

A new characterization of q -convexifiable Banach spaces

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Abstract. In this paper an atomic decomposition theorem for Banach-space-valued weak Hardy regular martingale space $wH_p(X)$ is given. As an application, we show that a Banach space X is q -convexifiable if and only if $\|S^{(q)}(f)\|_{wL_p} \leq C\|f^*\|_{wL_p}$ ($0 < p < \infty$, $2 \leq q < \infty$) for each X -valued regular martingale $f = (f_n)_{n \geq 0}$.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(\mathcal{F}_n)_{n \geq 0}$ a non-decreasing sequence of sub- σ -algebras of $\mathcal{F}$ satisfying $\mathcal{F} = \sigma(\bigcup_{n \geq 0} \mathcal{F}_n)$. The expectation operator and the conditional expectation operators relative to $\mathcal{F}_n$ are denoted by $\mathbb{E}$ and $\mathbb{E}_n$, respectively. We say that $(\mathcal{F}_n)_{n \geq 0}$ satisfies the regular condition, if $\chi(F) \leq d\mathbb{E}_{n-1}[\chi(F)]$, $\forall F \in \mathcal{F}_n$, $n = 1, 2, 3, \dots$, where $\chi(A)$ denotes the characteristic function of the set A and $d \geq 1$. In the following we always assume that $(\mathcal{F}_n)_{n \geq 0}$ satisfies the regular condition and d denotes the constant in the definition above.

Let X be a Banach space with norm $\|\cdot\|$ and $f = (f_n)_{n \geq 0}$ an X -valued regular martingale relative to $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n \geq 0})$. For convenience, we assume

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that $f_0 = 0$, denote $df_i = f_i - f_{i-1}$ ($i \geq 1$), and

$$f_n^* = \sup_{0 \leq i \leq n} \|f_i\|, \quad f^* = \sup_n f_n^* = \sup_{n \geq 0} \|f_n\|,$$

$$S_n^{(p)}(f) = \left(\sum_{i=1}^n \|df_i\|^p \right)^{\frac{1}{p}}, \quad S^{(p)}(f) = \sup_n S_n^{(p)}(f) = \left(\sum_{i=1}^{\infty} \|df_i\|^p \right)^{\frac{1}{p}}, \quad (0 < p < \infty).$$

Let $0 < p < \infty$, the space of all X -valued measurable functions f satisfying

$$\|f\|_{wL_p(X)} = \sup_{y>0} y \mathbb{P}(\|f\| > y)^{\frac{1}{p}} < \infty$$

is called a weak $L_p(X)$ -space, denoted by $wL_p(X)$. We write $wL_p(X)$ as wL_p if X is the scalar field. It is well-known that $\|\cdot\|_{wL_p(X)}$ is a quasi-norm on $wL_p(X)$. Notice that $L_p(X) \subset wL_p(X)$ since $\|f\|_{wL_p(X)} \leq \|f\|_{L_p(X)}$. If $0 < p < \infty$, we define X -valued weak regular martingale Hardy space $wH_p(X)$ as follows:

$$wH_p(X) = \{f = (f_n)_{n \geq 0} : \|f\|_{wH_p(X)} = \|f^*\|_{wL_p} < \infty\}.$$

It is clear that if X is the scalar field and $\|\cdot\|_{wL_p}$ is replaced by $\|\cdot\|_{L_p}$ we get the familiar martingale Hardy spaces H_p in the definition above (see F. WEISZ [2]).

Definition 1.1 ([5]). Let $0 < \alpha, r \leq \infty$. An X -valued measurable function a is called a $(3, \alpha, r)$ -atom if there exists a stopping time ν (ν is called the stopping time associated with a) such that

- (i) $a_n = \mathbb{E}_n[a] = 0$ if $\nu \geq n$,
- (ii) $\|a^*\|_r \leq \mathbb{P}(\nu \neq \infty)^{\frac{1}{r} - \frac{1}{\alpha}}$.

Throughout this paper, we denote the set of integers and the set of non-negative integers by $\mathbb{Z}$ and $\mathbb{N}$, respectively. We use C_α to denote constants which depend only on α and may denote different constants at different occurrences.

It is well-known that the inequalities of Banach-space-valued martingales are closely connected with the geometrical properties of Banach spaces. For the related definitions see refs [6], [7]. In particular, we need the following lemmas:

Lemma 1.2 ([10]). *If $\{\mathcal{F}\}_{n \geq 0}$ satisfies the regular condition, then there exists stopping times $\nu_\lambda = \inf \{n \in \mathbb{N} : \mathbb{E}_n[\chi(\gamma_{n+1} > \lambda)] \geq \frac{1}{d}\}$ such that*

$$\mathbb{P}(\nu_\lambda \neq \infty) \leq d \mathbb{P}(\sup_n \gamma_n > \lambda) \quad (1.1)$$

for all nonnegative, adapted sequences $(\gamma)_{n \geq 0}$ and $\lambda \geq \|\gamma_0\|_\infty$, and $\nu_{\lambda_1} \leq \nu_{\lambda_2}$ when $\lambda_1 \leq \lambda_2$.

Lemma 1.3 ([8]). *Let X be a Banach space and $2 \leq q < \infty$. Then the following statements are equivalent:*

- (i) X is q -convexifiable;
- (ii) There exists a constant $C > 0$ such that for all X -valued regular martingales:

$$\|S^{(q)}(f)\|_r \leq C\|f^*\|_r, \quad 0 < r < \infty;$$

- (iii) There exists a constant $C > 0$ such that for all X -valued regular martingales:

$$\lambda^q \mathbb{P}(S^{(q)}(f) > \lambda) \leq C\|f^*\|_q^q, \quad \lambda > 0.$$

2. Atomic decomposition

In the scalar-valued case, atomic decomposition for regular martingale Hardy space H_p was given by LONG [10]. In this section we give a weak type vector-valued parallelism.

Theorem 2.1. *Let $0 < p < \infty$ and X a Banach space with Radon–Nikodym property. Then $f = (f_n)_{n \geq 0} \in wH_p(X)$ if and only if there exist a sequence $(a^k)_{k \in \mathbb{Z}}$ of $(3, p, \infty)$ -atoms and the corresponding stopping times $(\nu_k)_{k \in \mathbb{Z}}$ such that*

$$f_n = \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}_n[a^k], \quad \forall n \in \mathbb{N}, \quad (2.1)$$

where $0 \leq \mu_k \leq A \cdot 2^k \mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}}$ for some constant $A > 0$ and $\sup_{k \in \mathbb{Z}} 2^k \mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}} < \infty$.

Moreover, the following equivalence of norms holds:

$$\|f\|_{wH_p(X)} \sim \inf \sup_{k \in \mathbb{Z}} 2^k \mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}}, \quad (2.2)$$

where the infimum is taken over all the preceding decompositions of f .

PROOF. Assume that $f = (f_n)_{n \geq 0} \in wH_p(X)$. We define the following stopping times for all $k \in \mathbb{Z}$:

$$\nu_k = \inf \left\{ n \in \mathbb{N} : \mathbb{E}_n[\chi(F_k)] \geq \frac{1}{d} \right\} \quad (\inf \emptyset = \infty).$$

where $F_k = \{\|f_{n+1}\| > 2^k\}$. Obviously by Lemma 1.2, $\nu_k \uparrow \infty$ ($k \rightarrow \infty$). Let $f^{\nu_k} = (f_{n \wedge \nu_k})_{n \geq 0}$ be the stopping martingale. Then

$$\begin{aligned} \sum_{k \in \mathbb{Z}} (f_n^{\nu_{k+1}} - f_n^{\nu_k}) &= \sum_{k \in \mathbb{Z}} \left(\sum_{m=0}^n \chi(m \leq \nu_{k+1}) df_m - \sum_{m=0}^n \chi(m \leq \nu_k) df_m \right) \\ &= \sum_{m=0}^n \left(\sum_{k \in \mathbb{Z}} \chi(\nu_k < m \leq \nu_{k+1}) df_m \right) = f_n \end{aligned} \quad (2.3)$$

Now let

$$\mu_k = 2^k \cdot 3\mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}}, \quad a_n^k = \mu_k^{-1}(f_n^{\nu_{k+1}} - f_n^{\nu_k}), \quad k \in \mathbb{Z}, n \in \mathbb{N} \quad (2.4)$$

($a_n^k = 0$ if $\mu_k = 0$). It is clear that for any fixed $k \in \mathbb{Z}$, $a^k = (a_n^k)_{n \geq 0}$ is an X -valued martingale. Since $\chi(F_k) \leq d\mathbb{E}_n[\chi(F_k)] < 1$ on the set $\{\nu_k > n\}$, we have $\|f_{n+1}\| \leq 2^k$. i.e., $\|f_{\nu_k}\| \leq 2^k$. So by (2.4) we have

$$\begin{aligned} \|a_n^k\| &= \mu_k^{-1} \|f_n^{\nu_{k+1}} - f_n^{\nu_k}\| \leq \mu_k^{-1} (\|f_{\nu_{k+1} \wedge n}\| + \|f_{\nu_k \wedge n}\|) \\ &\leq \mathbb{P}(\nu_k \neq \infty)^{-\frac{1}{p}}. \end{aligned} \quad (2.5)$$

Furthermore, $\|a^{k*}\|_\infty \leq \mathbb{P}(\nu_k \neq \infty)^{-\frac{1}{p}}$. By assumption X has the Radon-Nikodym property, so there exists an X -valued integrable function a^k such that $\mathbb{E}_n[a^k] = a_n^k$ ($n \geq 0$). It is clear that $a_n^k = 0$ if $n \leq \nu_k$. And by (2.5) we have $\|a^{k*}\|_\infty < \mathbb{P}(\nu_k \neq \infty)^{-\frac{1}{p}}$. So a^k is a $(3, p, \infty)$ -atom. It follows from the definition of μ_k that $0 \leq \mu_k \leq A \cdot 2^k \mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}}$ with $A = 3$. By (2.3) we get

$$f_n = \sum_{k \in \mathbb{Z}} \mu_k (\mu_k^{-1} (f_n^{\nu_{k+1}} - f_n^{\nu_k})) = \sum_{k \in \mathbb{Z}} \mu_k a_n^k = \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}_n[a^k].$$

Hence (2.1) holds. Since $\{\|f_n\|\}$ is a nonnegative and adapted sequence, by (1.1) we have $\mathbb{P}(\nu_k \neq \infty) \leq d\mathbb{P}(f^* > 2^k)$. It follows that for any $k \in \mathbb{Z}$ we have

$$\begin{aligned} 2^k \mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}} &\leq d2^k \mathbb{P}(f^* > 2^k)^{\frac{1}{p}} \leq d\|f^*\|_{wL_p} \\ &= d\|f\|_{wH_p(X)} < \infty, \end{aligned} \quad (2.6)$$

which implies $\sup_{k \in \mathbb{Z}} 2^k \mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}} < \infty$.

Conversely, assume that $f = (f_n)_{n \geq 0}$ has a decomposition of the form (2.1). Let $M = \sup_{k \in \mathbb{Z}} 2^k \mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}}$. For any fixed $y > 0$ choose $j \in \mathbb{Z}$ such that $2^j \leq y < 2^{j+1}$. Let

$$f_n = \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}_n[a^k] = \sum_{k=-\infty}^{j-1} \mu_k a_n^k + \sum_{k=j}^{\infty} \mu_k a_n^k =: g_n + h_n, \quad n \in \mathbb{N}.$$

Thus $f^* \leq g^* + h^*$, and it follows that

$$\mathbb{P}(f^* > 2Ay) \leq \mathbb{P}(g^* > Ay) + \mathbb{P}(h^* > Ay).$$

Since $(a^k)_{k \in \mathbb{Z}}$ are $(3, p, \infty)$ -atoms, $\|a^{k*}\|_\infty \leq \mathbb{P}(\nu_k \neq \infty)^{-\frac{1}{p}}$ and we have

$$g^* \leq \sum_{k=-\infty}^{j-1} \mu_k a^{k*} \leq \sum_{k=-\infty}^{j-1} \mu_k \|a^{k*}\|_\infty \leq \sum_{k=-\infty}^{j-1} A \cdot 2^k \leq A 2^j.$$

Follows which we have

$$\mathbb{P}(g^* > Ay) \leq \mathbb{P}(g^* > A 2^j) = 0.$$

On the other hand, since $a_n^k = \mathbb{E}_n[a^k] = 0$ if $n \leq \nu_k$, we know $a^{k*} = 0$ on the set $\{\nu_k = \infty\}$. Moreover, $h^* \leq \sum_{k=j}^\infty \mu_k a^{k*}$. So we have $\{h^* > 0\} \subset \bigcup_{k=j}^\infty \{\nu_k \neq \infty\}$. Consequently,

$$\begin{aligned} \mathbb{P}(f^* > 2Ay) &\leq \mathbb{P}(h^* > Ay) \leq \mathbb{P}(h^* > 0) \leq \sum_{k=j}^\infty \mathbb{P}(\nu_k \neq \infty) \leq \sum_{k=j}^\infty M^p 2^{-kp} \\ &\leq C_p M^p 2^{-(j+1)p} \leq C_p M^p y^{-p}. \end{aligned}$$

It follows that

$$\|f\|_{wH_p(X)}^p \leq C_p M^p = C_p \sup_{k \in \mathbb{Z}} 2^{kp} \mathbb{P}(\nu_k \neq \infty) < \infty, \quad (2.7)$$

which shows that $f \in wH_p(X)$. Combine (2.6) with (2.7) we obtain that

$$C_p \sup_{k \in \mathbb{Z}} 2^k \mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}} \leq \|f\|_{wH_p(X)} \leq C_p \sup_{k \in \mathbb{Z}} 2^k \mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}}.$$

Thus (2.2) holds. The proof of the theorem is completed. $\square$

3. A characterization of q -convexifiable spaces

In this section we give an application of Theorem 2.1. A characterization for q -convexifiable Banach spaces is given.

Theorem 3.1. *Let $2 \leq q < \infty$ and X a Banach space. Then the following statements are equivalent:*

- (i) X is q -convexifiable;

(ii) *There exists a constant $C > 0$ such that for all X -valued regular martingales:*

$$\|S^{(q)}(f)\|_{wL_p} \leq C\|f^*\|_{wL_p}, \quad 0 < p < \infty. \quad (3.1)$$

PROOF. (i) $\Rightarrow$ (ii). Assume that X is a q -convexifiable Banach space and $f \in wH_p(X)$. Since q -convexity of X implies that X has Radon–Nikodym property, then f can be decomposed into the sum of a sequence of $(3, p, \infty)$ -atoms by Theorem 2.1. For any fixed $y > 0$ choose $j \in \mathbb{Z}$ such that $2^j \leq y < 2^{j+1}$ and let

$$f = \sum_{k \in \mathbb{Z}} \mu_k a^k = \sum_{k=-\infty}^{j-1} \mu_k a^k + \sum_{k=j}^{\infty} \mu_k a^k =: g + h.$$

Thus $\mathbb{P}(S^{(q)}(f) > 2y) \leq \mathbb{P}(S^{(q)}(g) > y) + \mathbb{P}(S^{(q)}(h) > y)$. For $0 < p < \infty$, choose $\max\{p, 2\} < q < \infty$, by Lemma 1.3, Theorem 2.1 and notice that $a^{k^*} = 0$ on the set $\{\nu_k = \infty\}$, we have

$$\begin{aligned} \|S^{(q)}(g)\|_q &\leq \sum_{k=-\infty}^{j-1} \mu_k \|S^{(q)}(a^k)\|_q \leq C_q \sum_{k=-\infty}^{j-1} \mu_k \|a^{k^*}\|_q \\ &\leq C_q \sum_{k=-\infty}^{j-1} A \cdot 2^{k(1-\frac{p}{q})} 2^{\frac{kp}{q}} \mathbb{P}(\nu_k \neq \infty)^{\frac{1}{q}} \\ &\leq C_q \sum_{k=-\infty}^{j-1} A \cdot 2^{k(1-\frac{p}{q})} \|f\|_{wH_p(X)}^{\frac{p}{q}} \leq C_{p,q} y^{1-\frac{p}{q}} \|f\|_{wH_p(X)}^{\frac{p}{q}}. \end{aligned}$$

It follows that

$$\mathbb{P}(S^{(q)}(g) > y) \leq y^{-q} \mathbb{E}[S^{(q)}(g)^q] \leq C_{p,q} y^{-p} \|f\|_{wH_p(X)}^p. \quad (3.2)$$

On the other hand, we have

$$\begin{aligned} \mathbb{P}(S^{(q)}(h) > y) &\leq \mathbb{P}(S^{(q)}(h) > 0) \leq \sum_{k=j}^{\infty} \mathbb{P}(S^{(q)}(a^k) > 0) \\ &\leq \sum_{k=j}^{\infty} \mathbb{P}(\nu_k \neq \infty) \leq \sum_{k=j}^{\infty} 2^{-kp} \cdot 2^{kp} \mathbb{P}(\nu_k \neq \infty) \\ &\leq C_p y^{-p} \|f\|_{wH_p(X)}^p. \end{aligned} \quad (3.3)$$

Combine (3.2) with (3.3) we get $\mathbb{P}(S^{(q)}(f) > 2y) \leq C_{p,q} y^{-p} \|f\|_{wH_p(X)}^p$. Thus (3.1) is obtained.

(ii) $\Rightarrow$ (i). Assume that (3.1) holds. Since $\|f^*\|_{wL_q} \leq \|f^*\|_q$, it is easy to obtain that

$$\lambda^q \mathbb{P}(S^{(q)}(f) > \lambda) \leq C \|f^*\|_q^q, \quad \lambda > 0.$$

By Lemma 1.3 we know that X is q -convexifiable. The proof is completed. $\square$

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