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Subordination properties of subclasses of p-valent functions involving certain operators

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Abstract. Using of the methods of differential subordinations, we investigate inclusion relationships among certain classes of analytic and p-valent functions, which are defined here by means of a recently defined operator.

1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in \mathbb{N} = \{1, 2, \dots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$. If f and g are analytic functions in U, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a *Schwarz function* w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathrm{U}$, such that $f(z) = g(w(z))$, $z \in \mathrm{U}$. Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathrm{U}) \subset g(\mathrm{U}).$$

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Motivated essentially by Jung et al. [JuKiSr], Shams et al. [ShKuJa] introduced the operator $\mathrm{I}_p^\alpha : A(p) \to A(p)$ as follows:

$$\text{(i)} \qquad \mathrm{I}_p^\alpha f(z) = \frac{(p+1)^\alpha}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)\mathrm{d}t, \ z \in \mathrm{U}, \quad (\alpha > 0;\ p \in \mathbb{N})$$

and

$$\text{(ii)} \quad \mathrm{I}_p^0 f(z) = f(z), \ z \in \mathrm{U}, \quad (\alpha = 0;\ p \in \mathbb{N}).$$

Note that the one-parameter family of integral operator $\mathrm{I}^\alpha \equiv \mathrm{I}_1^\alpha$ was defined by Jung et al. [JuKiSr].

For $f \in A(p)$ given by (1.1), it was shown that (see [ShKuJa])

$$\mathrm{I}_p^\alpha f(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{p+1}{k+p+1}\right)^\alpha a_{k+p} z^{k+p} \quad (\alpha \geq 0;\ p \in \mathbb{N}). \tag{1.2}$$

Using the definition (1.2), it is easy to verify the identity (see [ShKuJa])

$$z(\mathrm{I}_p^\alpha f(z))' = (p+1)\mathrm{I}_p^{\alpha-1} f(z) - \mathrm{I}_p^\alpha f(z). \tag{1.3}$$

Definition 1.1. For fixed parameters A and B, with $-1 \leq B < A \leq 1$ and $p > \eta$, we say that the function $f \in A(p)$ is in the class $S_p^\alpha(\eta; A, B)$ if it satisfies the following subordination condition:

$$\frac{1}{p-\eta}\left(\frac{z(\mathrm{I}_p^\alpha f(z))'}{\mathrm{I}_p^\alpha f(z)} - \eta\right) \prec \frac{1+Az}{1+Bz}. \tag{1.4}$$

A function f analytic in U is called to be a *convex function of order* α, $\alpha < 1$, if $f'(0) \neq 0$ and

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathrm{U}.$$

If $\alpha = 0$, then the function f is called to be *convex*.

It is easy to check that, if $h(z) = \frac{1+Az}{1+Bz}$, then $h'(0) \neq 0$ and $\mathrm{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] = \mathrm{Re}\frac{1-Bz}{1+Bz} > 0$, $z \in \mathrm{U}$, whenever $|B| \leq 1$ and $A \neq B$, hence h is convex in the unit disc.

If $B \neq -1$, from the fact that $h(\overline{z}) = \overline{h(z)}$, $z \in \mathrm{U}$, we deduce that the image $h(\mathrm{U})$ is symmetric with respect to the real axis, and that h maps the unit disc U onto the disc $\left|w - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}$. If $B = -1$, the function h maps the unit disc U onto the half plane $\mathrm{Re}\,w > \frac{1-A}{2}$, hence we obtain:

Remark 1.1. The function $f \in A(p)$ is in the class $S_p^\alpha(\eta; A, B)$ if and only if

$$\left|\frac{1}{p-\eta}\left(\frac{z(\mathrm{I}_p^\alpha f(z))'}{\mathrm{I}_p^\alpha f(z)} - \eta\right) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}, \quad z \in \mathrm{U}, \text{ for } B \neq -1, \tag{1.5}$$

and

$$\mathrm{Re}\frac{1}{p-\eta}\left(\frac{z(\mathrm{I}_p^\alpha f(z))'}{\mathrm{I}_p^\alpha f(z)} - \eta\right) > \frac{1-A}{2}, \qquad z \in \mathrm{U}, \text{ for } B = -1. \tag{1.6}$$

Denoting by $K_p^\alpha(\eta;\rho)$ the class of functions $f \in A(p)$ that satisfy the inequality

$$\mathrm{Re}\frac{1}{p-\eta}\left(\frac{z(\mathrm{I}_p^\alpha f(z))'}{\mathrm{I}_p^\alpha f(z)} - \eta\right) > \rho, \quad z \in \mathrm{U}, \tag{1.7}$$

where $\rho < 1$, from (1.5) and (1.6) it follows respectively that

$$S_p^\alpha(\eta; A, B) \subset K_p^\alpha\left(\eta, \frac{1-A}{1-B}\right),$$

and

$$S_p^\alpha(\eta; A, B) = K_p^\alpha\left(\eta, \frac{1-A}{1-B}\right) \Longleftrightarrow B = -1.$$

Let us consider the first-order differential subordination

$$H(\varphi(z), z\varphi'(z)) \prec h(z).$$

A univalent function q is called its dominant, if $\varphi(z) \prec q(z)$ for all analytic functions φ that satisfy this differential subordination. A dominant $\widetilde{q}$ is called the best dominant, if $\widetilde{q}(z) \prec q(z)$ for all dominants q. For the general theory of the first-order differential subordination and its applications, we refer the reader to [Bu] and [MiMo].

The object of the present paper is to obtain several inclusion relationships and other interesting properties of functions belonging to the subclass $S_p^\alpha(\eta; A, B)$ and $K_p^\alpha(\eta;\rho)$ by using the method of differential subordination.

2. Preliminaries

To establish our main results, we shall require the following lemmas. The first one deals with the *Briot–Bouquet differential subordinations.*

Lemma 2.1 ([EeMiMoRe]). *Let $\beta, \gamma \in \mathbb{C}$, and let h be a convex function with*

$$\mathrm{Re}[\beta h(z) + \gamma] > 0, \ z \in \mathrm{U}.$$

If p is analytic in U, *with $p(0) = h(0)$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

The class of *starlike (and normalized) functions of order* α in U, $\alpha < 1$, is

$$S^*(\alpha) = \left\{ f \in A(1) : \mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha, \ z \in \mathrm{U} \right\}.$$

In particular, the class $S^*(0) \equiv S^*$ is the class of *starlike (normalized) functions.*

For complex numbers a, b and c, the *Gauss hypergeometric function* is defined by

$$\begin{aligned} {}_2F_1(a, b, c; z) &= 1 + \frac{a \cdot b}{c}\frac{z}{1!} + \frac{a(a+1) \cdot b(b+1)}{c(c+1)}\frac{z^2}{2!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k}\frac{z^k}{k!}, \quad a, b \in \mathbb{C}, \ c \in \mathbb{C} \setminus \{0, -1, -2, \dots\}, \end{aligned} \tag{2.1}$$

where $(d)_k = d(d+1)\dots(d+k-1)$ and $(d)_0 = 1$. The series (2.1) converges absolutely for $z \in \mathrm{U}$, hence it represents an analytic function in U (see [WhWa, Chapter 14]).

Lemma 2.2 ([MoRiSe]). *Let $\beta > 0$, $\beta + \gamma > 0$ and consider the integral operator $\mathrm{J}_{\beta,\gamma}$ defined by*

$$\mathrm{J}_{\beta,\gamma}(f)(z) = \left[\frac{\beta + \gamma}{z^\gamma} \int_0^z f^\beta(t) t^{\gamma - 1} \mathrm{d}t \right]^{\frac{1}{\beta}},$$

where the powers are the principal ones.

If $\sigma \in \left[-\frac{\gamma}{\beta}, 1 \right)$ then the order of starlikeness of the class $\mathrm{J}_{\beta,\gamma}(S^*(\sigma))$, i.e. the largest number $\delta(\sigma; \beta, \gamma)$ such that $\mathrm{J}_{\beta,\gamma}(S^*(\sigma)) \subset S^*(\delta)$, is given by the number $\delta(\sigma; \beta, \gamma) = \inf\{\mathrm{Re} q(z) : z \in \mathrm{U}\}$, where

$$q(z) = \frac{1}{\beta Q(z)} - \frac{\gamma}{\beta} \quad \text{and} \quad Q(z) = \int_0^1 \left(\frac{1-z}{1-tz} \right)^{2\beta(1-\sigma)} t^{\beta + \gamma - 1} \mathrm{d}t.$$

Moreover, if $\sigma \in [\sigma_0, 1)$, *where* $\sigma_0 = \max\left\{\frac{\beta-\gamma-1}{2\beta}; -\frac{\gamma}{\beta}\right\}$ *and* $g = \mathrm{J}_{\beta,\gamma}(f)$ *with* $f \in S^*(\sigma)$, *then*

$$\mathrm{Re}\frac{zg'(z)}{g(z)} > \delta(\sigma;\beta,\gamma), \quad z \in \mathrm{U},$$

where

$$\delta(\sigma;\beta,\gamma) = \frac{1}{\beta}\left[\frac{\beta+\gamma}{{}_2F_1(1, 2\beta(1-\sigma), \beta+\gamma+1; \frac{1}{2})} - \gamma\right].$$

Lemma 2.3 ([ObOw]). *Let* ϕ *be analytic in* U *with* $\phi(0) = 1$ *and* $\phi(z) \neq 0$ *for* $0 < |z| < 1$, *and let* $A, B \in \mathbb{C}$ *with* $A \neq B$, $|B| \leq 1$.

(i) *Let* $B \neq 0$ *and* $\gamma \in \mathbb{C} \setminus \{0\}$ *satisfy either* $\left|\frac{\gamma(A-B)}{B} - 1\right| \leq 1$ *or* $\left|\frac{\gamma(A-B)}{B} + 1\right| \leq 1$. *If* ϕ *satisfies*

$$1 + \frac{z\phi'(z)}{\gamma\phi(z)} \prec \frac{1+Az}{1+Bz}$$

then

$$\phi(z) \prec (1+Bz)^{\frac{\gamma(A-B)}{B}},$$

and this is the best dominant.

(ii) *Let* $B = 0$ *and* $\gamma \in \mathbb{C} \setminus \{0\}$ *be such that* $|\gamma A| < \pi$, *and if* ϕ *satisfies*

$$1 + \frac{z\phi'(z)}{\gamma\phi(z)} \prec \frac{1+Az}{1+Bz},$$

then

$$\phi(z) \prec e^{\gamma A z}$$

and this is the best dominant.

3. Inclusion relationships for the classes $S_p^\alpha(\eta; A, B)$ and $K_p^\alpha(\eta; \rho)$

Unless otherwise mentioned, we assume throughout this paper that $-1 \leq B < A \leq 1$, $p > \eta$, $\alpha \geq 0$ and $p \in \mathbb{N}$.

Theorem 3.1. *Let* $p - \eta > 0$, *and*

$$(p-\eta)(1-A) + (1+\eta)(1-B) \geq 0. \tag{3.1}$$

1. Supposing that $\mathrm{I}_p^\alpha f(z) \neq 0$ for all $z \in \dot{\mathrm{U}} = \mathrm{U} \setminus \{0\}$, then

$$S_p^{\alpha-1}(\eta; A, B) \subset S_p^\alpha(\eta; A, B).$$

2. Moreover, if we suppose in addition that

$$A \leq 1 + \frac{1-B}{p-\eta} \min\left\{\frac{2+2\eta-p}{2}; \eta+1\right\}, \tag{3.2}$$

then

$$S_p^{\alpha-1}(\eta; A, B) \subset K_p^{\alpha}(\eta; \rho(A, B)),$$

where the bound

$$\rho(A, B) = \frac{1}{p-\eta}\left[\frac{p+1}{{}_2F_1(1, 2(p-\eta)(A-B)/(1-B), p+2; \frac{1}{2})} - (\eta+1)\right], \tag{3.3}$$

is the best possible.

Proof. Let $f \in S_p^{\alpha-1}(\eta; A, B)$, and put

$$g(z) = z\left(\frac{\mathrm{I}_p^{\alpha} f(z)}{z^p}\right)^{\frac{1}{p-\eta}}, \quad z \in \mathrm{U}, \tag{3.4}$$

where the power is the principal one. Since $\mathrm{I}_p^{\alpha} f(z) \neq 0$ for all $z \in \dot{\mathrm{U}}$, the function g is analytic in U, with $g(0) = 0$ and $g'(0) = 1$. Taking the logarithmic differentiation in (3.4), we have

$$\phi(z) = \frac{zg'(z)}{g(z)} = \frac{1}{p-\eta}\left(\frac{z(\mathrm{I}_p^{\alpha} f(z))'}{\mathrm{I}_p^{\alpha} f(z)} - \eta\right), \quad z \in \mathrm{U}, \tag{3.5}$$

then, using the identity (1.3) in (3.5), we obtain

$$(p+1)\frac{\mathrm{I}_p^{\alpha-1} f(z)}{\mathrm{I}_p^{\alpha} f(z)} = (p-\eta)\phi(z) + (1+\eta). \tag{3.6}$$

Logarithmical differentiating in both sides of (3.6), and multiplying by z, we have

$$\frac{1}{p-\eta}\left(\frac{z(\mathrm{I}_p^{\alpha-1} f(z))'}{\mathrm{I}_p^{\alpha-1} f(z)} - \eta\right) = \phi(z) + \frac{z\phi'(z)}{(p-\eta)\phi(z)+1+\eta}. \tag{3.7}$$

Combining (3.7) together with $f \in S_p^{\alpha-1}(\eta; A, B)$, we obtain that the function ϕ satisfies the Briot–Bouquet differential subordination

$$\phi(z) + \frac{z\phi'(z)}{(p-\eta)\phi(z)+\eta+1} \prec \frac{1+Az}{1+Bz} \equiv h(z).$$

Now we will use Lemma 2.1 for the special case $\beta = p - \eta$ and $\gamma = \eta + 1$. Since h is a convex function in U, a simple computation shows that

$$\mathrm{Re}\left[(p-\eta)h(z) + \eta + 1\right] > 0, \quad z \in \mathrm{U},$$

whenever (3.1) holds, then we have $\phi(z) \prec h(z)$, i.e. $f \in S_p^{\alpha}(\eta; A, B)$. If, in addition, we suppose that the inequality (3.2) holds, then all the assumptions of Lemma 2.2 are verified for the above values of β, γ and $\sigma = \frac{1-A}{1-B}$. Then it follows the inclusion $S_p^{\alpha-1}(\eta; A, B) \subset K_p^{\alpha}(\eta; \rho(A, B))$, where the bound $\rho(A, B)$ given by (3.3) is the best possible. □

From the above Theorem, according to the definitions (1.4) and (1.7), we deduce the next inclusions:

Corollary 3.1. *Let $p > \eta$ such that* (3.1)) *holds.*

1. *Suppose that* $\mathrm{I}_p^{\alpha} f(z) \neq 0$ *for all* $z \in \dot{\mathrm{U}}$, *then*

$$S_p^{\alpha-1}(\eta; A, B) \subset S_p^{\alpha}(\eta; A, B) \subset K_p^{\alpha}\left(\eta; \frac{1-A}{1-B}\right).$$

2. *If we suppose in addition that* (3.2) *holds, then*

$$S_p^{\alpha-1}(\eta; A, B) \subset S_p^{\alpha}(\eta; A, B) \subset K_p^{\alpha}(\eta; \rho(A, B)),$$

where $\rho(A, B)$ *is given by* (3.3). *As a consequence of the last inclusion, we have* $\rho(A, B) \geq \frac{1-A}{1-B}$.

For the special case $B = -1$, Theorem 3.1 reduces to:

Corollary 3.2. *Let $p > \eta$ and $a > -\frac{\eta+1}{p-\eta}$.*

1. *Suppose that* $\mathrm{I}_p^{\alpha} f(z) \neq 0$ *for all* $z \in \dot{\mathrm{U}}$, *then*

$$K_p^{\alpha-1}(\eta; a) \subset K_p^{\alpha}(\eta; a).$$

2. *If we suppose in addition that*

$$a \geq \max\left\{-\frac{2+2\eta-p}{2(p-\eta)}; -\frac{\eta+1}{p-\eta}\right\},$$

then

$$K_p^{\alpha-1}(\eta; a) \subset K_p^{\alpha}(\eta; \rho(a)),$$

where the bound

$$\rho(a) = \frac{1}{p-\eta}\left[\frac{p+1}{{}_2F_1\left(1, 2(p-\eta)(1-a), p+2; \frac{1}{2}\right)} - (\eta+1)\right],$$

is the best possible.

Theorem 3.2. *If $f \in K_p^{\alpha}(\eta;\rho)$, where $\rho < 1$ and $p > \eta$, then $f \in K_p^{\alpha-1}(\eta;\rho)$ for $|z| < R(p,\alpha,\eta;\rho)$, where*

$$R(p,\alpha,\eta;\rho) = \min\{r > 0 : \theta(r) = 0\} \tag{3.8}$$

and

$$\theta(r) = \frac{2r}{(1-r)(p-\eta)\left|(1-\rho)(1-r) - \left|\rho + \frac{\eta+1}{p-\eta}\right|(1+r)\right|}.$$

Proof. Since $f \in K_p^{\alpha}(\eta;\rho)$, the function u given by

$$u(z) = \frac{1}{p-\eta}\left(\frac{z(\mathrm{I}_p^{\alpha} f(z))'}{\mathrm{I}_p^{\alpha} f(z)} - \eta\right), \tag{3.9}$$

is analytic in U with $u(0) = 1$ and $\mathrm{Re}\,u(z) > \rho$. Using the identity (1.3) in (3.9) and taking the logarithmic differentiation in the resulting equation, we obtain

$$\frac{1}{p-\eta}\left(\frac{z(\mathrm{I}_p^{\alpha-1} f(z))'}{\mathrm{I}_p^{\alpha-1} f(z)} - \eta\right) = u(z) + \frac{zu'(z)}{(\eta+1) + (p-\eta)u(z)}. \tag{3.10}$$

If we denote $k(z) = \frac{u(z)-\rho}{1-\rho}$, then $k(0) = 1$ and $\mathrm{Re}\,k(z) > 0$, $z \in$ U, and substituting in (3.10) we obtain

$$\frac{1}{p-\eta}\left(\frac{z(\mathrm{I}_p^{\alpha-1} f(z))'}{\mathrm{I}_p^{\alpha-1} f(z)} - \eta\right) - \rho = (1-\rho)\left[k(z) + \frac{zk'(z)}{(p-\eta)(1-\rho)k(z) + \rho(p-\eta) + \eta + 1}\right],$$

hence

$$\mathrm{Re}\frac{1}{p-\eta}\left(\frac{z(\mathrm{I}_p^{\alpha-1} f(z))'}{\mathrm{I}_p^{\alpha-1} f(z)} - \eta\right) - \rho \geq (1-\rho)\left[\mathrm{Re}\,k(z) - \frac{|zk'(z)|}{(p-\eta)\left|(1-\rho)k(z) - \left|\rho + \frac{\eta+1}{p-\eta}\right|\right|}\right]. \tag{3.11}$$

By using the well-known results [McGr]

$$|zk'(z)| \leq \frac{2r}{1-r^2}\mathrm{Re}\,k(z) \text{ and } \mathrm{Re}\,k(z) \geq \frac{1-r}{1+r}, \quad |z| = r < 1,$$

together with the inequality (3.11), we get

$$\mathrm{Re}\frac{1}{p-\eta}\left(\frac{z(\mathrm{I}_p^{\alpha-1}f(z))'}{\mathrm{I}_p^{\alpha-1}f(z)}-\eta\right)-\rho\geq(1-\rho)[1-\theta(r)]\mathrm{Re}k(z),\quad |z|=r. \tag{3.12}$$

Since the right hand side term of the inequality (3.12) is nonnegative whenever $|z|\leq R(p,\alpha,\eta;\rho)$, where $R(p,\alpha,\eta;\rho)$ is given by (3.8), using the fact that the real part of an analytic function is harmonic, we deduce that $f\in K_p^{\alpha-1}(\eta;\rho)$ for $|z|<R(p,\alpha,\eta;\rho)$. □

For a function $f\in A(p)$, let the integral operator $\mathrm{F}_{\delta,p}:A(p)\to A(p)$ defined by [ChSaSr]

$$\begin{aligned}\mathrm{F}_{\delta,p}(f(z))&=\frac{\delta+p}{z^{\delta}}\int_0^z t^{\delta-1}f(t)\mathrm{d}t=z^p+\sum_{k=1}^{\infty}\frac{\delta+p}{\delta+p+k}a_{p+k}z^{p+k}\\&=\left(z^p+\sum_{k=1}^{\infty}\frac{\delta+p}{\delta+p+k}z^{p+k}\right)*f(z)\\&=z^p{}_2\mathrm{F}_1(1,\delta+p;\delta+p+1;z)*f(z),\quad z\in\mathrm{U},\ (\delta>-p),\end{aligned} \tag{3.13}$$

where the symbol "$*$" represents the well-known *Hadamard convolution product* of the power series.

From (1.2) and (3.13), we have

$$z\left(\mathrm{I}_p^{\alpha}\mathrm{F}_{\delta,p}(f(z))\right)'=(\delta+p)\mathrm{I}_p^{\alpha}f(z)-\delta\mathrm{I}_p^{\alpha}\mathrm{F}_{\delta,p}(f(z)),\ z\in\mathrm{U}, \tag{3.14}$$

and

$$\mathrm{I}_p^{\alpha}\mathrm{F}_{\delta,p}(f(z))=\mathrm{F}_{\delta,p}(\mathrm{I}_p^{\alpha}f(z)),\quad f\in A(p).$$

We now prove the next result:

Theorem 3.3. *Let* $p>\eta$, $p+\delta>0$ *and*

$$(1-B)(\delta+\eta)+(1-A)(p-\eta)\geq 0. \tag{3.15}$$

(i) *Supposing that* $\mathrm{F}_{\delta,p}(f(z))\neq 0$ *for all* $z\in\dot{\mathrm{U}}$, *then*

$$\mathrm{F}_{\delta,p}(S_p^{\alpha}(\eta;A,B))\subset S_p^{\alpha}(\eta;A,B).$$

(ii) *Moreover, if we suppose in addition that*

$$A\leq 1+\frac{1-B}{p-\eta}\min\left\{\frac{\delta+2\eta-p+1}{2};\delta+\eta\right\}, \tag{3.16}$$

then

$$\mathrm{F}_{\delta,p}(S_p^\alpha(\eta;A,B)) \subset K_p^\alpha(\eta;r(A,B)),$$

where the bound

$$r(A,B) = \frac{1}{p-\eta}\left[\frac{\delta+p}{{}_2F_1\left(1, \frac{2(p-\eta)(A-B)}{1-B}, \delta+p+1; \frac{1}{2}\right)} - (\delta+\eta)\right], \tag{3.17}$$

is the best possible.

Proof. Let $f \in S_p^\alpha(\eta;A,B)$, and suppose that $\mathrm{F}_{\delta,p}(f(z)) \neq 0$ for all $z \in \dot{\mathrm{U}}$. Let

$$g(z) = z\left(\frac{\mathrm{I}_p^\alpha \mathrm{F}_{\delta,p}(f(z))}{z^p}\right)^{\frac{1}{p-\eta}}, \quad z \in \mathrm{U}, \tag{3.18}$$

where the power is the principal one, then g is analytic in U, with $g(0) = 0$ and $g'(0) = 1$. Differentiating (3.18) logarithmically with respect to z, we have

$$\phi(z) = \frac{zg'(z)}{g(z)} = \frac{1}{p-\eta}\left(\frac{z\left(\mathrm{I}_p^\alpha \mathrm{F}_{\delta,p}(f(z))\right)'}{\mathrm{I}_p^\alpha \mathrm{F}_{\delta,p}(f(z))} - \eta\right), \quad z \in \mathrm{U}. \tag{3.19}$$

Now, by using the differential formula (3.14) in (3.19), we obtain

$$(p+\delta)\frac{\mathrm{I}_p^\alpha f(z)}{\mathrm{I}_p^\alpha \mathrm{F}_{\delta,p}(f(z))} = (p-\eta)\phi(z) + (\delta+\eta). \tag{3.20}$$

Differentiating logarithmically the relation (3.20) and multiplying by z, we have

$$\frac{1}{p-\eta}\left(\frac{z\left(\mathrm{I}_p^\alpha f(z)\right)'}{\mathrm{I}_p^\alpha f(z)} - \eta\right) = \phi(z) + \frac{z\phi'(z)}{(p-\eta)\phi(z) + (\delta+\eta)}. \tag{3.21}$$

Since $f \in S_p^\alpha(\eta;A,B)$, from (3.21), we obtain that the function ϕ satisfies the Briot–Bouquet differential subordination

$$\phi(z) + \frac{z\phi'(z)}{(p-\eta)\phi(z)+(\delta+\eta)} \prec \frac{1+Az}{1+Bz} \equiv h(z).$$

The function h is convex in U, and it is easy to check that

$$\operatorname{Re}\left[(p-\eta)h(z) + (\delta+\eta)\right] > 0, \quad z \in \mathrm{U},$$

whenever (3.15) holds, then from Lemma 2.1, with $\beta = p-\eta$ and $\gamma = \delta+\eta$, we have $\phi(z) \prec h(z)$, that is, that $\mathrm{F}_{\delta,p} \in S_p^\alpha(\eta;A,B)$. If we suppose in addition that the inequality (3.16) holds, then all the assumptions of the Lemma 2.2 are satisfied for β,γ and $\sigma = \frac{1-A}{1-B}$, hence it follows the inclusion $\mathrm{F}_{\delta,p}(S_p^\alpha(\eta;A,B)) \subset K_p^\alpha(\eta;r(A,B))$, and the bound $r(A,B)$ given by (3.17) is the best possible. $\square$

Taking $B = -1$ in Theorem 3.3, we obtain the next Corollary:

Corollary 3.3. *Let* $p > \eta$, $p + \delta > 0$ *and* $a \geq -\frac{\delta+\eta}{p-\eta}$.

(i) *Supposing that* $\mathrm{F}_{\delta,p} f(z) \neq 0$ *for all* $z \in \dot{\mathrm{U}}$, *then*

$$\mathrm{F}_{\delta,p}(K_p^\alpha(\eta; a)) \subset K_p^\alpha(\eta; a).$$

(ii) *If we suppose in addition that*

$$a \geq \max\left\{-\frac{\delta + 2\eta - p + 1}{2(p-\eta)}; -\frac{\delta+\eta}{p-\eta}\right\},$$

then

$$\mathrm{F}_{\delta,p}(K_p^\alpha(\eta; a)) \subset K_p^\alpha(\eta; r(a)),$$

where the bound

$$r(a) = \frac{1}{p-\eta}\left[\frac{\delta+p}{{}_2F_1\left(1, 2(p-\eta)(1-a), \delta+p+1; \frac{1}{2}\right)} - (\delta+\eta)\right],$$

is the best possible.

Theorem 3.4. *Let* $\alpha \geq 0$, $\nu \in \mathbb{C} \setminus \{0\}$ *and let* $A, B \in \mathbb{C}$ *with* $A \neq B$ *and* $|B| \leq 1$. *Suppose that*

$$\begin{array}{ll} \left|\dfrac{\nu(p+1)(A-B)}{B} - 1\right| \leq 1 \quad \text{or} \quad \left|\dfrac{\nu(p+1)(A-B)}{B} + 1\right| \leq 1, & \text{if } B \neq 0, \\ |\nu| \leq \dfrac{\pi}{p+1}, & \text{if } B = 0. \end{array}$$

If $f \in A(p)$ *with* $\mathrm{I}_p^\alpha f(z) \neq 0$ *for all* $z \in \dot{\mathrm{U}}$, *then*

$$\frac{\mathrm{I}_p^{\alpha-1} f(z)}{\mathrm{I}_p^\alpha f(z)} \prec \frac{1+Az}{1+Bz}$$

implies

$$\left(\frac{\mathrm{I}_p^\alpha f(z)}{z^p}\right)^\nu \prec q_1(z),$$

where

$$q_1(z) = \begin{cases} (1+Bz)^{\nu(p+1)(A-B)/B}, & \text{if } B \neq 0, \\ e^{\nu(p+1)Az}, & \text{if } B = 0, \end{cases}$$

is the best dominant. (All the powers are the principal ones).

Proof. Let us put

$$\phi(z)=\left(\frac{\mathrm{I}_p^{\alpha}f(z)}{z^p}\right)^{\nu},\quad z\in\mathrm{U},\tag{3.22}$$

where the power is the principal one. Then ϕ is analytic in U, $\phi(0)=1$ and $\phi(z)\neq 0$ for all $z\in\mathrm{U}$. Taking the logarithmic derivatives in both sides of (3.22) and using the identity (1.3) we have

$$1+\frac{z\phi'(z)}{\nu(p+1)\phi(z)}=\frac{\mathrm{I}_p^{\alpha-1}f(z)}{\mathrm{I}_p^{\alpha}f(z)}\prec\frac{1+Az}{1+Bz}.$$

Now the assertions of Theorem 3.4 follows by using Lemma 2.3 for the special case $\gamma=\nu(p+1)$. □

Putting $B=-1$ and $A=1-2\rho$, $0\le\rho<1$, in Theorem 3.4, we obtain the following result:

Corollary 3.4. *Assume that $\alpha\ge 0$ and $\nu\in\mathbb{C}\setminus\{0\}$ satisfies either*

$$|2\nu(p+1)(1-\rho)-1|\le 1\quad \text{or}\quad |2\nu(p+1)(1-\rho)+1|\le 1.$$

If $f\in A(p)$ with $\mathrm{I}_p^{\alpha}f(z)\neq 0$ for $z\in\dot{\mathrm{U}}$, then

$$\operatorname{Re}\frac{\mathrm{I}_p^{\alpha-1}f(z)}{\mathrm{I}_p^{\alpha}f(z)}>\rho,\quad z\in\mathrm{U},$$

implies

$$\left(\frac{\mathrm{I}_p^{\alpha}f(z)}{z^p}\right)^{\nu}\prec q_2(z)=(1-z)^{-2\nu(p+1)(1-\rho)},$$

and q_2 is the best dominant. (The power is the principal one).

4. Properties involving the operator $\mathbf{I}_p^{\alpha}$

Theorem 4.1. *If $f\in S_p^{\alpha}(\eta;A,B)$, then, for all $s,t\in\mathbb{C}$ with $|s|\le 1$, $|t|\le 1$, and $s\neq t$, the next subordination holds:*

$$\frac{t^p\mathrm{I}_p^{\alpha}f(zs)}{s^p\mathrm{I}_p^{\alpha}f(zt)}\prec\begin{cases}\left(\dfrac{1+Bzs}{1+Bzt}\right)^{(p-\eta)(A-B)/B}, & \text{for } B\neq 0,\\ \exp[(p-\eta)Az(s-t)], & \text{for } B=0.\end{cases}\tag{4.1}$$

Proof. If $f \in S_p^{\alpha}(\eta; A, B)$, from (1.4) it follows that

$$\frac{z(\mathrm{I}_p^{\alpha} f(z))'}{\mathrm{I}_p^{\alpha} f(z)} \prec \frac{p + [pB + (A-B)(p-\eta)]z}{1+Bz} \equiv k(z). \tag{4.2}$$

Moreover, the function k defined by (4.2) and the function h given by

$$h(z) \equiv h(z; s, t) = \int_0^z \left(\frac{s}{1-su} - \frac{t}{1-tu} \right) \mathrm{d}u$$

are convex in U. By combining a general subordination theorem [RuShSm, Theorem 4] with (4.2), we get

$$\left(\frac{z(\mathrm{I}_p^{\alpha} f(z))'}{\mathrm{I}_p^{\alpha} f(z)} - p \right) * h(z) \prec \frac{(p-\eta)(A-B)z}{1+Bz} * h(z). \tag{4.3}$$

For every analytic function ϕ in U with $\phi(0) = 0$, we have

$$\phi(z) * h(z) = \int_{tz}^{sz} \frac{\phi(u)}{u} \mathrm{d}u, \tag{4.4}$$

and thus, from (4.3) and (4.4), we deduce

$$\int_{tz}^{sz} \left(\frac{(\mathrm{I}_p^{\alpha} f(u))'}{\mathrm{I}_p^{\alpha} f(u)} - \frac{p}{u} \right) \mathrm{d}u \prec (p-\eta)(A-B) \int_{tz}^{sz} \frac{\mathrm{d}u}{1+Bu}.$$

This last subordination implies

$$\exp \left(\int_{tz}^{sz} \left(\frac{(\mathrm{I}_p^{\alpha} f(u))'}{\mathrm{I}_p^{\alpha} f(u)} - \frac{p}{u} \right) \mathrm{d}u \right) \prec \exp \left((p-\eta)(A-B) \int_{tz}^{sz} \frac{\mathrm{d}u}{1+Bu} \right),$$

and by simplification, we get the assertion of Theorem 4.1. □

Corollary 4.1. *If $f \in S_p^{\alpha}(\eta; A, B)$, then for $|z| = r < 1$, the next inequalities hold:*

$$\left| \mathrm{I}_p^{\alpha} f(z) \right| \le \begin{cases} r^p (1+Br)^{(p-\eta)(A-B)/B}, & \text{for } B \neq 0, \\ r^p \exp[(p-\eta)Ar], & \text{for } B = 0, \end{cases} \tag{4.5}$$

$$\left| \mathrm{I}_p^{\alpha} f(z) \right| \ge \begin{cases} r^p (1-Br)^{(p-\eta)(A-B)/B}, & \text{for } B \neq 0, \\ r^p \exp[-(p-\eta)Ar], & \text{for } B = 0, \end{cases} \tag{4.6}$$

and

$$\left| \arg \frac{\mathrm{I}_p^{\alpha} f(z)}{z^p} \right| \le \begin{cases} \dfrac{(p-\eta)(A-B)}{|B|} \sin^{-1}(|B|\, r), & \text{for } B \neq 0, \\ (p-\eta)Ar, & \text{for } B = 0. \end{cases} \tag{4.7}$$

All of the estimates asserted here are sharp.

Proof. Taking $s=1$ and $t=0$ in (4.1), and using the definition of subordination, we obtain

$$\frac{\mathrm{I}_p^\alpha f(z)}{z^p}=\begin{cases}(1+Bw(z))^{(p-\eta)(A-B)/B}, & \text{for } B\neq 0,\\ \exp[(p-\eta)Aw(z)], & \text{for } B=0,\end{cases}\tag{4.8}$$

where w is analytic function in U with $w(0)=0$ and $|w(z)|\leq 1$ for $z\in$ U. According to the well-known *Schwarz's Theorem*, we have $|w(z)|\leq |z|$ for all $z\in$ U.

(i) If $B>0$, then we find from (4.8) that

$$\left|\frac{\mathrm{I}_p^\alpha f(z)}{z^p}\right|=\exp\left[\frac{(p-\eta)(A-B)}{B}\log|1+Bw(z)|\right]$$
$$=|1+Bw(z)|^{\frac{(p-\eta)(A-B)}{B}}\leq(1+Br)^{\frac{(p-\eta)(A-B)}{B}}.$$

(ii) If $B<0$, we can easily obtain

$$\left|\frac{\mathrm{I}_p^\alpha f(z)}{z^p}\right|=|1+Bw(z)|^{\frac{(p-\eta)(A-B)}{-B}}\leq\left[(1+Br)^{-1}\right]^{\frac{(p-\eta)(A-B)}{-B}}=(1+Br)^{\frac{(p-\eta)(A-B)}{B}}.$$

This proves the inequality (4.5) for $B\neq 0$. Similarly, we can prove the other inequalities in (4.5) and (4.6). Now, for $|z|=r$ and $B\neq 0$, we observe from (4.8) that

$$\left|\arg\frac{\mathrm{I}_p^\alpha f(z)}{z^p}\right|=\frac{(p-\eta)(A-B)}{|B|}\left|\arg(1+Bw(z))\right|\leq\frac{(p-\eta)(A-B)}{|B|}\sin^{-1}(|B|\,r),$$

and, for $B=0$, (4.7) is a direct consequence of (4.8).

It is easy to see that all of the estimates in Corollary 4.1 are sharp, being attained by the function f_0 defined by

$$\mathrm{I}_p^\alpha f_0(z)=\begin{cases}z^p(1+Bz)^{(p-\eta)(A-B)/B}, & \text{for } B\neq 0,\\ z^p\exp\left[(p-\eta)Az\right], & \text{for } B=0.\end{cases}\tag{4.9}$$

$\square$

Corollary 4.2. *If $f\in S_p^\alpha(\eta;A,B)$, then, for all $|z|=r<1$, the next inequalities hold:*

$$\left|(\mathrm{I}_p^\alpha f(z))'\right|$$
$$\leq\begin{cases}r^{p-1}\left\{p+[\eta B+(p-\eta)A]r\right\}(1+Br)^{\frac{(p-\eta)(A-B)}{B}-1}, & \text{for } B\neq 0,\\ r^{p-1}\left[p+(p-\eta)Ar\right]\exp((p-\eta)Ar), & \text{for } B=0,\end{cases}\tag{4.10}$$

$$\left|(\mathrm{I}_p^{\alpha} f(z))'\right|$$

$$\geq \begin{cases} r^{p-1}\left\{p-[\eta B+(p-\eta)A]r\right\}(1-Br)^{\frac{(p-\eta)(A-B)}{B}-1}, & \text{for } B \neq 0, \\ r^{p-1}\left[p-(p-\eta)Ar\right]\exp(-(p-\eta)Ar), & \text{for } B = 0, \end{cases} \tag{4.11}$$

and

$$\left|\arg \frac{(\mathrm{I}_p^{\alpha} f(z))'}{z^{p-1}}\right| \leq \begin{cases} \dfrac{(p-\eta)(A-B)}{|B|}\sin^{-1}(|B|r) \\ \quad + \sin^{-1}\left[\dfrac{(p-\eta)(A-B)r}{p-[\eta B+(p-\eta)A]Br^2}\right], & \text{for } B \neq 0, \\ (p-\eta)Ar + \sin^{-1}\left[\dfrac{A(p-\eta)r}{p}\right], & \text{for } B = 0. \end{cases} \tag{4.12}$$

All of the estimates asserted here are sharp.

Proof. If we let

$$g(z) = \frac{z(\mathrm{I}_p^{\alpha} f(z))'}{\mathrm{I}_p^{\alpha} f(z)}, \quad z \in \mathrm{U}, \tag{4.13}$$

then g is analytic in U with $g(0) = p$, and

$$g(z) \prec \frac{p+[pB+(A-B)(p-\eta)]z}{1+Bz}.$$

It is known from [Ao] that the function g satisfies the following sharp inequalities:

$$\frac{p-[\eta B+(p-\eta)A]r}{1-Br} \leq |g(z)| \leq \frac{p+[\eta B+(p-\eta)A]r}{1+Br}, \quad |z| = r < 1, \tag{4.14}$$

$$\left|g(z) - \frac{p-[\eta B+(p-\eta)A]Br^2}{1-B^2r^2}\right| \leq \frac{(A-B)(p-\eta)r}{1-B^2r^2}, \quad |z| = r < 1, \tag{4.15}$$

and

$$|\arg g(z)| \leq \sin^{-1}\left[\frac{(A-B)(p-\eta)r}{p-[\eta B+(p-\eta)A]Br^2}\right], \quad |z| = r < 1. \tag{4.16}$$

Using (4.14), (4.15) and (4.16), in conjunction with the estimates given by Corollary 4.2, in (4.13), we deduce the estimates (4.10), (4.11) and (4.12) of Corollary 4.2. All of the estimates are sharp for the function f_0 defined by (4.9). $\square$

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