval $[0;1]$, there is a $\delta \in (0;1)$ such that for all $t \in (\delta;1)$ we have

$$y := \frac{1}{t}x + \left(1 - \frac{1}{t}\right) x_0 \in U.$$

Note that $x = ty + (1-t)x_0$, the set $B(x) := ty + (1-t)B(x_0, r)$ is a neighbourhood of x and, by the convexity of U, $ty + (1-t)B(x_0, r) \subset U$.
By the continuity of α there are $r > 0$ and $t \in (0;\delta)$ such that

$$\alpha(u) \neq \alpha(y), \quad u \in B(x_0, r).$$

Now, applying Lemma 1, we get

$$\begin{aligned} f(ty + (1-t)u) &\leq \frac{\alpha(ty + (1-t)u) - \alpha(u)}{\alpha(y) - \alpha(u)} f(y) + \frac{\alpha(y) - \alpha(ty + (1-t)u)}{\alpha(y) - \alpha(u)} f(u) \\ &\leq \max(f(y), M) \left[\frac{\alpha(ty + (1-t)u) - \alpha(u)}{\alpha(y) - \alpha(u)} + \frac{\alpha(y) - \alpha(ty + (1-t)u)}{\alpha(y) - \alpha(u)} \right] \\ &= \max(f(y), M), \end{aligned}$$

for all $u \in B(x_0, r)$, which shows that

$$\sup f(ty + (1-t)B(x_0, r)) \leq \max(f(y), M).$$

Thus f is bounded above in a neighbourhood $B(x)$ of the point x. Since x is arbitrarily chosen, the function f is bounded above in a neighbourhood of every point of U.

To show that f is locally bounded from below in U suppose, for an indirect argument, that there is an $x \in U$ and a sequence (x_n) such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} f(x_n) = -\infty$. Put $y_n := 2x - x_n$, $n \in \mathbb{N}$. Modifying, if necessary, the sequence (x_n) with the aid of the continuity of α and the conditions (6) and (7), we may assume that $\alpha(x_n) \neq \alpha(y_n)$, $n \in \mathbb{N}$. Since $\lim_{n\to\infty} y_n = x_0$, in view of what we have already shown, there is $M \in \mathbb{R}$ such that $f(y_n) \leq M$, $n \in \mathbb{N}$. From (2) we obtain

$$\begin{aligned} f(x) = f\left(\frac{x_n + y_n}{2}\right) &\leq \frac{\alpha\left(\frac{x_n+y_n}{2}\right) - \alpha(y_n)}{\alpha(x_n) - \alpha(y_n)} f(x_n) + \frac{\alpha(x_n) - \alpha\left(\frac{x_n+y_n}{2}\right)}{\alpha(x_n) - \alpha(y_n)} f(y_n) \\ &\leq f(x_n) + f(y_n) \end{aligned}$$

and, consequently,

$$f(x) \leq f(x_n) + M, \quad n \in \mathbb{N},$$

which contradicts to the relation $\lim_{n\to\infty} f(x_n) = -\infty$. This completes the proof. □

Theorem 4. *Let, for some $m \in \mathbb{N}$, a set $U \subset \mathbb{R}^m$ be open and convex. Suppose that a continuous $\alpha : U \to \mathbb{R}$ satisfies condition* (6) *and* (7)*. If $f : U \to \mathbb{R}$ is Jensen convex with respect to α and Lebesgue measurable, then it is locally bounded in U.*

Proof. Put

$$U_n := \{x \in U : f(x) \leq n\}, \quad n \in \mathbb{N},$$

and note that, for some $k \in \mathbb{N}$, the Lebesgue measure $l_m(U_k)$ of U_k is positive. In fact, in the opposite case

$$l_m\left(\bigcup_{n=1}^{\infty} U_n\right) = 0,$$

and we would have $f = +\infty$ a.e. in U, which is a contradiction. From (2), for all $x, y \in U_k$ we have

$$f\left(\frac{x+y}{2}\right) \leq k\frac{\alpha\left(\frac{x+y}{2}\right) - \alpha(y)}{\alpha(x) - \alpha(y)} + k\frac{\alpha(x) - \alpha\left(\frac{x+y}{2}\right)}{\alpha(x) - \alpha(y)} = k.$$

Since, by the Steinhaus Theorem (cf. M. Kuczma [6], p. 69), the interior of the set $\frac{1}{2}(U_k + U_k)$ is nonempty, we infer that f is bounded above in a neighbourhood of a point. Now the assertion is a consequence of Theorem 3. □

The next result is a generalization of Bernstein–Doetsch Theorem [4] and Sierpinski Theorem [9] for Jensen convex functions (cf. also M. Kuczma [6]).

Theorem 5. *Let $U \subset \mathbb{R}^m$ be open and convex and let a mapping $A : U \to \mathbb{R}^m$, $A = (\alpha_1, \dots, \alpha_m)$ be a local C^1-diffeomorphism in U such that at least one of the coordinate functions α_j satisfies conditions* (6) *and* (7). *If a function $f : U \to \mathbb{R}$ is Jensen convex with respect to α_j for each $j \in \{1, \dots, m\}$ and Lebesgue measurable or bounded above in a neighbourhood of a point, then it is continuous.*

Proof. Suppose that $f : U \to \mathbb{R}$ satisfies the assumptions of our theorem. Take an $x \in U$, an arbitrary sequence $u_n \in U$ such that $\lim_{n\to\infty} u_n = x$ and put

$$C := \lim_{n\to\infty} \sup f(u_n), \quad c := \lim_{n\to\infty} \inf f(u_n).$$

The numbers C and c are finite because, in view of Theorems 3 and 4, the function f is locally bounded. Suppose that $c < C$ and choose some one-to-one sequences (x_n), (y_n) such that

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = x \quad \lim_{n\to\infty} f(x_n) = C, \quad \lim_{n\to\infty} f(y_n) = c,$$

Putting

$$z_n := 2x_n - y_n, \qquad n \in \mathbb{N}.$$

we have

$$x_n = \frac{y_n + z_n}{2}, \quad n \in \mathbb{N}; \quad \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = x.$$

Of course we can assume that

$$x_n \neq y_n \neq z_n \neq x_n, \quad n \in \mathbb{N}.$$

The compactness of the unique sphere in $\mathbb{R}^m$ implies that we can choose a subsequence of the sequence

$$\left(\frac{y_n - z_n}{\|y_n - z_n\|}\right)_{n\in\mathbb{N}}$$

to a point $v \in \mathbb{R}^m$, $\|v\| = 1$. We may assume, without any loss of generality, that

$$\lim_{n\to\infty} \frac{y_n - z_n}{\|y_n - z_n\|} = v.$$

Since $A = (\alpha_1, \dots, \alpha_m)$ is a diffeomorphism, by the Local Inverse Mapping Theorem (for instance [10], p. 172), there is a $k \in \{1, \dots, m\}$ such that

$$\alpha_k'(x)v \neq 0.$$

Note that, for every $n \in \mathbb{N}$, the functions $\varphi_n, \psi_n : [0,1] \to \mathbb{R}$ defined by

$$\varphi_n(t) := \alpha_k\left(z_n + t\left(y_n - z_n\right)\right), \quad \psi_n(t) := \alpha_k\left(z_n + t\frac{y_n - z_n}{2}\right), \quad t \in [0,1],$$

are continuously differentiable in $[0,1]$,

$$\varphi_n'(t) = \alpha_k'\left(z_n + t(y_n - z_n)\right)(y_n - z_n), \quad t \in [0,1],$$

$$\psi_n'(t) = \frac{1}{2}\alpha_k'\left(z_n + t\frac{y_n - z_n}{2}\right)(y_n - z_n),$$

and, for sufficiently large n,

$$\varphi_n'(t) \neq 0, \quad t \in [0,1],$$

Hence, for sufficiently large n, by the Cauchy Mean–Value Theorem, there is $t_n \in (0,1)$ such that

$$\frac{\alpha_k\left(\frac{y_n+z_n}{2}\right) - \alpha_k(z_n)}{\alpha_k(y_n) - \alpha_k(z_n)} = \frac{\psi_n(1) - \psi_n(0)}{\varphi_n(1) - \varphi_n(0)} = \frac{\psi_n'(t_n)}{\varphi_n'(t_n)}$$

$$= \frac{1}{2}\frac{\alpha_k'\left(z_n + t_n\frac{y_n - z_n}{2}\right)(y_n - z_n)}{\alpha'(z_n + t_n(y_n - z_n))(y_n - z_n)} = \frac{1}{2}\frac{\alpha'\left(z_n + t_n\frac{y_n - z_n}{2}\right)\frac{y_n - z_n}{\|y_n - z_n\|}}{\alpha_k'(z_n + t_n(y_n - z_n))\frac{y_n - z_n}{\|y_n - z_n\|}}.$$

Letting $n \to \infty$ we hence get

$$\lim_{n\to\infty} \frac{\alpha_k\left(\frac{y_n+z_n}{2}\right) - \alpha_k(z_n)}{\alpha_k(y_n) - \alpha_k(z_n)} = \frac{1}{2}\frac{\alpha_k'(x)\,v}{\alpha_k'(x)v} = \frac{1}{2}. \tag{8}$$

The convexity of f with respect to α_k implies that

$$f(x_n) = f\left(\frac{y_n + z_n}{2}\right) \leq \frac{\alpha_k\left(\frac{y_n+z_n}{2}\right) - \alpha_k(z_n)}{\alpha_k(y_n) - \alpha_k(z_n)} f(y_n) + \frac{\alpha_k(y_n) - \alpha_k\left(\frac{y_n+z_n}{2}\right)}{\alpha_k(y_n) - \alpha_k(z_n)} f(z_n)$$

for all $n \in \mathbb{N}$. Since

$$\frac{\alpha_k\left(\frac{y_n+z_n}{2}\right) - \alpha_k(z_n)}{\alpha_k(y_n) - \alpha_k(z_n)} + \frac{\alpha_k(y_n) - \alpha_k\left(\frac{y_n+z_n}{2}\right)}{\alpha_k(y_n) - \alpha_k(z_n)} = 1, \quad n \in \mathbb{N}.$$

Letting $n \to \infty$ in the last inequality and making use of (8), we hence get

$$C = \lim_{n\to\infty} f(x_n) \leq \frac{1}{2}\lim_{n\to\infty} f(x_n) + \frac{1}{2}\lim_{n\to\infty} \sup f(z_n) = \frac{1}{2}c + \frac{1}{2}C,$$

whence $C \leq c$. This contradiction completes the proof. □

Remark 8. The measurability of the function f can be replaced by the existence of a Lebesgue measurable set $T \subset U$ of a positive measure and a Lebesgue measurable function $g : T \to \mathbb{R}$ such that $f(x) \leq g(x)$ for all $x \in T$ (cf. M. Kuczma [6], p. 218, Theorem 1 where the classical Jensen convex functions are considered).

4. Generalized Jensen affine functions

In this section we examine the Jensen affine functions with respect to a function α.

Theorem 6. *Let U be an open and convex set in a real normed space, and $\alpha : U \to \mathbb{R}$ be a locally non-constant continuous function such that for all $x, y \in U$ the function*

$$[0;1] \ni s \to \alpha(sx + (1-s)y)$$

is monotonic, and strictly monotonic in the case when $\alpha(x) \neq \alpha(y)$.

Suppose that $f : U \to \mathbb{R}$ is continuous at least at one point. Then f is Jensen affine with respect to α, that is f satisfies the functional equation (2)*:*

$$f\left(\frac{x+y}{2}\right) = \frac{\alpha\left(\frac{x+y}{2}\right) - \alpha(y)}{\alpha(x) - \alpha(y)} f(x) + \frac{\alpha(x) - \alpha\left(\frac{x+y}{2}\right)}{\alpha(x) - \alpha(y)} f(y)$$

for all $x, y \in U$ such that $\alpha(x) \neq \alpha(y)$ if, and only if, there are $b, c \in \mathbb{R}$ such that

$$f(x) = b\alpha(x) + c, \quad x \in U.$$

Proof. Suppose that f satisfies equation (2) and f is continuous at a point $z_0 \in U$. Making use of an idea presented in [7] we shall prove that f is continuous in U. Take arbitrary $x \in \frac{1}{2}(z_0 + U)$. Then there is a unique $y \in U$ such that

$$x = \frac{z_0 + y}{2}.$$

Assume first that $\alpha(y) \neq \alpha(z_0)$. For an arbitrary sequence $x_n \in U$, $n \in N$, such that $\lim_{n\to\infty} x_n = x$ put

$$z_n := 2x_n - y$$

and note that $\lim_{n\to\infty} z_n = z_0$. As U is open, $z_n \in U$, for sufficiently large $n \in \mathbb{N}$. Hence, by the continuity of f at z_0, and the continuity of α, we have

$$\begin{aligned}\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} f\left(\frac{z_n + y}{2}\right) \\ &= \lim_{n\to\infty} \left[\frac{\alpha\left(\frac{z_n+y}{2}\right) - \alpha(y)}{\alpha(z_n) - \alpha(y)} f(z_n) + \frac{\alpha(z_n) - \alpha\left(\frac{z_n+y}{2}\right)}{\alpha(z_n) - \alpha(y)} f(y)\right]\end{aligned}$$

$$= \frac{\alpha\left(\frac{z_0+y}{2}\right)-\alpha(y)}{\alpha(z_0)-\alpha(y)} f(z_0) + \frac{\alpha(z_0)-\alpha\left(\frac{z_0+y}{2}\right)}{\alpha(z_0)-\alpha(y)} f(y) = f\left(\frac{z_0+y}{2}\right) = f(x)$$

which proves that f is continuous at the point x. Thus we have shown that f is continuous at each point of the set

$$C := \frac{1}{2}(z_0+U) \setminus \{y \in U : \alpha(y) = \alpha(z_0)\}.$$

Now we consider the case when $\alpha(y) = \alpha(z_0)$. Since, by assumption, $\operatorname{int}\alpha^{-1}(\{\alpha(z_0)\}) = \emptyset$, by the previous step of the proof, the set of the continuity points of f is dense in $\frac{1}{2}(z_0+U)$. Take a ball $B(z_0,\varepsilon) \subset \frac{1}{2}(z_0+U)$. Since U is open and $y := 2x - z_0 \in U$, we can choose an $\varepsilon > 0$ such that $2x - B(z_0,\varepsilon) \subset U$. Note that there is a point $z \in B(z_0,\varepsilon)$ such that $\alpha(2x-z) \neq \alpha(z)$. Indeed, in the opposite case $\alpha(2x-z) = \alpha(z)$ for all $z \in U$, whence, by the monotonicity condition (3), the function $\gamma_z : [0;1] \to \mathbb{R}$,

$$\gamma_z(s) := \alpha(s(2x-z) + (1-s)z)$$

would be constant for every $z \in B(z_0,\varepsilon)$. Consequently, $\gamma_z(0) = \gamma_z(\frac{1}{2})$ for every $z \in B(z_0,\varepsilon)$ that is $\alpha(z) = \alpha(x)$ for every $z \in B(z_0,\varepsilon)$, which contradicts to the assumption that $\operatorname{int}\alpha^{-1}(\{\alpha(z_0)\}) = \emptyset$. Thus we have shown that there is a point $z \in B(z_0,\varepsilon)$ such that $\alpha(2x-z) \neq \alpha(z)$. Hence, taking into account the continuity of α and the density of the set C in $B(z_0,\varepsilon)$, we infer that there is a point $z_1 \in B(z_0,\varepsilon)$ such that f is continuous at z_1 and $\alpha(2x-z_1) \neq \alpha(z_1)$. Setting $y := 2x - z_1$ we have $\alpha(y) \neq \alpha(z_1)$ and

$$x = \frac{z_1+y}{2}.$$

Now, repeating the argument of the first step of the proof with z_0 replaced by z_1, we show that f is continuous at the point x.

Thus we have proved that f is continuous in the set $\frac{1}{2}(z_0+U)$. Replacing here z_0 by an arbitrary point of the set $\frac{1}{2}(z_0+U)$ we infer that f is continuous in the set

$$\frac{1}{2}\left(\frac{1}{2}(z_0+U)+U\right) = \frac{z_0}{2^2} + \frac{1}{2^2}U + \frac{1}{2}U,$$

and, by inductive argument, for every positive integer n, the function f is continuous in the set

$$\frac{z_0}{2^n} + \frac{1}{2^n}U + \frac{1}{2^{n-1}}U + \cdots + \frac{1}{2}U,$$

which by the convexity of U, coincides with the set

$$\frac{z_0}{2^n}+\left(\frac{1}{2^n}+\frac{1}{2^{n-1}}+\cdots+\frac{1}{2}\right)U=\frac{z_0}{2^n}+\left(1-\frac{1}{2^n}\right)U.$$

Since

$$U=\bigcup_{n=1}^{\infty}\left[\frac{z_0}{2^n}+\left(1-\frac{1}{2^n}\right)U\right],$$

we conclude that f is continuous in U.

Take arbitrary $x,y\in U$ such that $\alpha(x)\neq\alpha(y)$. There are unique $b(x,y),c(x,y)\in\mathbb{R}$ such that

$$f(x)=b(x,y)\alpha(x)+c(x,y),\qquad f(y)=b(x,y)\alpha(y)+c(x,y).$$

Put

$$A:=\{t\in[0,1]:f(tx+(1-t)y)=b(x,y)\alpha(tx+(1-t)y)+c(x,y)\}.$$

Since $0,1\in A$, the set A is nonempty. The continuity of α and f implies that the set A is closed. We shall show that $A=[0;1]$. For an indirect argument assume that the set $[0,1]\backslash A$ is nonempty. Since $[0,1]\backslash A$ is open in $\mathbb{R}$, it is at most countable union of nonempty pairwise disjoint open intervals. Let $I=(r,s)\subset[0,1]\backslash A$, $r<s$, be one of these intervals. Then $r,s\in A$ and

$$\frac{r+s}{2}\notin A.$$

On the other hand, as $r,s\in A$, we have

$$f(rx+(1-r)y)=b\alpha(rx+(1-r)y)+c,$$

$$f(sx+(1-s)y)=b\alpha(sx+(1-s)y)+c,$$

where $b=b(x,y)$, $c=c(x,y)$. Hence, making use of (5), we have

$$\begin{aligned}
f\left(\frac{r+s}{2}x+\left(1-\frac{r+s}{2}\right)y\right)&=f\left(\frac{[rx+(1-r)y]+[sx+(1-s)y]}{2}\right)\\
&=\frac{\alpha\left(\frac{r+s}{2}x+\left(1-\frac{r+s}{2}\right)y\right)-\alpha(sx+(1-s)y)}{\alpha(rx+(1-r)y)-\alpha(sx+(1-s)y)}f(rx+(1-r)y)\\
&\quad+\frac{\alpha(rx+(1-r)y)-\alpha\left(\frac{r+s}{2}x+\left(1-\frac{r+s}{2}\right)y\right)}{\alpha(rx+(1-r)y)-\alpha(sx+(1-s)y)}f(sx+(1-s)y)\\
&=\frac{\alpha\left(\frac{r+s}{2}x+\left(1-\frac{r+s}{2}\right)y\right)-\alpha(sx+(1-s)y)}{\alpha(rx+(1-r)y)-\alpha(sx+(1-s)y)}[b\alpha(rx+(1-r)y)+c]\\
&\quad+\frac{\alpha(rx+(1-r)y)-\alpha\left(\frac{r+s}{2}x+\left(1-\frac{r+s}{2}\right)y\right)}{\alpha(rx+(1-r)y)-\alpha(sx+(1-s)y)}[b\alpha(sx+(1-s)y)+c]
\end{aligned}$$

$$= b\alpha\left(\frac{r+s}{2}x + \left(1 - \frac{r+s}{2}\right)y\right) + c,$$

which shows that $\frac{r+s}{2} \in A$. This contradiction proves that $A = [0,1]$.

Thus we have shown that for all $x, y \in U$ such that $\alpha(x) \neq \alpha(y)$ there are uniquely determined $b(x,y)$, $c(x,y)$ such that for all $t \in [0;1]$,

$$f(tx + (1-t)y) = b(x,y)\alpha(tx + (1-t)y) + c(x,y).$$

Replacing here x by $rx + (1-r)y$ and y by $sx + (1-s)y$, and taking into account the continuity of α and f, we obtain

$$\begin{aligned} &f([tr + (1-t)s]x + (1 - [tr + (1-t)s])y)) \\ &\quad = f(t[rx + (1-r)y] + (1-t)[sx + (1-s)y]) \\ &\quad = b(rx + (1-r)y, sx + (1-s)y)\alpha([tr + (1-t)s]x + (1 - [tr + (1-t)s])y)) \\ &\qquad + c(rx + (1-r)y, sx + (1-s)y), \end{aligned}$$

for all $x, y \in U$ such that $\alpha(x) \neq \alpha(y)$ and for all $r, s, t \in [0;1]$. On the other hand, according to what has been already shown, we have

$$\begin{aligned} &f([tr + (1-t)s]x + (1 - [tr + (1-t)s])y)) \\ &\qquad = b(x,y)\alpha([tr + (1-t)s]x + (1 - [tr + (1-t)s])y)) + c(x,y) \end{aligned}$$

for all $x, y \in U$ such that $\alpha(x) \neq \alpha(y)$ and for all $r, s, t \in [0;1]$. Since the numbers b and c are uniquely determined, it follows that

$$\begin{aligned} b(rx + (1-r)y, sx + (1-s)y) &= b(x,y) \\ \text{and} \quad c(rx + (1-r)y, sx + (1-s)y) &= c(x,y) \end{aligned}$$

for all $x, y \in U$ such that $\alpha(x) \neq \alpha(y)$ and for all $r, s, t \in [0;1]$. These relations imply that b and c do not depend on x and y. Consequently, there are $b, c \in \mathbb{R}$ such that for all $x, y \in U$ such that $\alpha(x) \neq \alpha(y)$ we have

$$f(tx + (1-t)y) = b\alpha(tx + (1-t)y) + c, \qquad t \in [0;1]. \tag{9}$$

If f is constant in U then, of course, $b = 0$ and $f(z) = c$ for all $z \in U$. In the opposite case, for an arbitrary fixed $z \in U$ there is an $x \in U$ such that $\alpha(x) \neq \alpha(z)$. Since U is open, for all $s \in (0;1)$ close enough to 1 we have

$$y := \frac{1}{s}z + \left(1 - \frac{1}{s}\right)x \in U.$$

The continuity of f allows to choose an s such that $\alpha(y) \neq \alpha(x)$. Since

$$z = sy + (1-s)y$$

in view of (9) we have

$$f(z) = f(sy + (1-s)y) = b\alpha(sx + (1-s)y) + c = b\alpha(z) + c,$$

which completes the "only if" part of the proof. The remaining part of the proof is easy to verify. □

From Theorems 5 and 6 we obtain the following

Proposition 2. *Let $U \subset \mathbb{R}^m$ be open and convex and let $A : U \to \mathbb{R}^m$, $A = (\alpha_1, \dots, \alpha_m)$ be a local C^1-diffeomorphism in U such that, in a neighbourhood of every point of U, at least one of the coordinate functions α_j satisfies conditions* (6) *and* (7). *Suppose that a function $f : U \to \mathbb{R}$ is Jensen convex with respect to α_j for each $j \in \{1, \dots, m\}$, and Lebesgue measurable or bounded above in a neighbourhood of a point. If f is Jensen affine with respect to a non-constant continuous function $\alpha : U \to \mathbb{R}$ such that for all $x, y \in U$ the function*

$$[0;1] \ni s \to \alpha(sx + (1-s)y)$$

is monotonic and strictly monotonic in the case when $\alpha(x) \neq \alpha(y)$, then there are $b, c \in \mathbb{R}$ such that

$$f(x) = b\alpha(x) + c, \quad x \in U.$$

5. Existence of discontinuous affine functions and a related functional equation

The following result plays a crucial role in this paper.

Theorem 7. *Let U be an open and convex set in a real normed space X. Suppose that $\alpha : U \to \mathbb{R}$ is a non-constant continuous function such that for all $x, y \in U$ the function*

$$[0;1] \ni s \to \alpha(sx + (1-s)y)$$

is monotonic, and strictly monotonic in the case when $\alpha(x) \neq \alpha(y)$.

If there exists a function $f : U \to \mathbb{R}$ that is Jensen affine with respect to α and discontinuous at least at one point, then α satisfies the functional equation

$$\frac{\alpha\left(\frac{tx+y+(1-t)z}{2}\right) - \alpha(y)}{\alpha(tx + (1-t)z) - \alpha(y)} \frac{\alpha\left(tx + (1-t)z\right) - \alpha(z)}{\alpha(x) - \alpha(z)}$$
$$= \frac{\alpha\left(\frac{tx+y+(1-t)z}{2}\right) - \alpha(ty + (1-t)z)}{\alpha(tx + (1-t)y) - \alpha(ty + (1-t)z)} \frac{\alpha\left(tx + (1-t)y\right) - \alpha(y)}{\alpha(x) - \alpha(y)}, \quad (10)$$

for all $t \in (0,1)$, *and all* $x, y, z \in I$, $x \neq y \neq z \neq x$.

PROOF. Suppose that $f : U \to \mathbb{R}$ is Jensen affine with respect to α and f is discontinuous at a point. In view of Theorem 6, the function f must be discontinuous at every point. Since

$$\frac{[tx+(1-t)z]+y}{2} = \frac{tx+y+(1-t)z}{2} = \frac{[tx+(1-t)y]+[ty+(1-t)z]}{2},$$

applying Lemma 1.2 twice, we get

$$\begin{aligned}
f\left(\frac{tx+y+(1-t)z}{2}\right) &= \frac{\alpha\left(\frac{tx+y+(1-t)z}{2}\right)-\alpha(y)}{\alpha(tx+(1-t)z)-\alpha(y)} f(tx+(1-t)z) \\
&\quad + \frac{\alpha(tx+(1-t)z)-\alpha\left(\frac{tx+y+(1-t)z}{2}\right)}{\alpha(tx+(1-t)z)-\alpha(y)} f(y) \\
&= \frac{\alpha\left(\frac{tx+y+(1-t)z}{2}\right)-\alpha(y)}{\alpha(tx+(1-t)z)-\alpha(y)} \frac{\alpha\left(tx+(1-t)z\right)-\alpha(z)}{\alpha(x)-\alpha(z)} f(x) \\
&\quad + \frac{\alpha\left(\frac{tx+y+(1-t)z}{2}\right)-\alpha(y)}{\alpha(tx+(1-t)z)-\alpha(y)} \frac{\alpha(z)-\alpha\left(tx+(1-t)z\right)}{\alpha(x)-\alpha(z)} f(z) \\
&\quad + \frac{\alpha(tx+(1-t)z)-\alpha\left(\frac{tx+y+(1-t)z}{2}\right)}{\alpha(tx+(1-t)z)-\alpha(z)} f(y)
\end{aligned}$$

and, similarly,

$$\begin{aligned}
f\left(\frac{tx+y+(1-t)z}{2}\right) &= f\left(\frac{[tx+(1-t)y]+[ty+(1-z)z}{2}\right) \\
&= \frac{\alpha\left(\frac{tx+y+(1-t)z}{2}\right)-\alpha(ty+(1-t)z)}{\alpha(tx+(1-t)y)-\alpha(ty+(1-t)z)} \frac{\alpha\left(tx+(1-t)y\right)-\alpha(y)}{\alpha(x)-\alpha(y)} f(x) \\
&\quad + \frac{\alpha\left(\frac{tx+y+(1-t)z}{2}\right)-\alpha(ty+(1-t)z)}{\alpha(tx+(1-t)y)-\alpha(ty+(1-t)z)} \frac{\alpha(x)-\alpha\left(tx+(1-t)y\right)}{\alpha(x)-\alpha(y)} f(y) \\
&\quad + \frac{\alpha(tx+(1-t)y)-\alpha\left(\frac{tx+y+(1-t)z}{2}\right)}{\alpha(tx+(1-t)y)-\alpha(ty+(1-t)z)} \frac{\alpha\left(ty+(1-t)z\right)-\alpha(y)}{\alpha(y)-\alpha(z)} f(y) \\
&\quad + \frac{\alpha(tx+(1-t)y)-\alpha\left(\frac{tx+y+(1-t)z}{2}\right)}{\alpha(tx+(1-t)y)-\alpha(ty+(1-t)z)} \frac{\alpha(y)-\alpha\left(ty+(1-t)z\right)}{\alpha(y)-\alpha(z)} f(z)
\end{aligned}$$

for all $t \in T$ and for all $x, y, z \in U$ such that $x \neq y \neq z \neq x$.

Subtracting these equations by sides we obtain

$$F_\alpha(t,x,y,z)f(x) + G_\alpha(t,x,y,z)f(y) + H_\alpha(t,x,y,z)f(z) = 0$$

for all $t \in T$, and all $x, y, z \in U$, $x \neq y \neq z \neq x$, where

$$\begin{aligned}
F_\alpha(t,x,y,z) &:= \frac{\alpha\left(\frac{tx+y+(1-t)z}{2}\right) - \alpha(y)}{\alpha(tx+(1-t)z) - \alpha(y)} \frac{\alpha\left(tx+(1-t)z\right) - \alpha(z)}{\alpha(x) - \alpha(z)} \\
&\quad - \frac{\alpha\left(\frac{tx+y+(1-t)z}{2}\right) - \alpha(ty+(1-t)z)}{\alpha(tx+(1-t)y) - \alpha(ty+(1-t)z)} \frac{\alpha\left(tx+(1-t)y\right) - \alpha(y)}{\alpha(x) - \alpha(y)}, \\
G_\alpha(t,x,y,z) &:= \frac{\alpha(tx+(1-t)z) - \alpha\left(\frac{tx+y+(1-t)z}{2}\right)}{\alpha(tx+(1-t)z) - \alpha(z)} \\
&\quad - \frac{\alpha\left(\frac{tx+y+(1-t)z}{2}\right) - \alpha(ty+(1-t)z)}{\alpha(tx+(1-t)y) - \alpha(ty+(1-t)z)} \frac{\alpha(x) - \alpha\left(tx+(1-t)y\right)}{\alpha(x) - \alpha(y)} \\
&\quad - \frac{\alpha(tx+(1-t)y) - \alpha\left(\frac{tx+y+(1-t)z}{2}\right)}{\alpha(tx+(1-t)y) - \alpha(ty+(1-t)z)} \frac{\alpha\left(ty+(1-t)z\right) - \alpha(y)}{\alpha(y) - \alpha(z)}, \\
H_\alpha(t,x,y,z) &:= \frac{\alpha\left(\frac{tx+y+(1-t)z}{2}\right) - \alpha(y)}{\alpha(tx+(1-t)z) - \alpha(y)} \frac{\alpha(z) - \alpha\left(tx+(1-t)z\right)}{\alpha(x) - \alpha(z)} \\
&\quad - \frac{\alpha(tx+(1-t)y) - \alpha\left(\frac{tx+y+(1-t)z}{2}\right)}{\alpha(tx+(1-t)y) - \alpha(ty+(1-t)z)} \frac{\alpha(y) - \alpha\left(ty+(1-t)z\right)}{\alpha(y) - \alpha(z)}.
\end{aligned}$$

If, for some $t_0 \in T$ and $x_0, y_0, z_0 \in U$,

$$\left[F_\alpha(t_0,x_0,y_0,z_0)\right]^2 + \left[G_\alpha(t_0,x_0,y_0,z_0)\right]^2 + \left[H_\alpha(t_0,x_0,y_0,z_0)\right]^2 \neq 0,$$

then f would be continuous in a neighbourhood of one of the points x_0, y_0, z_0. Indeed, if, for instance, $F_\alpha(\ t_0,x_0,y_0,z_0) \neq 0$, then there would exist a neighbourhood V of the point x_0 such that

$$f(x) = -\frac{G_\alpha(t_0,x,y_0,z_0)}{F_\alpha(t_0,x,y_0,z_0)} f(y_0) - \frac{H_\alpha(t_0,x,y_0,z_0)}{F_\alpha(t_0,x,y_0,z_0)} f(z_0), \quad x \in V,$$

contrary to the assumption. This proves that

$$F_\alpha(t,x,y,z) = 0$$

for all $t \in T$ and all $x, y, z \in U$, $x \neq y \neq z \neq x$. The continuity of F_α and the density of T imply that this equation holds true for all $t \in (0,1)$ and all $x, y, z \in U$, $x \neq y \neq z \neq x$. The proof is completed. □

Remark 9. Under the assumptions of the above theorem, in the same way we can show that simultaneously

$$F_\alpha(t,x,y,z)=0,\quad G_\alpha(t,x,y,z)=0,\quad H_\alpha(t,x,y,z)=0$$

for all $t\in(0,1)$ and all $x,y,z\in U$, $x\neq y\neq z\neq x$. However each two of these equations are equivalent.

Taking $t=\frac{1}{2}$ in the above result we obtain the following:

Corollary 1. *Let U be an open and convex set in a real normed space, and let $\alpha:U\to\mathbb{R}$ be continuous and non-constant. Suppose that $f:U\to\mathbb{R}$ satisfies equation (2):*

$$f\left(\frac{x+y}{2}\right)=\frac{\alpha\left(\frac{x+y}{2}\right)-\alpha(y)}{\alpha(x)-\alpha(y)}f(x)+\frac{\alpha(x)-\alpha\left(\frac{x+y}{2}\right)}{\alpha(x)-\alpha(y)}f(y),$$

for all $x,y\in U$ such that $\alpha(x)\neq\alpha(y)$.

If there exist $x,y,z\in U$ such that

$$\frac{\alpha\left(\frac{x+2y+z}{4}\right)-\alpha\left(y\right)}{\alpha\left(\frac{x+z}{2}\right)-\alpha\left(y\right)}\frac{\alpha\left(\frac{x+z}{2}\right)-\alpha\left(z\right)}{\alpha\left(x\right)-\alpha\left(z\right)}\neq\frac{\alpha\left(\frac{x+2y+z}{4}\right)-\alpha\left(\frac{y+z}{2}\right)}{\alpha\left(\frac{x+y}{2}\right)-\alpha\left(\frac{y+z}{2}\right)}\frac{\alpha\left(\frac{x+y}{2}\right)-\alpha\left(y\right)}{\alpha\left(x\right)-\alpha\left(y\right)}$$

then f is continuous in a neighbourhood of $x\in U$.

Corollary 2. *Let $U\subset\mathbb{R}$ be an open interval and $\alpha:U\to\mathbb{R}$ be continuous and strictly monotonic. If $f:U\to\mathbb{R}$ is discontinuous at every point and satisfies equation (2) then α satisfies the functional equation*

$$\frac{\alpha\left(\frac{x+2y+z}{4}\right)-\alpha\left(y\right)}{\alpha\left(\frac{x+z}{2}\right)-\alpha\left(y\right)}\frac{\alpha\left(\frac{x+z}{2}\right)-\alpha\left(z\right)}{\alpha\left(x\right)-\alpha\left(z\right)}=-\frac{\alpha\left(\frac{x+2y+z}{4}\right)-\alpha\left(\frac{y+z}{2}\right)}{\alpha\left(\frac{x+y}{2}\right)-\alpha\left(\frac{y+z}{2}\right)}\frac{\alpha\left(\frac{x+y}{2}\right)-\alpha\left(y\right)}{\alpha\left(x\right)-\alpha\left(y\right)}$$

for all $x,y,z\in U$, $y\neq x\neq z$, $x+z\neq 2y$.

If moreover α is differentiable and $\alpha'(x)\neq 0$ for all $x\in U$, then

$$\frac{1}{2}\frac{\alpha\left(\frac{x+y}{2}\right)-\alpha\left(y\right)}{\alpha\left(x\right)-\alpha\left(y\right)}=\frac{\alpha\left(\frac{x+y}{2}\right)-\alpha\left(\frac{x+3y}{4}\right)}{\alpha\left(\frac{3x+y}{4}\right)-\alpha\left(\frac{x+3y}{4}\right)}\frac{\alpha\left(\frac{3x+y}{4}\right)-\alpha\left(\frac{x+y}{2}\right)}{\alpha\left(x\right)-\alpha\left(\frac{x+y}{2}\right)}$$

for all $x,y\in U$, $x\neq y$.

Proof. The first part is an immediate consequence of the previous corollary. Letting $y\to\frac{x+z}{2}$ and then replacing z by y we obtain the second part. □

6. Irregular generalized Jensen affine functions in one-dimensional case and homographic functions

An important result of this paper reads as follows:

Theorem 8. *Let $I \subset \mathbb{R}$ be an open interval and $\alpha : I \to \mathbb{R}$ be continuous and strictly monotonic.*

(1) *There exists a discontinuous at least at one point function $f : I \to \mathbb{R}$ and Jensen affine with respect to α, i.e. such that for all $x, y \in I, x \neq y$,*

$$f\left(\frac{x+y}{2}\right) = \frac{\alpha\left(\frac{x+y}{2}\right) - \alpha(y)}{\alpha(x) - \alpha(y)} f(x) + \frac{\alpha(x) - \alpha\left(\frac{x+y}{2}\right)}{\alpha(x) - \alpha(y)} f(y)$$

if, and only if, there are $p, q, r, s \in \mathbb{R}$, $ps \neq rq$, such that

$$\alpha(x) = \frac{px+q}{rx+s}, \quad x \in I,$$

i.e., α is a homographic function.

(2) *If, for some $p, q, r, s \in \mathbb{R}$ such that $ps - rq \neq 0$,*

$$\alpha(x) = \frac{px+q}{rx+s}, \quad x \in I,$$

then the above generalized Jensen equation becomes

$$f\left(\frac{x+y}{2}\right) = \frac{rx+s}{r(x+y)+2s} f(x) + \frac{ry+s}{r(x+y)+2s} f(y), \quad x, y \in I, x \neq y, \tag{11}$$

and its general solution is of the form

$$f(x) = \frac{A(x)+b}{rx+s}, \quad x \in I,$$

where $A : \mathbb{R} \to \mathbb{R}$ is an arbitrary additive function and $b \in \mathbb{R}$.

(3) *If α is not homographic, then every Jensen-affine function with respect to α is of the form*

$$f(x) = b\alpha(x) + c, \quad x \in I,$$

for some $b, c \in \mathbb{R}$.

Proof. Suppose that $f : I \to \mathbb{R}$ is Jensen affine with respect to α and f is discontinuous at a point. In view of Theorem 7 the function α must satisfy equation (10) (with $U = I$). Let $z \in I$ be a differentiability point of α. (In view

of the Lebesque Theorem, the monotonicity of α implies its differentiability a.e. in I.) Dividing both sides of equation (10) by t we obtain

$$\frac{\alpha\left(\frac{tx+y+(1-t)z}{2}\right)-\alpha(y)}{\alpha(tx+(1-t)z)-\alpha(y)}\,\frac{\alpha(z+t(x-z)-\alpha(z)}{t(x-z)}\,\frac{x-z}{\alpha(x)-\alpha(z)}$$
$$=\frac{\alpha\left(\frac{tx+y+(1-t)z}{2}\right)-\alpha(ty+(1-t)z)}{\alpha(tx+(1-t)y)-\alpha(ty+(1-t)z)}\,\frac{\alpha(y+t(x-y))-\alpha(y)}{t(x-y)}\,\frac{(x-y)}{\alpha(x)-\alpha(y)}$$

for all $y,z\in I$, $y\neq z$. Since

$$\lim_{t\to 0}\frac{\alpha\left(tx+(1-t)z\right)-\alpha(z)}{t(x-z)}=\alpha'(z),$$

letting $t\to 0$ and taking into account the continuity of α, we infer that α is one-sided differentiable at y and, if $x-y>0$, we get

$$\frac{\alpha\left(\frac{y+z}{2}\right)-\alpha(y)}{\alpha(z)-\alpha(y)}\,\frac{\alpha'(z)(x-z)}{\alpha(x)-\alpha(z)}=\frac{\alpha\left(\frac{y+z}{2}\right)-\alpha(z)}{\alpha(y)-\alpha(z)}\,\frac{\alpha'_+\left(y\right)(x-y)}{\alpha(x)-\alpha(y)}$$

that is

$$\left(\alpha\left(\frac{y+z}{2}\right)-\alpha(y)\right)\frac{\alpha'(z)(x-z)}{\alpha(x)-\alpha(z)}=-\left(\alpha\left(\frac{y+z}{2}\right)-\alpha(z)\right)\frac{\alpha'_+\left(y\right)(x-y)}{\alpha(x)-\alpha(y)},\quad(12)$$

and, similarly, if $x-y<0$, we get

$$\left(\alpha\left(\frac{y+z}{2}\right)-\alpha(y)\right)\frac{\alpha'(z)(x-z)}{\alpha(x)-\alpha(z)}=-\left(\alpha\left(\frac{y+z}{2}\right)-\alpha(z)\right)\frac{\alpha'_-\left(y\right)(x-y)}{\alpha(x)-\alpha(y)}\quad(13)$$

where $\alpha'_+(y)$, $\alpha'_-(y)$ denote, respectively, the right and the left derivative of α at y. The relations (12) and (13) imply that α'_+ and α'_- are continuous. It follows that α is differentiable and α' is continuous in I. If $\alpha'(z)=0$ for some $z\in I$ then, by (11) we would have $\alpha'(y)=0$ for all $y\in I$. This cannot happen as, by assumption, α is strictly monotonic. Thus

$$\alpha'(x)\neq 0,\quad x\in I.$$

Letting $z\to x$ in (12) and making use of the just proved differentiability of α, we infer that

$$\alpha\left(\frac{x+y}{2}\right)-\alpha(y)=-\left(\alpha\left(\frac{x+y}{2}\right)-\alpha(x)\right)\frac{\alpha'\left(y\right)(x-y)}{\alpha(x)-\alpha(y)},$$

whence

$$\alpha'(y) = \frac{\alpha(y)-\alpha(x)}{x-y}\frac{\alpha\left(\frac{x+y}{2}\right)-\alpha(y)}{\alpha\left(\frac{x+y}{2}\right)-\alpha(x)}$$

for all $x, y \in I$ such that $\alpha(x) \neq \alpha(y)$. Interchanging here x and y we hence get

$$\alpha'(x) = \frac{\alpha(x)-\alpha(y)}{y-x}\frac{\alpha\left(\frac{x+y}{2}\right)-\alpha(x)}{\alpha\left(\frac{x+y}{2}\right)-\alpha(y)}$$

for all $x, y \in I$ such that $x \neq y$. Multiplying the respective sides in the last two equations we obtain

$$\alpha'(x)\,\alpha'(y) = \left(\frac{\alpha(x)-\alpha(y)}{x-y}\right)^2, \quad x, y \in I,\ x \neq y.$$

Writing this equation in the form

$$\frac{\alpha'(x)}{(\alpha(x)-\alpha(y))^2} = \frac{1}{\alpha'(y)}\frac{1}{(x-y)^2}$$

and integrating with respect to x we obtain

$$\frac{1}{\alpha(x)-\alpha(y)} = \frac{1}{\alpha'(y)}\frac{1}{x-y} + c(y)$$

for some $c(y)$ and for all $x, y \in I,\ x \neq y$, whence

$$\alpha(x) = \frac{p(y)x+q(y)}{r(y)x+s(y)}, \quad x \in J,$$

for some $p(y), q(y), r(y), s(y) \in \mathbb{R}$. Since the right-hand side does not depend on y, we infer that

$$\alpha(x) = \frac{px+q}{rx+s}, \quad x \in I,$$

for some $p, q, r, s \in \mathbb{R}$ such that $ps - rq \neq 0$.

Setting this function into equation (2) we obtain

$$2\left(r\frac{x+y}{2}+s\right)f\left(\frac{x+y}{2}\right) = (rx+s)f(x)+(ry+s)f(y), \quad x, y \in I, x \neq y,$$

It follows that the function $g : I \to \mathbb{R}$,

$$g(x) := (rx+s)f(x), \quad x \in I,$$

satisfies the classical Jensen equation

$$g\left(\frac{x+y}{2}\right) = \frac{g(x)+g(y)}{2}, \quad x \in I.$$

Consequently, (cf. for instance M. Kuczma [6], p. 315), there are an additive function $A : \mathbb{R} \to \mathbb{R}$ and a constant $b \in \mathbb{R}$ such that $g = A + b$. The proof is complete. □

Remark 10. Note that for $r = 0$ equation (11) reduces to the classical Jensen functional equation.

Remark 11. Let $I \subset \mathbb{R}$ be an interval and $\alpha_1, \alpha_2 : I \to \mathbb{R}$ some nonlinear homographic functions, i.e.

$$\alpha_1(x) = \frac{px+q}{rx+s}, \qquad \alpha_2(x) = \frac{Px+O}{Rx+S}, \quad x \in I,$$

for some $p, q, r.s, P, Q, R, S \in \mathbb{R}$, $ps - qr \neq 0 \neq PS - QR$. If $f : I \to \mathbb{R}$ is Jensen-affine with respect to α_1 and α_2 then, in view of the above theorem,

$$f(x) = \frac{a(x)+b}{rx+s} \quad \text{and} \quad f(x) = \frac{A(x)+B}{Rx+S} \quad \text{for all} \quad x \in I,$$

for some additive functions $a, A : \mathbb{R} \to \mathbb{R}$ and $b, B \in \mathbb{R}$, whence

$$A(x) = (Rx+S)\,\frac{a(x)+b}{rx+s} - B, \quad x \in I.$$

Making use of the rational homogeneity of additive functions, we hence get

$$Rs - rS = Sb - sB = 0, \quad rA = Ra$$

It follows that, in general, the class of all Jensen affine functions with respect to α_1 is different than the class of all Jensen affine functions with respect to α_2.

7. An application to more general Beckenbach affine functions and a solution of a problem of Páles

We begin this section with recalling a special case of a more general definition of the convexity (concavity, affinity) with respect to a two-parameter family of functions of E. F. Beckenbach [2] (cf. also M. Bessenyei and Zs. Páles [3]).

Definition 2. Let $I \subset \mathbb{R}$ be an interval $f : I \to \mathbb{R}$ and $t \in (0,1)$. Given the continuous and strictly monotonic functions $\varphi, \psi : I \to \mathbb{R}$ and $\gamma : \mathbb{R} \to \mathbb{R}$ put $\mathcal{F} := \{\gamma \circ (b\varphi + c\psi : b, c \in \mathbb{R}\}$. Suppose that for all $x, y \in I$, $x \neq y$, there exist unique real numbers $b = b(x,y)$, $c = c(x,y)$ such that

$$\gamma\left(b(x,y)\varphi(x) + c(x,y)\psi(x)\right) = f(x), \ \gamma\left(b(x,y)\varphi(y) + c(x,y)\psi(y)\right) = f(y). \tag{14}$$

The function f is said to be:

(1) t-affine with respect to family $\mathcal{F}$ if, for all $x, y \in I$, $x \neq y$,

$$f(tx + (1-t)y) = \gamma\left(b(x,y)\varphi(tx + (1-t)y) + c(x,y)\psi(tx + (1-t)y)\right),$$

(2) t-convex with respect to family $\mathcal{F}$ if, for all $x, y \in I$, $x \neq y$,

$$f(tx + (1-t)y) \leq \gamma\left(b(x,y)\varphi(tx + (1-t)y) + c(x,y)\psi(tx + (1-t)y)\right),$$

(3) t-concave with respect to family $\mathcal{F}$ if, for all $x, y \in I$, $x \neq y$,

$$f(tx + (1-t)y) \geq \gamma\left(b(x,y)\varphi(tx + (1-t)y) + c(x,y)\psi(tx + (1-t)y)\right).$$

For $t = \frac{1}{2}$ the function f is called Jensen affine (convex, concave) with respect to $\mathcal{F}$.

The function f is called affine (convex, concave) with respect to family $\mathcal{F}$, if the respective condition holds true for all $t \in (0,1)$.

Since (14) can be written in the form

$$b(x,y)\varphi(x) + c(x,y)\psi(x) = \gamma^{-1}(f(x)),\ b(x,y)\varphi(y) + c(x,y)\psi(y) = \gamma^{-1}(f(y)), \quad (15)$$

the condition of the above definition is satisfied if the functions $\varphi, \psi : I \to \mathbb{R}$ form a Tchebycheff system, i.e. if

$$\begin{vmatrix} \varphi(x) & \psi(x) \\ \varphi(y) & \psi(y) \end{vmatrix} \neq 0, \quad x, y \in I,\ x \neq y.$$

Applying Theorem 8, the main result of the previous section, we prove the following:

Theorem 9. *Let $I \subset \mathbb{R}$ be an interval, $\gamma : \mathbb{R} \to \mathbb{R}$ be a continuous and strictly monotonic function and let $t \in (0,1)$. Suppose that the continuous functions $\varphi, \psi : I \to \mathbb{R}$ form a Tchebycheff system.*

(1) *If a function $f : I \to \mathbb{R}$ is discontinuous at least at one point, and t-affine with respect to the family $\mathcal{F} := \{\gamma \circ (b\varphi + c\psi : b, c \in \mathbb{R}\}$, then there are $p, q, r, s \in \mathbb{R}$, $ps \neq qr$, an additive function $A : \mathbb{R} \to \mathbb{R}$ and $b \in \mathbb{R}$ such that*

$$\varphi(x) = \frac{px + q}{rx + s}\psi(x), \quad x \in I, \quad (16)$$

and

$$f(x) = \gamma\left(\frac{A(x) + b}{rx + s}\psi(x)\right), \quad x \in I. \quad (17)$$

(2) *If* (16) *holds true then* f *is Jensen with respect to the family* $\mathcal{F}$ *iff* f *is of the form* (17).

(3) *If* (16) *does not hold (i.e. if* $\frac{\varphi}{\psi}$ *is not a homographic function) then, without any regularity assumption, every function* $f : I \to \mathbb{R}$ *which is* t*-affine with respect to the family* $\mathcal{F}$ *belongs to the family* $\mathcal{F}$*, i.e. there are* $b, c \in \mathbb{R}$ *such that*

$$f(x) = \gamma \circ (b\varphi(x) + c\psi(x)), \quad x \in I.$$

PROOF. Since φ and ψ form a Tchebycheff system, from (15) we have

$$b(x,y) = \frac{\begin{vmatrix} \gamma^{-1}(f(x)) & \psi(x) \\ \gamma^{-1}(f(y)) & \psi(y) \end{vmatrix}}{\begin{vmatrix} \varphi(x) & \psi(x) \\ \varphi(y) & \psi(y) \end{vmatrix}}, \quad c(x,y) = \frac{\begin{vmatrix} \varphi(x) & \gamma^{-1}(f(x)) \\ \varphi(y) & \gamma^{-1}(f(y)) \end{vmatrix}}{\begin{vmatrix} \varphi(x) & \psi(x) \\ \varphi(y) & \psi(y) \end{vmatrix}}, \quad x, y \in I,\ x \neq y.$$

According to the above definition, f is t-affine with respect to $\mathcal{F}$ if

$$\begin{aligned} &\left(\gamma^{-1} \circ f\right)(tx + (1-t)y) \\ &= \frac{\begin{vmatrix} \gamma^{-1}(f(x)) & \psi(x) \\ \gamma^{-1}(f(y)) & \psi(y) \end{vmatrix}}{\begin{vmatrix} \varphi(x) & \psi(x) \\ \varphi(y) & \psi(y) \end{vmatrix}} \varphi(tx + (1-t)y) + \frac{\begin{vmatrix} \varphi(x) & \gamma^{-1}(f(x)) \\ \varphi(y) & \gamma^{-1}(f(y)) \end{vmatrix}}{\begin{vmatrix} \varphi(x) & \psi(x) \\ \varphi(y) & \psi(y) \end{vmatrix}} \psi(tx + (1-t)y) \end{aligned}$$

for all $x, y \in I$, $x \neq y$, which can be written in the form

$$\begin{vmatrix} \gamma^{-1}(f(x)) & \gamma^{-1}(f(tx + (1-t)y)) & \gamma^{-1}(f(y)) \\ \varphi(x) & \varphi(tx + (1-t)y) & \varphi(y) \\ \psi(x) & \psi(tx + (1-t)y) & \psi(y) \end{vmatrix} = 0, \quad x, y \in I, x \neq y.$$

Since $\varphi, \psi : I \to \mathbb{R}$ form a Tchebycheff system, we have either $\varphi(x) \neq 0$ for all $x \in I$ or $\psi(x) \neq 0$ for all $x \in I$. Suppose, for instance that the second case holds true. Then we can write this equation in the form

$$\begin{aligned} &\frac{\gamma^{-1} \circ f}{\psi}(tx + (1-t)y) \\ &= \frac{\frac{\varphi}{\psi}(tx + (1-t)y) - \frac{\varphi}{\psi}(y)}{\frac{\varphi}{\psi}(x) - \frac{\varphi}{\psi}(y)} \frac{\gamma^{-1} \circ f}{\psi}(x) + \frac{\frac{\varphi}{\psi}(x) - \frac{\varphi}{\psi}(tx + (1-t)y)}{\frac{\varphi}{\psi}(x) - \frac{\varphi}{\psi}(y)} \frac{\gamma^{-1} \circ f}{\psi}(y) \end{aligned}$$

for all $x, y \in I, x \neq y$. This shows that f is t-affine with respect to the family $\mathcal{F}$ iff the function $\frac{\gamma^{-1} \circ f}{\psi}$ is t-affine with respect to the function $\frac{\varphi}{\psi}$. Now our result is a consequence of Theorem (8). □

Remark 12. Taking in this theorem $\gamma = \mathrm{id}|_{\mathbb{R}}$ and $t = \frac{1}{2}$ we get a complete solution of the following problem posed by Zs. Páles (cf. [8]):

Given a Tchebycheff system $\varphi, \psi : I \to \mathbb{R}$, describe the noncontinuous functions $f : I \to \mathbb{R}$ satisfying the functional equation

$$\begin{vmatrix} f(x) & f\left(\frac{x+y}{2}\right) & f(y) \\ \varphi(x) & \varphi\left(\frac{x+y}{2}\right) & \varphi(y) \\ \psi(x) & \psi\left(\frac{x+y}{2}\right) & \psi(y) \end{vmatrix} = 0, \quad x, y \in I.$$

8. An application in solving some functional equations without any regularity conditions

Theorem 7 as well as Theorem 8 can be applied to determine all solutions of some functional equations without any regularity conditions. For instance, from the third part of Theorem 8 we obtain the following

Proposition 3. *Let $I \subseteq (0, \infty)$ be an open interval and let $p \in \mathbb{R}$, $0 \neq p \neq 1$. A function $f : I \to \mathbb{R}$ satisfies the functional equation*

$$f\left(\frac{x+y}{2}\right) = \frac{\left(\frac{x+y}{2}\right)^p - y^p}{x^p - y^p} f(x) + \frac{x^p - \left(\frac{x+y}{2}\right)^p}{x^p - y^p} f(y),$$

for all $x, y \in I$, $x \neq y$, if, and only if, there are $b, c \in \mathbb{R}$ such that

$$f(x) = bx^p + c, \quad x \in I.$$

Proof. Put $\alpha(t) := t^p$ $(t \in I)$ and note that f satisfies the functional equation iff f is Jensen affine with respect to α. Now it is enough to apply Theorem 8(3). □

Taking $p = 2$ we hence get

Corollary 3. *Let $I \subseteq (0, \infty)$ be an open interval. A function $f : I \to \mathbb{R}$ satisfies the equation*

$$f\left(\frac{x+y}{2}\right) = \frac{x+3y}{4(x+y)} f(x) + \frac{3x+y}{4(x+y)} f(y), \quad x, y \in I,$$

if, and only if, there are $b, c \in \mathbb{R}$ such that

$$f(x) = bx^2 + c, \quad x \in I.$$

9. Monotonic solutions of the generalized Jensen equation

In Theorem 6 we assume that the function α is continuous. In this section we show that in one-dimensional case the continuity of α can be omitted.

We begin with the following obvious

Remark 13. Let $I \subset \mathbb{R}$ be an interval and let A be a dense subset in I. Suppose that the functions $f, g : I \to \mathbb{R}$ are monotonic and $f(x) = g(x)$ for all $x \in A$. If $x \in I$ is a point of the continuity of f or g, then $f(x) = g(x)$.

Theorem 10. *Let $I \subset \mathbb{R}$ be an interval and $\alpha : I \to \mathbb{R}$ be a strictly monotonic function. Then a monotonic function $f : I \to \mathbb{R}$ satisfies equation* (2)*:*

$$f\left(\frac{x+y}{2}\right) = \frac{\alpha\left(\frac{x+y}{2}\right) - \alpha(y)}{\alpha(x) - \alpha(y)} f(x) + \frac{\alpha(x) - \alpha\left(\frac{x+y}{2}\right)}{\alpha(x) - \alpha(y)} f(y), \quad x, y \in I,$$

if, and only if, there are $b, c \in \mathbb{R}$ such that

$$f(x) = b\alpha(x) + c, \quad x \in I.$$

Proof. Suppose that $f : I \to \mathbb{R}$ is monotonic and satisfies equation (2). Take $x_0, y_0 \in I$, $x_0 < y_0$. Since α is one-to-one there are unique $b, c \in \mathbb{R}$ such that

$$f(x_0) = b\alpha(x_0) + c, \quad f(y_0) = b\alpha(y_0) + c.$$

Put

$$A := \{x \in [x_0, y_0] : f(x) = b\alpha(x) + c\}.$$

Of course $x_0, y_0 \in A$. Note that for all $x_1, y_1 \in A$ we have

$$\begin{aligned} f\left(\frac{x_1+y_1}{2}\right) &= \left(\frac{\alpha\left(\frac{x_1+y_1}{2}\right) - \alpha(y_1)}{\alpha(x_1) - \alpha(y_1)}\right)[b\alpha(x_1) + c] \\ &\quad + \left(1 - \frac{\alpha\left(\frac{x_1+y_1}{2}\right) - \alpha(y_1)}{\alpha(x_1) - \alpha(y_1)}\right)[b\alpha(y_1) + c] \\ &= b\left(\frac{\alpha\left(\frac{x_1+y_1}{2}\right) - \alpha(y_1)}{\alpha(x_1) - \alpha(y_1)}\right)[\alpha(x_1) - \alpha(y_1)] + b\alpha(y_1) + c \\ &= b\left(\alpha\left(\frac{x_1+y_1}{2}\right) - \alpha(y_1)\right) + b\alpha(y_1) + c = b\alpha\left(\frac{x_1+y_1}{2}\right) + c, \end{aligned}$$

and, consequently,

$$\frac{x_1+y_1}{2} \in A$$

Thus the set A is dense in the interval $[x_0, y_0]$. It follows that the monotonic functions f and $b\alpha + c$ coincide on the dense set A. By the definition of the set A and Remark 8 we infer that $f(x) = b\alpha(x) + c$ for all $x \in I$ such that α is continuous at x (or f is continuous at x). It follows that f and $b\alpha + c$ coincide on I except for at most countable set. If there where a point $\bar{y} \in I$ such that $f(z) \neq b\alpha(z) + c$, then replacing for instance y_0 by $\bar{y}$ we could find some real constant $\bar{b}$, $\bar{c}$ such that $\bar{b} \neq b$ or $\bar{c} \neq c$ and arguing in the same way we would get that $f = \bar{b}\alpha + \bar{c}$ on I except for at most countable subset of I which, of course, is impossible. This contradiction proves that $f = b\alpha + c$ in I.

Since the converse implication is obvious the proof is complete. □

10. A generalized Cauchy functional equation

Assume that U is a cone in a real linear space X , that is $U + U \subset U$, $tU \subset U$ for $t > 0$. If $\alpha, f : U \to \mathbb{R}$ are homogeneous of order p and q, respectively, that is

$$\alpha(tx) = t^p \alpha(x), \quad f(tx) = t^q t(x), \quad x \in U,\ t > 0,$$

for some $p, q \in \mathbb{R}$, then the generalized Jensen functional equation (2) becomes

$$f(x+y) = 2^{q-p}\left(\frac{\alpha(x+y) - \alpha(2y)}{\alpha(x) - \alpha(y)} f(x) + \frac{\alpha(2x) - \alpha(x+y)}{\alpha(x) - \alpha(y)} f(y)\right)$$

for all $x, y \in U$, $\alpha(x) \neq \alpha(y)$. In the case $q = p$ we get

$$f(x+y) = \frac{\alpha(x+y) - \alpha(2y)}{\alpha(x) - \alpha(y)} f(x) + \frac{\alpha(2x) - \alpha(x+y)}{\alpha(x) - \alpha(y)} f(y), \tag{18}$$

for all $x, y \in U$ such that $\alpha(x) \neq \alpha(y)$.

Note that this functional equation can be treated as a generalization of the classical Cauchy equation, as we have the following obvious

Proposition 4. *Let U be a cone in a real linear space X and let $f : U \to \mathbb{R}$. Then f is additive, that is*

$$f(x+y) = f(x) + f(y), \quad x, y \in U,$$

iff f satisfies equation (18) *for every linear functional $\alpha \in X'$ and all for all $x, y \in U$ such that $\alpha(x) \neq \alpha(y)$, (that is f is additive with respect to α for each $\alpha \in X'$ with respect to α).*

As a consequence of Theorem 6 we obtain

Theorem 11. *Let U be an open cone in a real normed space. Suppose that $\alpha : U \to \mathbb{R}$ is non-constant, continuous, homogeneous of a finite order and such that, for all $x, y \in U$, the function*

$$[0;1] \ni s \to \alpha(sx + (1-s)y)$$

is monotonic and strictly monotonic in the case when $\alpha(x) \neq \alpha(y)$.

Then a continuous at least at one point function $f : U \to \mathbb{R}$ satisfies the functional equation (18)*:*

$$f(x+y) = \frac{\alpha(x+y) - \alpha(2y)}{\alpha(x) - \alpha(y)} f(x) + \frac{\alpha(2x) - \alpha(x+y)}{\alpha(x) - \alpha(y)} f(y),$$

for all $x, y \in U$ such that $\alpha(x) \neq \alpha(y)$, if, and only if, there is $b \in \mathbb{R}$ such that

$$f(x) = b\alpha(x), \quad x \in U.$$

Proof. A similar reasoning as in the proof of Theorem 6 shows that f is continuous in U. For an arbitrary $x \in U$ take $y \in U$ such that $\alpha(y) \neq \alpha(x)$. Then, by the assumed monotonicity properties of α, we have $\alpha(tx) \neq \alpha(x)$ for all $t > 0$ and $t \neq 1$. Setting $y := tx$ with $t > 0$, $t \neq 1$ in (18) and making use of the homogeneity of α, we get

$$f(x+tx) = \frac{(1+t)^p - (2t)^p}{1-t^p} f(x) + \frac{2^p - (1+t)^p}{1-t^p} f(tx),$$

whence, letting $t \to 1$ and using the continuity of f, we get $f(2x) = 2^p f(x)$, i.e.

$$f\left(\frac{x}{2}\right) = \frac{f(x)}{2^p}, \quad x \in U.$$

Hence, making use of (18),

$$f\left(\frac{x+y}{2}\right) = \frac{f(x+y)}{2^p} = \frac{\alpha\left(\frac{x+y}{2}\right) - \alpha(y)}{\alpha(x) - \alpha(y)} f(x) + \frac{\alpha(x) - \alpha\left(\frac{x+y}{2}\right)}{\alpha(x) - \alpha(y)} f(y),$$

for all $x, y \in U$ such that $\alpha(y) \neq \alpha(x)$, which means that f satisfies the generalized Jensen equation (2). From Theorem 6 we get $f = b\alpha + c$ for some $b, c \in \mathbb{R}$. The homogeneity of α implies that $f = b\alpha$. The remaining part of the proof is obvious. □

Remark 14. Note that a counterpart of Proposition 2 for equation (18) also holds true.

Theorem 12. *Let U be an open cone in a real normed space and $\alpha : U \to \mathbb{R}$ be continuous and locally non-constant. Suppose that $f : U \to \mathbb{R}$ satisfies the functional equation*

$$f(x+y) = \frac{\alpha(x+y) - \alpha(2y)}{\alpha(x) - \alpha(y)} f(x) + \frac{\alpha(2x) - \alpha(x+y)}{\alpha(x) - \alpha(y)} f(y),$$

for all $x, y \in U$ such that $\alpha(x) \neq \alpha(y)$.

Then

(1) *If α is neither affine nor of the form*

$$\alpha = b\beta + c$$

where β is an exponential function and $b, c \in \mathbb{R}$, $b \neq 0$, nor homogeneous of order 0, then f is continuous on some open subset of U.

(2) *If α is affine or of the form*

$$\alpha = b\beta + c$$

where β is an exponential function and b, c are some real numbers, then f can be discontinuous everywhere in U.

Proof. Suppose that $f : U \to \mathbb{R}$ satisfies equation (18). Applying twice (18) we get

$$\begin{aligned}
f(x+y+z) &= f((x+y)+z) \\
&= \frac{\alpha(x+y+z) - \alpha(2z)}{\alpha(x+y) - \alpha(z)} f(x+y) + \frac{\alpha(2(x+y)) - \alpha(x+y+z)}{\alpha(x+y) - \alpha(z)} f(z) \\
&= \frac{\alpha(x+y+z) - \alpha(2z)}{\alpha(x+y) - \alpha(z)} \left(\frac{\alpha(x+y) - \alpha(2y)}{\alpha(x) - \alpha(y)} f(x) + \frac{\alpha(2x) - \alpha(x+y)}{\alpha(x) - \alpha(y)} f(y) \right) \\
&\quad + \frac{\alpha(2(x+y)) - \alpha(x+y+z)}{\alpha(x+y) - \alpha(z)} f(z)
\end{aligned}$$

for all $x, y, z \in U$ such that $\alpha(x+y) \neq \alpha(z)$ and $\alpha(x) \neq \alpha(y)$. Interchanging x and z we hence get

$$\begin{aligned}
f(x+y+z) &= f(z+(y+x)) \\
&= \frac{\alpha(x+y+z) - \alpha(2x)}{\alpha(x+y) - \alpha(x)} \left(\frac{\alpha(z+y) - \alpha(2y)}{\alpha(z) - \alpha(y)} f(z) + \frac{\alpha(2z) - \alpha(z+y)}{\alpha(z) - \alpha(y)} f(y) \right) \\
&\quad + \frac{\alpha(2(z+y)) - \alpha(x+y+z)}{\alpha(z+y) - \alpha(x)} f(x)
\end{aligned}$$

for all $x, y, z \in U$ such that $\alpha(z+y) \neq \alpha(x)$ and $\alpha(z) \neq \alpha(y)$. Subtracting these equations by sides we obtain

$$A_\alpha(x,y,z)f(x) + B_\alpha(x,y,z)f(y) + C_\alpha(x,y,z)f(z) = 0, \quad x,y,z \in U,$$

where

$$A_\alpha(x,y,z) := \frac{\alpha(x+y)-\alpha(2y)}{\alpha(x)-\alpha(y)}\frac{\alpha(x+y+z)-\alpha(2z)}{\alpha(x+y)-\alpha(z)} - \frac{\alpha(2(z+y))-\alpha(x+y+z)}{\alpha(z+y)-\alpha(x)};$$

$$\begin{aligned} B_\alpha(x,y,z) := {} & \frac{\alpha(x+y+z)-\alpha(2z)}{\alpha(x+y)-\alpha(z)}\frac{\alpha(2x)-\alpha(x+y)}{\alpha(x)-\alpha(y)} \\ & - \frac{\alpha(x+y+z)-\alpha(2x)}{\alpha(x+y)-\alpha(x)}\frac{\alpha(2z)-\alpha(z+y)}{\alpha(z)-\alpha(y)} \end{aligned}$$

$$\begin{aligned} C_\alpha(x,y,z) := {} & \frac{\alpha(2(x+y))-\alpha(x+y+z)}{\alpha(x+y)-\alpha(z)} \\ & - \frac{\alpha(x+y+z)-\alpha(2x)}{\alpha(x+y)-\alpha(x)}\frac{\alpha(z+y)-\alpha(2y)}{\alpha(z)-\alpha(y)} \end{aligned}$$

for all $x, y, z \in U$ such that $\alpha(x+y) \neq \alpha(z)$, $\alpha(x) \neq \alpha(y)$, $\alpha(z+y) \neq \alpha(x)$ and $\alpha(z) \neq \alpha(y)$.

If there exist $x_0, y_0, z_0 \in U$ such that $A_\alpha(x_0, y_0, z_0) \neq 0$, then there exists an open neighbourhood V of the point x_0 such that

$$f(x) = -\frac{B_\alpha(x,y_0,z_0)}{A_\alpha(x,y_0,z_0)}f(y_0) - \frac{C_\alpha(x,y_0,z_0)}{A_\alpha(x,y_0,z_0)}f(z_0), \quad x \in V,$$

and, consequently, f is continuous in V.

Suppose that A_α disappears everywhere, i.e. that

$$\frac{\alpha(x+y)-\alpha(2y)}{\alpha(x)-\alpha(y)}\frac{\alpha(x+y+z)-\alpha(2z)}{\alpha(x+y)-\alpha(z)} - \frac{\alpha(2(z+y))-\alpha(x+y+z)}{\alpha(z+y)-\alpha(x)} = 0,$$

for all $x, y, z \in U$ such that $\alpha(x+y) \neq \alpha(z)$, $\alpha(x) \neq \alpha(y)$, $\alpha(z+y) \neq \alpha(x)$ and $\alpha(z) \neq \alpha(y)$. It follows that

$$\begin{aligned} & [\alpha(x+y+z)-\alpha(2z)][\alpha(x+y)-\alpha(2y)][\alpha(z+y)-\alpha(x)] \\ & \qquad = [\alpha(2(z+y))-\alpha(x+y+z)][\alpha(x+y)-\alpha(z)][\alpha(x)-\alpha(y)] \end{aligned}$$

for all $x, y, z \in U$. Setting here $z = 2y$ gives,

$$\begin{aligned} & [\alpha(x+3y)-\alpha(4y)][\alpha(x+y)-\alpha(2y)][\alpha(3y)-\alpha(x)] \\ & \qquad = [\alpha(6y)-\alpha(x+3y)][\alpha(x+y)-\alpha(2y)][\alpha(x)-\alpha(y)], \quad x,y \in U. \end{aligned}$$

Since α is continuous and locally non-constant, $\alpha(x+y)-\alpha(2y)\neq 0$ in a dense set in U. It follows that, for all $x,y\in U$,

$$[\alpha(x+3y)-\alpha(4y)][\alpha(3y)-\alpha(x)]=[\alpha(6y)-\alpha(x+3y)][\alpha(x)-\alpha(y)],$$

which reduces to the equation

$$\alpha(x+3y)[\alpha(3y)-\alpha(y)]=\alpha(x)[\alpha(6y)-\alpha(4y)]+\alpha(3y)\alpha(4y)-\alpha(y)\alpha(6y) \quad (19)$$

for all $x,y\in U$. Hence, if there were a point $y\in U$ such that

$$\alpha(3y)-\alpha(y)=0 \quad \text{or} \quad \alpha(6y)-\alpha(4y)=0,$$

then α would be a constant function, as in the opposite case we would have respectively,

$$\alpha(x)=-\frac{\alpha(3y)\alpha(4y)-\alpha(y)\alpha(6y)}{\alpha(6y)-\alpha(4y)}, \quad x\in U,$$

or

$$\alpha(x+3y)=\frac{\alpha(3y)\alpha(4y)-\alpha(y)\alpha(6y)}{\alpha(3y)-\alpha(y)}, \quad x\in U,$$

which is impossible as, by assumption α is locally non-constant. It follows that either

$$\alpha(3y)-\alpha(y)\neq 0, \quad y\in U,$$

or

$$\alpha(3y)-\alpha(y)=0, \qquad \alpha(6y)-\alpha(4y)=0, \quad y\in U.$$

In the first case, replacing y by $\frac{y}{3}$ in (19), we get

$$\alpha(x+y)=\alpha(x)\beta(y)+\gamma(y), \quad x,y\in U. \quad (20)$$

where

$$\beta(y):=\frac{\alpha(6y)-\alpha(4y)}{\alpha(3y)-\alpha(y)}, \quad \gamma(y):=\frac{\alpha(3y)\alpha(4y)-\alpha(y)\alpha(6y)}{\alpha(3y)-\alpha(y)}, \quad y\in U.$$

From (20) we have

$$\alpha(x+y+z)=\alpha(x)\beta(y+z)+\gamma(y+z), \quad x,y,z\in U.$$

On the other hand, applying (20) twice, we have

$$\begin{aligned}\alpha(x+y+z)&=\alpha(x+y)\beta(z)+\gamma(z)=[\alpha(x)\beta(y)+\gamma(y)]\beta(z)+\gamma(z)\\&=\alpha(x)\beta(y)\beta(z)+\beta(z)\gamma(y)]+\gamma(z)\end{aligned}$$

for all $x, y, z \in U$. It follows that $\beta(x) \neq 0$ for all $x \in U$. Comparing the above equations and taking into account that α is not constant we infer that

$$\beta(y+z) = \beta(y)\beta(z), \quad \gamma(y+z) = \beta(z)\gamma(y)] + \gamma(z). \quad y, z \in U.$$

Thus β is an exponential function.

Suppose that there is a point $z \in U$ such that $\beta(z) \neq 1$. By the symmetry of the right-hand side of the second of these equations we infer that

$$\gamma(y) = c[\beta(y) - 1], \quad y \in U,$$

where $k := \frac{\gamma(z)}{\beta(z)-1}$. Now from (20) we have

$$\alpha(x+y) = [\alpha(x) + k]\beta(y) - k, \quad x, y \in U.$$

The symmetry of the right-hand side implies that

$$[\alpha(x) + k]\beta(y) - k = [\alpha(y) + k]\beta(x) - k, \quad x, y \in U.$$

Consequently,

$$\alpha(x) = \frac{\alpha(y) + k}{\beta(y)}\beta(x) - k, \quad x, y \in U.$$

It follows that

$$\alpha(x) = b\beta(x) + c, \quad x \in U,$$

for some real constants b and c, $b \neq 0$.

If $\beta(x) \equiv 1$ for all $x \in U$ then from (20) we have

$$\alpha(x+y) = \alpha(x) + \gamma(y), \quad x, y \in U.$$

It follows that $\alpha(x) + \gamma(y) = \alpha(y) + \gamma(x)$ for all $x, y \in U$, whence

$$\gamma(y) = \alpha(y) - c, \quad y \in U,$$

for some real c. It follows that

$$\alpha(x+y) = \alpha(x) + \alpha(y) - c, \quad x, y \in U,$$

or, equivalently,

$$\alpha(x+y) - c = [\alpha(x) - c] + [\alpha(y) - c], \quad x, y \in U.$$

Thus $\alpha - c$ is additive which means that α is an affine function.

Now we consider the remaining case when

$$\alpha(3y) - \alpha(y) = 0, \qquad \alpha(6y) - \alpha(4y) = 0, \quad y \in U.$$

Replacing y by $y/2$ in the second of these equations we hence get

$$\alpha(y) = \alpha(2y) = \alpha(3y), \quad y \in U$$

whence, by induction,

$$\alpha(2^n 3^m y) = \alpha(y), \quad y \in U,\ m, n \in \mathbb{Z},$$

where $\mathbb{Z}$ denotes the set of all integers. The density of the set $\{2^n 3^m y :\ m, n \in \mathbb{Z}\}$ in $(0, \infty)$ and continuity of α imply that

$$\alpha(ty) = \alpha(y), \quad y \in U,\ t > 0,$$

which means that α is homogeneous of the order 0. This completes the first part of the proof.

To prove the second part assume that assume first that α is affine. Then

$$\frac{\alpha(x+y) - \alpha(2y)}{\alpha(x) - \alpha(y)} = \frac{\alpha(2x) - \alpha(x+y)}{\alpha(x) - \alpha(y)} = \frac{\alpha(x-y)}{\alpha(x-y)} - 1,$$

for all $x, y \in U$ such that $\alpha(x) \neq \alpha(y)$, and equation (18) becomes

$$f(x+y) = f(x) + f(y), \quad x, y \in U,\ \alpha(x) \neq \alpha(y),$$

It is well know that there are solutions of the dense graphs (so discontinuous everywhere) (cf. M. Kuczma [6]).

If α is of the form $\alpha = b\beta + c$ where β is an exponential function and $b, c \in \mathbb{R}$, $b \neq 0$, then, for all $x, y \in U$ such that $\alpha(x) \neq \alpha(y)$, we have

$$\frac{\alpha(x+y) - \alpha(2y)}{\alpha(x) - \alpha(y)} = \frac{\beta(x+y) - \beta(2y)}{\beta(x) - \beta(x)} = \frac{\beta(x)\beta(y) - \beta(y)^2}{\beta(x) - \beta(x)} = \beta(y),$$

$$\frac{\alpha(2x) - \alpha(x+y)}{\alpha(x) - \alpha(y)} = \frac{\beta(2x) - \beta(x+y)}{\beta(x) - \beta(x)} = \frac{\beta(x)^2 - \beta(x)\beta(y)}{\beta(x) - \beta(x)} = \beta(x),$$

and equation (18) becomes

$$f(x+y) = \beta(y)f(x) + \beta(x)f(y), \quad x, y \in U,\ \alpha(x) \neq \alpha(y),$$

For $U = \mathbb{R}$ (or $U = (0, \infty)$) we have $\beta(x) = a^x$ for some $a > 0$, $a \neq 1$. Let $g : U \to (0, \infty)$ be an arbitrary additive function, i.e.

$$g(x+y) = g(x) + g(y), \quad x, y \in U,$$

of a dense graph in $U \times (0, \infty)$ (cf. [8]). Define $f : U \to \mathbb{R}$ by $f(x) := \beta(x)g(x)$. Then the graph of f is dense in $U \times (0, \infty)$ and, for all $x, y \in U$, we have

$$\begin{aligned} f(x+y) &= \beta(x+y)g(x+y) = \beta(y)\beta(x)g(x) + \beta(x)\beta(y)g(y) \\ &= \beta(y)f(x) + \beta(x)f(y) \end{aligned}$$

which proves that f satisfies equation (18). □

Remark 15. Note that the functional equations $A_\alpha = 0$, $B_\alpha = 0$, $C_\alpha = 0$ are equivalent.

Applying the above theorem and the counterpart of Proposition 2 we get

Corollary 4. *A function $f : (0,\infty) \to \mathbb{R}$ satisfies the functional equation*

$$f(x+y) = \frac{x+3y}{x+y} f(x) + \frac{3x+y}{x+y} f(y), \quad x, y > 0,\ x \neq y,$$

if, and only if, there exists $b \in \mathbb{R}$ such that

$$f(x) = bx^2, \quad x > 0.$$

Theorem 13. *Let $p \in \mathbb{R}$, $0 \neq p \neq 1$, be fixed. A function $f : (0,\infty) \to \mathbb{R}$ satisfies the functional equation*

$$f(x+y) = \frac{(x+y)^p - (2y)^p}{x^p - y^p} f(x) + \frac{(2x)^p (x+y)^p}{x^p - y^p} f(y), \quad x, y > 0,\ x \neq y,$$

if, and only if, there exists $b \in \mathbb{R}$ such that

$$f(x) = bx^p, \quad x > 0.$$

Remark 16. In the above two results we determine the solutions of the (generalized) Cauchy equation by reducing it to the respective (generalized) Jensen equation. Thus the procedure is converse to the classical one (cf. J. Aczél [1] , M. Kuczma [6]).

11. Remark on (M, N)-convexity and an open question

Let $I \subset \mathbb{R}$ be an interval. A function $M : I^2 \to I$ is said to be a *mean* in I if $M(K^2) \subset K$ for every subinterval $K \subset I$. Let M and N be some means in the intervals I and J, respectively. A function $f : I \to J$ is said to be (M, N)-*convex* if

$$f(M(x,y)) \leq N(f(x), f(y)), \quad x, y \in I.$$

To compare convexity considered in this paper and (M, N)-convexity, take an interval $U \subset \mathbb{R}$, a continuous strictly monotonic function $\alpha : U \to \mathbb{R}$ and observe that a function $f : U \to \mathbb{R}$ is Jensen convex with respect to α iff the function $g := f \circ \alpha^{-1}$ satisfies the inequality

$$g(M(x,y)) \leq N(x, y, g(x), g(y)), \quad x, y \in I := \alpha(U),$$

where $N : I^2 \times \mathbb{R}^2 \to \mathbb{R}$ is given by

$$N(x, y, u, v) := \frac{M(x, y) - y}{x - y} u + \frac{x - M(x, y)}{x - y} v, \quad x, y \in I,\ x \neq y;\ u, v \in \mathbb{R},$$

and $M : I \times I \to I$ is given by

$$M(x, y) := \alpha \left(\frac{\alpha^{-1}(x) + \alpha^{-1}(y)}{2} \right) \quad x, y \in I,$$

that is, M is a quasi-arithmetic mean of a generator α^{-1}.

Note that, for all $x, y \in I$, $x \neq y$; $u, v \in \mathbb{R}$,

$$\min(u, v) \leq N(x, y, u, v) \leq \max(u, v),$$

that is, for every $x, y \in I$, the function $(u, v) \to N(x, y, u, v)$ is a mean in $\mathbb{R}$.

Moreover, if N does not depend on x and y, then there is $a \in (0, 1)$ such that

$$\frac{M(x, y) - y}{x - y} = a, \quad x, y \in I,$$

and, consequently, $M(x, y) = ax + (1 - a)y$. The symmetry of M implies that $a = \frac{1}{2}$, whence $M(x, y) = \frac{x+y}{2}$ and $N(u, v) = \frac{u+v}{2}$. Now it is obvious that $\alpha = b \operatorname{id}|_U + c$ for some $b, c \in \mathbb{R}$. Thus the notions of (M, N)-convexity and the convexity with respect to α coincide only in the case when both are the ordinary ones.

This discussion suggests the following generalization of the notion of (M, N)-convexity.

Let $I, J \subset \mathbb{R}$ be intervals. Assume that $M : I^2 \to I$ is a mean in I, that is

$$\min(x, y) \leq M(x, y) \leq \max(x, y), \quad x, y \in I,$$

and $N : I^2 \times J^2 \to J$ is a function such that

$$\min(u, v) \leq N(x, y, u, v) \leq \max(u, v), \quad x, y \in I,\ u, v \in J, \tag{21}$$

that is, for all $x, y \in I$, the function $(u, v) \to N(x, y, u, v)$ is a mean in J. A function $f : I \to J$ is called (M, N)-*convex* if

$$f\left(M(x, y)\right) \leq N(x, y, f(x), f(y)), \quad x, y \in I.$$

Now the following question arises. Let $\mathcal{F}$ be a Beckenbach family of functions in an interval I. Do there exist a mean M in I and a function $N : I^2 \times J^2 \to J$ satisfying (21) such that f is (M, N)-convex?

References

[1] J. Aczél, Lectures on Functional Equations and Their Applications, *Academic Press*, *New York and London*, 1966.

[2] E. F. Beckenbach, Generalized convex functions, *Bull. Amer. Math. Soc.* **43** (1937), 363–371.

[3] M. Bessenyei and Zs. Páles, Higher order generalization of Hadamard's inequality, *Publ. Math. Debrecen* **61** (2002), 623–643.

[4] F. Bernstein and G. Doetsch, Zur Theorie der konvexen Funktionen, *Math. Ann.* **76** (1915), 514–526.

[5] Z. Daróczy and Zs. Páles, Convexity with given infinite weight sequences, *Stochastica* **11** (1987), 5–12.

[6] M. Kuczma, An Introduction to the Theory of Functional Equations and Inequalities, Cauchy's Equation and Jensen's Inequality, *P.W.N, Uniwersytet Ślaski*, *Warszawa – Kraków – Katowice*, 1985.

[7] J. Matkowski, Continuous solution of a functional equation, *Publ. Math. Debrecen* **52** (1998), 559–562. .

[8] Zs. Páles, 21. Problem in Report of The Forty-second International Symposium on Functional Equations, *Aequationes Math.* **69** (2005), 192.

[9] W. Sierpiński, Sur les fonctions convexes mesurables, *Fund. Math.* **1** (1920), 125–128.

[10] E. Zeidler, Nonlinear Functional Analysis and its Applications I, *Springer Verlag*, *New York*, 1986.

JANUSZ MATKOWSKI
FACULTY OF MATHEMATICS
COMPUTER SCIENCE AND ECONOMETRY
UNIVERSITY OF ZIELONA GÓRA
PODGÓRNA 50
PL-65246 ZIELONA GÓRA
POLAND,
AND
INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
PL-40-007 KATOWICE
POLAND

E-mail: J.Matkowski@wmie.uz.zgora.pl

(Received September 28, 2007; revised October 9, 2008)