

On the volume of the convex hull of $d + 1$ segments in $\mathbb{R}^d$

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Abstract. Let $d \geq 2$, $m \geq d$, and $u_1, \dots, u_m$ be non-zero vectors linearly spanning $\mathbb{R}^d$. The note is devoted to the problem of minimizing the volume of the polytopes $P := \text{conv}(I_1 \cup \dots \cup I_m)$, where, for $j = 1, \dots, m$, I_j is a translate of $\text{conv}\{o, u_j\}$. The solution of this problem for the case $m = d$ was previously known. For the case $m = d + 1$ the minimal volume is evaluated and the class of minimizing polytopes P is studied.

1. Introduction

Let $d \geq 2$. The origin in $\mathbb{R}^d$ is denoted by o . The abbreviations conv and vol stand for convex hull and volume, respectively. Let $d \geq 2$, $m \geq d$, and $u_1, \dots, u_m$ be non-zero vectors linearly spanning $\mathbb{R}^d$. We consider the class $\mathcal{P}(u_1, \dots, u_m)$ of convex polytopes P in $\mathbb{R}^d$ such that $P = \text{conv}(I_1 \cup \dots \cup I_m)$, where, for $j = 1, \dots, m$, the set I_j is a translate of the segment $\text{conv}\{o, u_j\}$. By $v(u_1, \dots, u_m)$ we denote the minimum among the volumes of all polytopes from $\mathcal{P}(u_1, \dots, u_m)$, and by $\mathcal{P}_0(u_1, \dots, u_m)$ the subclass of $\mathcal{P}(u_1, \dots, u_m)$ minimizing the volume. It is known that $v(u_1, \dots, u_d) = \frac{1}{d!} |\det(u_1, \dots, u_d)|$. In this note we evaluate $v(u_1, \dots, u_{d+1})$ and study the properties of $\mathcal{P}_0(u_1, \dots, u_{d+1})$. For the case $m = d$ the class $\mathcal{P}_0(u_1, \dots, u_m)$ and the corresponding quantity $v(u_1, \dots, u_m)$ were studied in [7] and [5]. In particular, in the above paper it was shown that

$$v(u_1, \dots, u_d) = \frac{1}{d!} |\det(u_1, \dots, u_d)|, \quad (1)$$

where $\det(u_1, \dots, u_d)$ denotes the determinant of the matrix with columns $u_1, \dots, u_d$. We perform an analogous study for the case $m = d + 1$.

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Theorem. *Let $d \geq 2$, and $u_1, \dots, u_{d+1}$ be non-zero vectors linearly spanning $\mathbb{R}^d$. Then the following statements hold true.*

- I. *The quantity $v(u_1, \dots, u_{d+1})$ is equal to the maximum of $v(u'_1, \dots, u'_d)$ over all possible subsets $\{u'_1, \dots, u'_d\}$ of $\{u_1, \dots, u_{d+1}\}$.*
- II. *The class $\mathcal{P}_0(u_1, \dots, u_{d+1})$ necessarily contains simplices.*
- III. *Every polytope from $\mathcal{P}_0(u_1, \dots, u_{d+1})$ has at most $2d$ vertices.*

The statement of Theorem can be reformulated in terms of *Minkowski geometry* (that is, the geometry of finite dimensional normed spaces); see [8] and [6]. In fact, if $\{\pm u_1, \dots, \pm u_m\}$ is a set of vertices of some o -symmetric d -dimensional convex polytope B and $\mathcal{M}^d$ is the normed space whose unit ball is B , then the class $\mathcal{P}_0(u_1, \dots, u_m)$ is precisely the class of convex bodies whose *minimum width*, measured with respect to $\mathcal{M}^d$, is equal to one and whose volume is minimal. It should also be mentioned that the elements of $\mathcal{P}(u_1, \dots, u_m)$ contain all $\mathcal{M}^d$ -*reduced bodies*, which were introduced in [4]. For more information on minimum width and reduced bodies in Minkowski spaces see [4], [3], and [2].

We ask about possible extensions of Theorem for the case of a larger number of segments. More precisely, we consider $d \geq 2$, $m \geq d$, and vectors $u_1, \dots, u_m$ linearly spanning $\mathbb{R}^d$, and we pose the following problems.

1. Describe m and d such that for all $u_1, \dots, u_m$ the quantity $v(u_1, \dots, u_m)$ is the maximum of $v(u'_1, \dots, u'_d)$ over all $\{u'_1, \dots, u'_d\} \subseteq \{u_1, \dots, u_m\}$.
2. Describe m and d such that for all $u_1, \dots, u_m \in \mathbb{R}^d$ the class $\mathcal{P}_0(u_1, \dots, u_m)$ necessarily contains simplices.
3. Describe m and d such that for all $u_1, \dots, u_m$ the class $\mathcal{P}_0(u_1, \dots, u_m)$ consists of polytopes with at most $2d$ vertices.

The solutions for the case $m \leq d+1$ are provided by Theorem and the results from [7] and [5]. Problems 2 and 3 can be solved for $d = 2$. Indeed, from the main result of [1] it follows that for $d = 2$ the class $\mathcal{P}_0(u_1, \dots, u_m)$ necessarily contains triangles and, furthermore, the polygons from $\mathcal{P}_0(u_1, \dots, u_m)$ distinct from triangles are necessarily quadrilaterals. To the best of our knowledge, all the remaining cases of Problems 1–3 are open.

2. The proof

PROOF OF THEOREM. For every subset $\{u'_1, \dots, u'_d\}$ of $\{u_1, \dots, u_{d+1}\}$ the inequality $v(u'_1, \dots, u'_d) \leq v(u_1, \dots, u_{d+1})$ is obvious. Let us show that for some $u'_1, \dots, u'_d$ as above the reverse inequality is fulfilled. Without loss of generality we assume that the maximum of $v(u'_1, \dots, u'_d)$ over $\{u'_1, \dots, u'_d\} \subseteq \{u_1, \dots, u_{d+1}\}$ is attained when $u'_i = u_i$ for $i = 1, \dots, d$.

For $i = 1, \dots, d$ we replace u_i by $-u_i$ (or equivalently, translate the original segment $\text{conv}\{o, u_i\}$ by $-u_i$) arriving at the representation $u_{d+1} = \sum_{i=1}^d \beta_i u_i$ with each $\beta_i \geq 0$. Moreover, we assume that $u_1, \dots, u_d$ are ordered so that $\beta_1 \leq \dots \leq \beta_d$. We have $\beta_d \leq 1$, since otherwise $v(u_1, \dots, u_{d-1}, u_{d+1})$ would be larger than $v(u_1, \dots, u_d)$. We define the points $p_1, p_2, \dots, p_d$ by

$$p_i := \sum_{j=i}^d u_j.$$

Let us show that the simplex $T := \text{conv}\{o, p_1, \dots, p_d\}$ belongs to $\mathcal{P}_0(u_1, \dots, u_{d+1})$. We introduce the values $\alpha_1, \dots, \alpha_d$ by the formulas

$$\alpha_1 := \beta_1, \quad \alpha_i := \beta_i - \beta_{i-1} \quad (2 \leq i \leq d).$$

It is easy to verify that for every $j = 1, \dots, d$ we have $0 \leq \alpha_j \leq 1$ and $\sum_{i=1}^j \alpha_i = \beta_j \leq 1$. Now let us show that $u_{d+1} \in T$. We have

$$\begin{aligned} \left(1 - \sum_{i=1}^d \alpha_i\right) o + \sum_{i=1}^d \alpha_i p_i &= \sum_{i=1}^d \alpha_i p_i = \sum_{i=1}^d \alpha_i \sum_{j=i}^d u_j \\ &= \sum_{1 \leq i \leq j \leq d} \alpha_i u_j = \sum_{j=1}^d \left(\sum_{i=1}^j \alpha_i\right) u_j = \sum_{j=1}^d \beta_j u_j = u_{d+1}. \end{aligned}$$

Thus, u_{d+1} is a convex combination of points $o, p_1, \dots, p_d$, and hence $u_{d+1} \in T$. Obviously, T can be represented by

$$T = \text{conv} \left(\bigcup_{i=1}^{d-1} \text{conv}\{p_i, p_{i+1}\} \cup \text{conv}\{o, p_d\} \right).$$

But $\text{conv}\{o, p_d\} = \text{conv}\{o, u_d\}$, while, for every $i = 1, \dots, d-1$, the segment $\text{conv}\{p_i, p_{i+1}\}$ is a translate of $\text{conv}\{o, u_i\}$. Since $\text{conv}\{o, u_{d+1}\} \subseteq T$, we see that $T \in \mathcal{P}(u_1, \dots, u_d)$. By construction,

$$\text{vol}(T) = \frac{1}{d!} |\det(p_1, \dots, p_d)| = \frac{1}{d!} |\det(u_1, \dots, u_d)| = v(u_1, \dots, u_d).$$

Hence $v(u_1, \dots, u_d) \geq v(u_1, \dots, u_{d+1})$. This shows Part I and (since T is a simplex) also Part II of the theorem.

We prove Part III by contradiction. Assume that P is a polytope from $\mathcal{P}_0(u_1, \dots, u_{d+1})$ with at least $2d + 1$ vertices. Let $J_1, \dots, J_{d+1}$ be segments in $\mathbb{R}^d$ such that, for $i = 1, \dots, d + 1$, the segment J_i is a translate of $\text{conv}\{o, u_i\}$ and $P = \text{conv}(J_1 \cup \dots \cup J_{d+1})$. Since P has at least $2d + 1$ vertices, at least one of the endpoints of J_{d+1} is also a vertex of P . Hence, taking into account (1),

$$\text{vol}(P) > \text{vol}(\text{conv}(J_1 \cup \dots \cup J_d)) \geq \frac{1}{d!} |\det(u_1, \dots, u_d)| = \text{vol}(T),$$

and we arrive at the contradiction. This finishes the proof of Part III. $\square$

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